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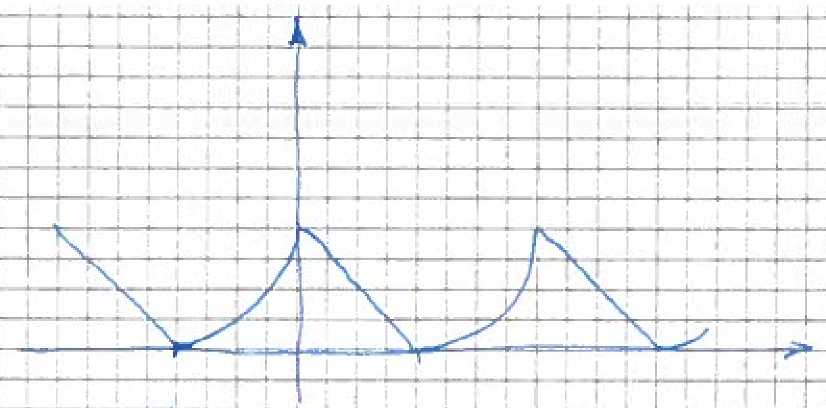

Grafico di f

Osserviamo che

$$\int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{1}{\pi^2}\int_{-\pi}^{0}(t+\pi)^4 dt + \int_0^{\pi}(\pi-t)^2 dt$$

$$= \frac{1}{\pi^2}\left.\frac{(t+\pi)^5}{5}\right|_{-\pi}^{0} - \left.\frac{(\pi-t)^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{\pi^2}\frac{\pi^5}{5} + \frac{\pi^3}{3} = \frac{3+5}{15}\pi^3 = \frac{8}{15}\pi^3 < +\infty$$

Quindi, poiché $f \in L^2(-\pi,\pi)$, concludiamo che f è sviluppabile in serie di Fourier

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\left[\frac{1}{\pi}\int_{-\pi}^{0}(t+\pi)^2 dt + \int_0^{\pi}(\pi-t)dt\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{\pi}\left.\frac{(t+\pi)^3}{3}\right|_{-\pi}^{0} - \frac{1}{2}(\pi-t)^2\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{3} + \frac{\pi^2}{2}\right] = \frac{5}{6}\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{1}{\pi}(t+\pi)^2\cos nt\,dt + \right.$$

$$\left. + \int_0^{\pi}(\pi-t)\cos nt\,dt\right] = \frac{1}{\pi}\left[\cancel{\frac{1}{\pi}(t+\pi)^2\frac{\sin nt}{n}\Big|_{-\pi}^{0}} + \right.$$

$$- \frac{2}{\pi}\int_{-\pi}^{0}(t+\pi)\frac{\sin nt}{n}dt + \cancel{(\pi-t)\frac{\sin nt}{n}\Big|_0^{\pi}} + \left.\int_0^{\pi}\frac{\sin nt}{n}dt\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{\pi}(t+\pi)\frac{\cos nt}{n^2}\Big|_{-\pi}^{0} - \cancel{\frac{2}{\pi}\int_{-\pi}^{0}\frac{\cos nt}{n^2}dt} - \frac{\cos nt}{n^2}\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{\cancel{\pi}}\cancel{\pi}\frac{1}{n^2} - \frac{(-)^n}{n^2} + \frac{1}{n^2}\right] = \frac{3-(-)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{(t+\pi)^2}{\pi}\sin nt\,dt +\right.$$

$$\left.+\int_0^{\pi}(\pi-t)\sin nt\,dt\right] = \frac{1}{\pi}\left[-\frac{(t+\pi)^2}{\pi}\frac{\cos nt}{n}\Big|_{-\pi}^{0} +\right.$$

$$\left.+\int_{-\pi}^{0}\frac{2}{\pi}(t+\pi)\frac{\cos nt}{n}\,dt - (\pi-t)\frac{\cos nt}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\cos nt}{n}\,dt\right]$$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\pi}{n}} + \cancel{\frac{2}{\pi}(t+\pi)\frac{\sin nt}{n^2}\Big|_{-\pi}^{0}} - \frac{2}{\pi}\int_{-\pi}^{0}\frac{\sin nt}{n^2}\,dt\right.$$

$$\left.+\cancel{\frac{\pi}{n}} - \cancel{\frac{\sin nt}{n^2}\Big|_0^{\pi}}\right] =$$

$$= \frac{1}{\pi}\left[\frac{2}{\pi}\frac{\cos nt}{n^3}\right]_{-\pi}^{0} = \frac{2}{\pi^2}\frac{1-(-)^n}{n^3}$$

Quindi

$$S(t) = \frac{5}{12}\pi + \sum_{n=1}^{\infty}\frac{3-(-)^n}{\pi n^2}\cos nt + \frac{2}{\pi^2}\frac{1-(-)^n}{n^3}\sin nt$$

Poiché $\forall \bar t \in \mathbb{R}$ f è in $\bar t$ continua e dotata di derivate destra e sinistra finite, concludiamo che $\forall t \in \mathbb{R}$

$$S(t) = f(t)$$

Abbiamo

$$S(0) = \frac{5}{12}\pi + \sum_{n=1}^{\infty}\frac{3-(-)^n}{\pi n^2} = f(0) = \pi$$

$$\pi\left(\pi - \frac{5}{12}\pi\right) = \sum_{n=1}^{\infty}\frac{3-(-)^n}{n^2}$$

$$\frac{7}{12}\pi^2 = \sum_{n=1}^{\infty}\frac{3-(-)^n}{n^2}$$

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(2) $y' = \dfrac{9-y^2}{(x+2)^2}$

$$f(x,y) = \frac{9-y^2}{(x+2)^2} \qquad f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$A = \{(x,y) \in \mathbb{R}^2 : x \neq -2\} \qquad A \text{ aperto}$$

Poiché $f \in C^\infty(A)$, concludiamo che $\forall\ (x_0, y_0) \in A$,

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$ ha un unica soluzione locale il cui grafico sta in A

I punti $(0,2)$ e $(0,3)$ sono esattamente in questa condizione.

Inoltre

$$y = 3 \qquad \text{e} \qquad y = -3$$

sono soluzioni il cui grafico è ~~sta~~ in A

Pertanto

$$\begin{cases} y' = \frac{9-y^2}{(x+2)^2} \\ y(0) = 3 \end{cases}$$

ha come unica soluzione proprio la retta $y=3$.

Quanto all'altro Problema di Cauchy, separiamo le variabili

$$\int \frac{dy}{9-y^2} = \int \frac{dx}{(x+2)^2}$$

$$\int \left(\frac{A}{y+3} + \frac{B}{y-3} \right) dy = -\frac{1}{x+2} + C$$

$$\frac{1}{6} \int \left(\frac{1}{y+3} - \frac{1}{y-3} \right) dy = -\frac{1}{x+2} + C$$

$$\frac{1}{6} \ln \left| \frac{y+3}{y-3} \right| = -\frac{1}{x+2} + C$$

$$\ln\left|\frac{y+3}{y-3}\right| = -\frac{6}{x+2} + K$$

$$\left|\frac{y+3}{y-3}\right| = H\exp\left(-\frac{6}{x+2}\right)$$

$$\frac{y+3}{y-3} = M\exp\left(-\frac{6}{x+2}\right)$$

$$y+3 = M\exp\left(-\frac{6}{x+2}\right)[y-3]$$

$$y\left(1 - M\exp\left(-\frac{6}{x+2}\right)\right) = -3M\exp\left(-\frac{6}{x+2}\right) - 3$$

$$y = 3\,\frac{M\exp\left(-\frac{6}{x+2}\right)+1}{M\exp\left(-\frac{6}{x+2}\right)-1}$$

Sostituendo la condizione iniziale

$$2 = 3\,\frac{Me^{-3}+1}{Me^{-3}-1}$$

$$2Me^{-3} - 2 = 3Me^{-3} + 3$$

$$Me^{-3} = -5$$

$$M = -5e^{3}$$

Sostituendo nell'integrale generale otteniamo la soluzione cercata

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(3) Si tratta di trovare il minimo di $f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$ con il vincolo di $x^2+y^2+z^2=1$

Costruiamo la lagrangiana ottenendo

$$\mathcal{L} = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda\left(x^2+y^2+z^2-1\right)$$

Abbiamo

$$\begin{cases} \mathcal{L}_x = 0 \\ \mathcal{L}_y = 0 \\ \mathcal{L}_z = 0 \\ \mathcal{L}_\lambda = 0 \end{cases} \Longrightarrow \begin{cases} \cancel{2}(x-1) + \cancel{2}\lambda x = 0 \\ \cancel{2}(y-2) + \cancel{2}\lambda y = 0 \\ \cancel{2}(z-3) + \cancel{2}\lambda z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$\begin{cases} x(1+\lambda) = 1 \\ y(1+\lambda) = 2 \\ z(1+\lambda) = 3 \\ x^2+y^2+z^2 = 1 \end{cases} \qquad \begin{aligned} x &= 1/1+\lambda \\ y &= 2/1+\lambda \\ z &= 3/1+\lambda \end{aligned}$$

$$\frac{1}{(\lambda+1)^2} + \frac{4}{(\lambda+1)^2} + \frac{9}{(1+\lambda)^2} = 1$$

$$\frac{14}{(\lambda+1)^2} = 1 \qquad (\lambda+1)^2 = 14 \qquad \lambda = -1 \pm \sqrt{14}$$

Corrispondentemente otteniamo

$$\begin{cases} x = \frac{1}{\sqrt{14}} \\ y = \frac{2}{\sqrt{14}} \\ z = \frac{3}{\sqrt{14}} \end{cases} \quad \text{oppure} \quad \begin{cases} x = -\frac{1}{\sqrt{14}} \\ y = -\frac{2}{\sqrt{14}} \\ z = -\frac{3}{14} \end{cases}$$

Poiché la sfera è un insieme chiuso e limitato, il minimo cercato esiste sicuramente per la continuità della funzione distanza. È, poi, immediato verificare che

$$m = \sqrt{\left(\frac{1}{\sqrt{14}} - 1\right)^2 + \left(\frac{2}{\sqrt{14}} - 2\right)^2 + \left(\frac{3}{\sqrt{14}} - 3\right)^2}$$

$$= \left(1 - \frac{1}{\sqrt{14}}\right)\sqrt{1+4+9}$$

$$= \frac{\sqrt{14}-1}{\sqrt{14}}\sqrt{14} = \sqrt{14} - 1.$$

(4) $\begin{cases} z = x^2+y^2 \\ 4x^2+9y^2+z^2 = 17 \end{cases}$

Osserviamo che, posto

$$f_1(x,y,z) = x^2+y^2-z$$

$$f_2(x,y,z) = 4x^2+9y^2+z^2-17$$

abbiamo $f_1, f_2 \in C^\infty(\mathbb{R}^3)$

Inoltre

$$f_1(1,1,2) = 0$$
$$f_2(1,1,2) = 0$$

Infine

$$\frac{\partial f_1}{\partial x} = 2x \qquad \frac{\partial f_1}{\partial y} = 2y \qquad \frac{\partial f_1}{\partial z} = -1$$

$$\frac{\partial f_2}{\partial x} = 8x \qquad \frac{\partial f_2}{\partial y} = 18y \qquad \frac{\partial f_2}{\partial z} = 2z$$

Pertanto

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}_{(1,1,2)} = \begin{bmatrix} 2 & 2 & -1 \\ 8 & 18 & 4 \end{bmatrix}$$

e $\det \begin{bmatrix} 2 & 2 \\ 8 & 18 \end{bmatrix} \neq 0$

Pertanto, possiamo applicare il Teorema di Dini e concludere che il sistema è univocamente risolubile rispetto a (x,y) in un intorno di $(1,1,2)$

Per quanto riguarda l'equazione della retta tangente, essa è data da

$$\begin{cases} \dfrac{\partial f_1}{\partial x}(1,1,2)\,(x-1) + \dfrac{\partial f_1}{\partial y}(1,1,2)\,(y-1) + \dfrac{\partial f_1}{\partial z}(1,1,2)\,(z-2) = 0 \\[2ex] \dfrac{\partial f_2}{\partial x}(1,1,2)\,(x-1) + \dfrac{\partial f_2}{\partial y}(1,1,2)\,(y-1) + \dfrac{\partial f_2}{\partial z}(1,1,2)\,(z-2) = 0 \end{cases}$$

cioè

$$\begin{cases} 2(x-1) + 2(y-1) - (z-2) = 0 \\ 8(x-1) + 18(y-1) + 4(z-2) = 0 \end{cases}$$