(1) $z' = \mathbb{A} z$ where

$$\mathbb{A} = \begin{bmatrix} 7 & -12 & 4 \\ 4 & -9 & 4 \\ 4 & -12 & 7 \end{bmatrix}$$

We solve

$$\det(\lambda I - \mathbb{A}) = 0 \qquad \begin{vmatrix} \lambda - 7 & 12 & -4 \\ -4 & \lambda + 9 & -4 \\ -4 & 12 & \lambda - 7 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda - 3 & 0 & 3 - \lambda \\ -4 & \lambda + 9 & -4 \\ -4 & 12 & \lambda - 7 \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 0 & 3 - \lambda \\ -8 & \lambda + 9 & -4 \\ \lambda - 11 & 12 & \lambda - 7 \end{vmatrix} = 0$$

$$(3-\lambda)\left[-96 - \lambda^2 + 2\lambda + 99\right] = 0$$

$$(-3+\lambda)(\lambda^2 - 2\lambda - 3) = 0 \qquad \lambda_1 = -1$$

$$\lambda_2 = \lambda_3 = 3$$

The general solution is given by

$$\underline{Z} = e^{-x}\underline{c}_1 + e^{3x}\underline{c}_2 + x e^{3x}\underline{c}_3$$

where $\underline{c}_1, \underline{c}_2, \underline{c}_3$ are proper vectors which depend on three arbitrary constants. We have

$$\underline{Z}' = -e^{-x}\underline{c}_1 + 3e^{3x}\underline{c}_2 + (e^{3x} + 3xe^{3x})\underline{c}_3$$

Therefore,

$$-e^{-x}\underline{c}_1 + 3e^{3x}\underline{c}_2 + e^{3x}(1+3x)\underline{c}_3 =$$

$$= e^{-x}\mathbb{A}\underline{c}_1 + e^{3x}\mathbb{A}\underline{c}_2 + xe^{3x}\mathbb{A}\underline{c}_3$$

and

$$\begin{cases} \mathbb{A}\underline{c}_1 = -\underline{c}_1 \\ \mathbb{A}\underline{c}_2 = 3\underline{c}_2 + \underline{c}_3 \\ \mathbb{A}\underline{c}_3 = 3\underline{c}_3 \end{cases}$$

If we let

$$\underline{c}_1 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\begin{bmatrix} -8 & +12 & -4 \\ -4 & 8 & -4 \\ -4 & 12 & -8 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} -2\alpha + 3\beta - \gamma = 0 \\ -\alpha + 2\beta - \gamma = 0 \end{cases}$$

$$\underline{c}_1 = \begin{bmatrix} L \\ L \\ L \end{bmatrix}$$

If we let $\underline{c}_3 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, then $\begin{bmatrix} -4 & 12 & -4 \\ -4 & 12 & -4 \\ -4 & 12 & -4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{cases} -\alpha + 3\beta - \gamma = 0 \end{cases} \qquad \underline{c}_3 = \begin{bmatrix} M \\ N \\ 3N - M \end{bmatrix}$$

Finally, if we let $\underline{c}_2 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, then

$$\begin{bmatrix} -4 & 12 & -4 \\ -4 & 12 & -4 \\ -4 & 12 & -4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} M \\ N \\ 3N - M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -4\alpha + 12\beta - 4\gamma = -M \\ -4\alpha + 12\beta - 4\gamma = -N \\ -4\alpha + 12\beta - 4\gamma = M - 3N \end{cases}$$

By the Rouché – Capelly Theorem, the only possibility to ensure resolutivity is to assume $M = N = 0$. Consequently $\underline{c}_3 = \underline{0}$, $\underline{c}_2 = \begin{bmatrix} H \\ K \\ 3K - H \end{bmatrix}$

Finally

$$\underline{z} = e^{-x} \begin{bmatrix} L \\ L \\ L \end{bmatrix} + e^{3x} \begin{bmatrix} H \\ K \\ 3K - H \end{bmatrix} = \begin{bmatrix} e^{-x} & e^{3x} & 0 \\ e^{-x} & 0 & e^{3x} \\ e^{-x} & -e^{3x} & 3e^{3x} \end{bmatrix} \begin{bmatrix} L \\ H \\ K \end{bmatrix}$$

② If we let

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : \ y > 0\}, \qquad f(x,y) = y \ln y$$

it is apparent that $f \in C^\infty(\mathcal{D})$. Therefore, for any $(x_0, y_0) \in \mathcal{D}$, by the local existence and uniqueness theorem there exists a unique solution $\bar{y}$ to the Cauchy Pb

$$\begin{cases} y' = y \ln y \\ y(x_0) = y_0 \end{cases}$$

Moreover, by the regularity theory, such a solution $\bar{y} \in C^\infty(I_{x_0})$.
The global existence and uniqueness theorem cannot be applied.

Moreover

$$y = 1$$

is a particular solution and

$$\begin{array}{lll} y' \geq 0 & \text{in} & \mathcal{D}_1 = \{(x,y) \in \mathbb{R}^2 : \ y > 1\} \\ y \leq 0 & \text{in} & \mathcal{D}_2 = \{(x,y) \in \mathbb{R}^2 : \ y \in (0,1)\} \end{array}$$

As for the second order derivative,

$$y'' = y' \ln y + y' = y \ln (y+1)$$

and

$$\begin{array}{ll} y'' > 0 & y > 1, \quad 0 < y < 1/e \\ y'' < 0 & 1/e < y < 1 \end{array}$$

Finally, pick (x_0, y_0) with $y_0 \in (0,1)$. If we separate the variables, we have

$$\int_{y_0}^{y} \frac{dt}{t \ln t} = x - x_0$$

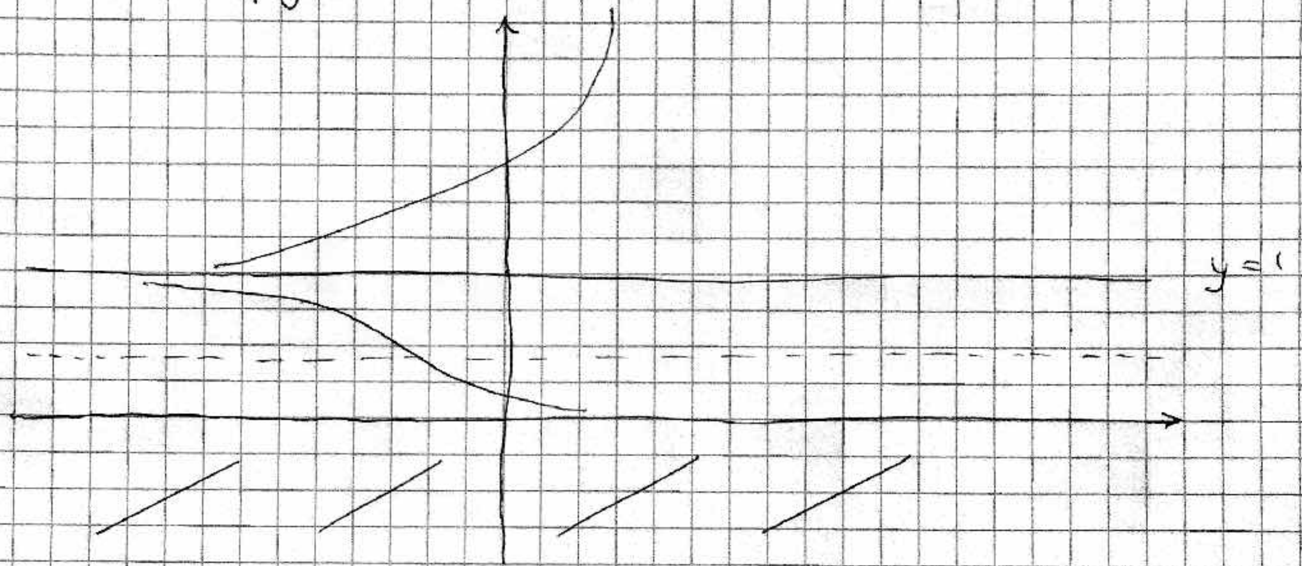
The left-hand side blows up when $y \to 0$ or $y \to 1$

Therefore, these two lines are horizontal asymptotes.

If we consider (x_0, y_0) with $y > 1$, once more

$$\int_{y_0}^{y} \frac{dt}{t \ln t} = x - x_0$$

and once more the left-hand side blows up both when $y \to \infty$ and $y \to 1$. Therefore, the qualitative graph is as in the figure below



---

(3) $u = x \sin x$

Since

$$|x| \geq |u|$$

i.e. u is bounded above by the absolute value of a polynomiale, we can conclude that $u \in \mathcal{S}'(\mathbb{R})$.

As for the Fourier transform

$$u = \frac{1}{2i} x e^{ix} - \frac{1}{2i} x e^{-ix}$$

Since

$$\hat{1} = 2\pi \delta,$$

we have

$$\hat{u} = \frac{1}{2i} 2\pi i \left(\delta'(\xi - 1) - \delta'(\xi + 1) \right)$$
$$= \pi \left[\delta'(\xi - 1) - \delta'(\xi + 1) \right]$$

$\hat{u} \in \mathcal{S}'(\mathbb{R})$ and $\operatorname{supp} \hat{u} = [-1, 1]$. This is in accordance with Paley-Wiener Theorem, since u is a real analytic function and

$$|u(z)| \leq |z| \, e^{|\operatorname{Im} z|}$$

---

(4)

$$u_1 = \frac{v_1}{\|v_1\|} \qquad \|v_1\|^2 = \int_{-\pi}^{\pi} v_1^2 \, dx + \int_{-\pi}^{\pi} (v_1')^2 dx$$
$$= 2\pi$$

$$u_1 = \frac{1}{\sqrt{2\pi}}$$

It is easy to check that v_1, v_2, v_3, v_4, v_5 are all mutually orthogonal. Therefore, it is enough to normalize them.

$$u_2 = \frac{v_2}{\|v_2\|} \qquad \|v_2\|^2 = \int_{-\pi}^{\pi} \sin^2 dx + \int_{-\pi}^{\pi} \cos^2 dx = 2\pi$$

$$u_2 = \frac{\sin x}{\sqrt{2\pi}} \qquad u_3 = \frac{\cos x}{\sqrt{2\pi}}$$

$$u_4 = \frac{v_4}{\|v_4\|} \qquad \|v_4\|^2 = \int_{-\pi}^{\pi} \sin^2 2x \, dx + 4 \int_{-\pi}^{\pi} \cos^2 2x \, dx$$
$$= \int_{-\pi}^{\pi} \frac{1 - \cos 4x}{2} + 4 \int_{-\pi}^{\pi} \frac{1 + \cos 4x}{2} dx$$
$$= \frac{1}{2} 5 \, 2\pi = 5\pi$$

$$u_4 = \frac{\sin 2x}{\sqrt{5\pi}} \qquad u_5 = \frac{\cos 2x}{\sqrt{5\pi}}$$

Therefore, the best approximation of $f(x) = x$ in Z is given by

$$g = (f, u_1)u_1 + (f, u_2)u_2 + (f, u_3)u_3 + (f, u_4)u_4 + (f, u_5)u_5$$

$$(f, u_1) = \int_{-\pi}^{\pi} x \cdot \frac{1}{\sqrt{2\pi}}\, dx + \int_{-\pi}^{\pi} 1.0\, dx = 0$$

$$(f, u_2) = \int_{-\pi}^{\pi} \frac{x \sin x}{\sqrt{2\pi}}\, dx + \int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{2\pi}}\, dx = \sqrt{2\pi}$$

$$(f, u_3) = \int_{-\pi}^{\pi} \frac{x \cos x}{\sqrt{2\pi}}\, dx + \int_{-\pi}^{\pi} \left(-\frac{\sin x}{\sqrt{2\pi}}\right) dx = 0$$

$$(f, u_4) = \int_{-\pi}^{\pi} \frac{x \sin 2x}{\sqrt{5\pi}}\, dx + \int_{-\pi}^{\pi} \frac{2 \cos 2x}{\sqrt{5\pi}}\, dx = -\sqrt{\frac{\pi}{5}}$$

$$(f, u_5) = \int_{-\pi}^{\pi} \frac{x \cos 2x}{\sqrt{5\pi}}\, dx - \int_{-\pi}^{\pi} \frac{2 \sin 2x}{\sqrt{5\pi}}\, dx = 0$$

We conclude

$$g = \sqrt{2\pi}\, \frac{\sin x}{\sqrt{2\pi}} - \sqrt{\frac{\pi}{5}}\, \frac{\sin 2x}{\sqrt{5\pi}}$$

$$= \sin x - \frac{1}{5} \sin 2x$$