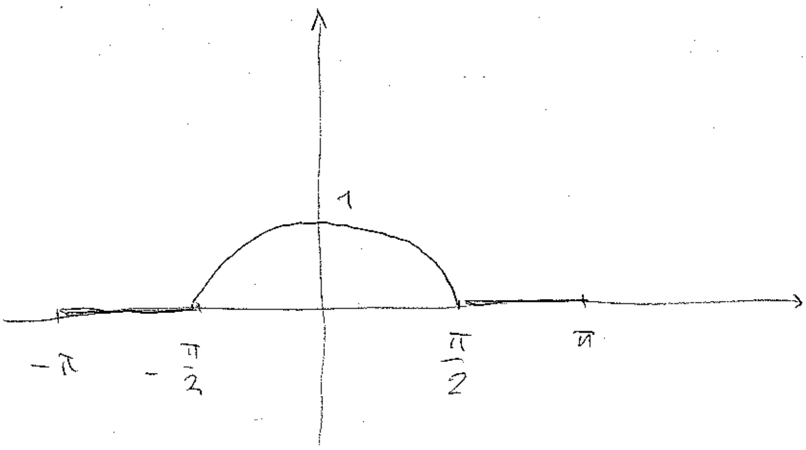
① $f : \mathbb{R} \to \mathbb{R}$, 2π-periodica pari



f è continua in $\mathbb{R}$, dunque è sicuramente sviluppabile

Poiché è pari, $b_n = 0$

$$a_0 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} f(t)\,dt = \frac{2}{\pi} \int_0^{\pi/2} \cos t\,dt = \frac{2}{\pi} \sin t \Big|_0^{\pi/2} = \frac{2}{\pi}$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi/2} f(t) \cos t\,dt = \frac{2}{\pi} \int_0^{\pi/2} \cos^2 t\,dt = \frac{2}{\pi} \int_0^{\pi/2} \frac{1+\cos 2t}{2}$$

$$= \frac{\cancel{2}}{\pi} \frac{1}{\cancel{2}} \left[t + \frac{\sin 2t}{2} \right]_0^{\pi/2} = \frac{1}{\pi} \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi/2} f(t) \cos nt\,dt = \frac{2}{\pi} \int_0^{\pi/2} \cos t \cos nt\,dt$$

$$\int_0^{\pi/2} \cos t \cos nt\,dt = \sin t \cos nt \Big|_0^{\pi/2} + n \int_0^{\pi/2} \sin t \sin nt\,dt$$

$$= \cos n\pi/2 - \cancel{n \cos t \sin nt}\Big|_0^{\pi/2} +$$

$$+ n^2 \int_0^{\pi/2} \cos t \cos nt\,dt$$

Dunque

$$(1-n^2) \int_0^{\pi/2} \cos t \cos nt = \cos n\pi/2$$

$$\int_0^{\pi/2} \cos t \cos nt = -\frac{\cos n\pi/2}{n^2-1}$$

$$S(t) = \frac{a_0}{2} + a_1 \cos t + \sum_{n=2}^{\infty} a_n \cos nt$$

$$= \frac{1}{\pi} + \frac{1}{2}\cos t - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{\cos n\pi/2}{n^2-1} \cos nt$$

Poiché $\forall t \in \mathbb{R}$ f è continua e derivabile, oppure continua con derivate dx e sx finite, concludiamo che

$$\forall t \in \mathbb{R} \qquad S(t) = f(t) \qquad (1)$$

Inoltre $f \in L^2(-\pi, \pi)$ e quindi

$$\int_{-\pi}^{\pi} |S(t)|^2\, dt = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right] \qquad (2)$$

e la serie converge alla funzione nel senso dell'energia
Se scegliamo $t = \pi/2$, dalla (1) abbiamo

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{\cos^2 n\pi/2}{n^2-1}$$

Osserviamo che

$$n = 2k \qquad \cos 2k\frac{\pi}{2} = \cos k\pi = (-)^k$$

$$n = 2k+1 \qquad \cos \frac{2k+1}{2}\pi = 0$$

Perciò

$$0 = \frac{1}{\pi} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1}$$

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}$$

Inoltre, dall'uguaglianza di Parseval abbiamo

$$2\int_0^{\pi/2} \cos^2 t\, dt = \pi \left[\frac{4}{\pi^2} + \frac{1}{4} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2} \right]$$

$$\cancel{2\int_0^{\pi/2}} \quad \frac{\pi}{2} = \frac{4}{\pi} + \frac{\pi}{4} + \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2} = \left[\pi/4 - 4/\pi\right]\frac{\pi^2}{4}.$$

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(2) $$y'' + 4y = x + \cot g\, 2x$$

$$\lambda^2 + 4 = 0 \qquad \lambda = \pm 2i$$

$$y = C_1 \cos 2x + C_2 \sin 2x + y_p$$

$$y_p = y_{p,1} + y_{p,2}$$

$$y_{p,1} = Ax + B \qquad y'_{p,1} = A \qquad y''_{p,1} = 0$$

Sostituendo

$$4Ax + B = x \implies \begin{array}{l} B = 0 \\ A = 1/4 \end{array}$$

$y_{p,2}$ va calcolato utilizzando di variazione delle costanti arbitrarie

$$y_{p,2} = \int^x \frac{\begin{vmatrix} \cos 2t & \sin 2t \\ \cos 2x & \sin 2x \end{vmatrix}}{\begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}} \frac{\cos 2t}{\sin 2t}\, dt = \int^x \frac{\cos^2 2t \sin 2x - \cos 2t \sin 2t \cos 2x}{2 \sin 2t}$$

$$= \frac{1}{2}\sin 2x \int^x \frac{\cos^2 2t}{\sin 2t}\, dt - \frac{1}{2}\int^x \cos 2t\, dt \cos 2x$$

$$= \frac{1}{2}\sin 2x \int^x \frac{1 - \sin^2 2t}{\sin 2t}\, dt - \frac{1}{2}\cos 2x \frac{\sin 2x}{2}$$

$$= \cancel{\frac{1}{2}\sin 2x \frac{\cos 2x}{2}} + \frac{1}{2}\sin 2x \int^x \frac{1}{\sin 2t}\, dt \cancel{- \frac{1}{2}\cos 2x \frac{\sin 2x}{2}}$$

$$= \frac{1}{4}\sin 2x \int^x \frac{\cos t}{\sin t} \frac{1}{\cos^2 t}\, dt = \frac{1}{4}\sin 2x \ln|\mathrm{tg}\, x|$$

Quindi l'integrale generale è

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4}x + \frac{1}{4}\sin 2x \ln|\operatorname{tg} x|$$

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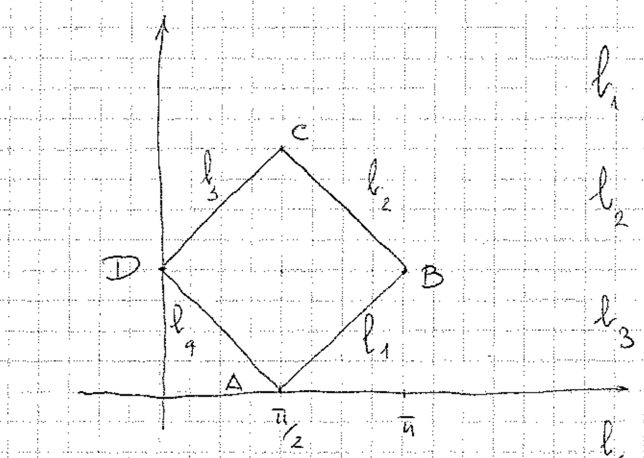


$l_1: \quad y = x - \frac{\pi}{2}$

$l_2: \quad y = -x + \frac{3}{2}\pi$

$l_3: \quad y = x + \frac{\pi}{2}$

$l_4: \quad y = -x + \frac{\pi}{2}$

$\nabla f = (\cos x, \cos y) \qquad \nabla f = (0,0) \implies \begin{cases} x = \frac{\pi}{2} + k\pi \\ y = \frac{\pi}{2} + n\pi \end{cases}$

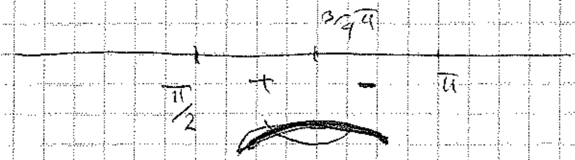
L'unico punto interno a Q è $P\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ dove $f = 2$ che è evidentemente il massimo assoluto della f.

Valutiamo il comportamento sui lati:

$f|_{l_1} = \sin x + \sin\left(x - \frac{\pi}{2}\right) = \sin x - \cos x \qquad x \in \left[\frac{\pi}{2}, \pi\right]$

$g_1 = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x\right) = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

$g_1' = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$



$g_1' \geq 0 \qquad -\frac{\pi}{2} \leq x - \frac{\pi}{4} \leq \frac{\pi}{2}$

$-\frac{\pi}{4} \leq x \leq \frac{3}{4}\pi$

Quindi $\quad m_1 = g_1\left(\frac{\pi}{2}\right) = g_1(\pi) = 1$

$M_1 = g_1\left(\frac{3}{4}\pi\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

$f|_{l_2} = \sin x + \sin\left(-x + \frac{3}{2}\pi\right) = \sin x - \cos x \qquad x \in \left[\frac{\pi}{2}, \pi\right]$

Quindi

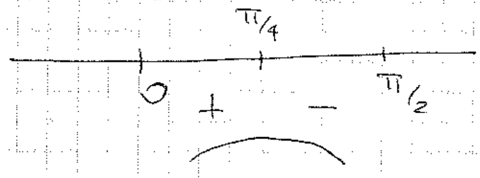
$m_2 = 1$

$M_2 = \sqrt{2}$

$$f|_{\ell_3} = \sin x + \sin\left(x + \frac{\pi}{2}\right) = \sin x + \cos x \qquad x \in \left[0, \frac{\pi}{2}\right]$$

$$g_3 = \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right)$$

$$g_3' = \sqrt{2}\,\cos\left(x + \frac{\pi}{4}\right)$$



$$g_3' \geq 0 \qquad -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$$

$$-\frac{3}{4}\pi \leq x \leq \frac{\pi}{4}$$

Quindi

$$m_3 = g_3(0) = g_3\left(\frac{\pi}{2}\right) = 1$$

$$M_3 = g_3\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f|_{\ell_4} = \sin x + \sin\left(\frac{\pi}{2} - x\right) = \sin x + \cos x$$

Quindi

$$m_4 = 1$$

$$M_4 = \sqrt{2}$$

e in definitiva $\qquad m_{ass} = 1, \qquad M_{ass} = 2$

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(4) $f \in C^\infty(\mathbb{R}^2)$ $\qquad$ Inoltre $\qquad f(-1,0) = 1 - (-1)^2 = 0$

$$f_y = 3x\,e^{3xy} + 4x^2 y \qquad f_y(-1,0) = -3$$

Quindi possiamo applicare il Teorema di Dini. Inoltre

$$f_x = 3y\,e^{3xy} - 2x(1 - 2y^2) \qquad f_x(-1,0) = 2$$

da cui otteniamo $\quad g'(-1) = -\dfrac{f_x(-1,0)}{f_y(-1,0)} = 2/3$

Poiché $f \in C^\infty$, $\quad g \in C^\infty(I_{x=-1})$. Derivando membro a membro

$$e^{3xg}(3g + 3xg') - 2x(1 - 2g^2) - x^2(-4gg') \equiv 0$$

$$e^{3xg}(3g + 3xg')^2 + e^{3xg}(3g' + 3g' + 3xg'') - 2(1 - 2g^2) +$$

$$-2x(-4gg') - 2x(-4gg') - x^2(-4g'^2 - 4gg'') = 0$$

da cui, tenendo conto che $x = -1$, $\; g(-1) = 0$, $\; g'(-1) = 2/3$

otteniamo $\quad g''(-1) = \dfrac{60}{27} > 0$