PARTE A

(1) $$I=\int_{-\pi}^{\pi}\frac{\sin t}{5+\sin t}\,dt=\int_{-\pi}^{\pi}\frac{\frac{e^{it}-e^{-it}}{2i}}{5+\frac{e^{it}-e^{-it}}{2i}}\,dt$$

$$=\int_{C_1(0)}\frac{\frac{z-z^{-1}}{2i}}{5+\frac{z-z^{-1}}{2i}}\,\frac{dz}{iz}=\int_{C_1(0)}\frac{\frac{z^2-1}{2i}}{\frac{z^2+10iz-1}{2i}}\,\frac{dz}{iz}$$

$$=\frac{1}{i}\int_{C_1(0)}\frac{z^2-1}{z(z^2+10iz-1)}\,dz$$

Se consideriamo $f(z)=\dfrac{z^2-1}{z(z^2+10iz-1)}$, la funzione ha poli di ordine 1 in $z_0=0$ e nelle radici di

$$z^2+10iz-1=0$$

cioè in

$$z=-5i\pm\sqrt{-25+1}$$

$$=-5i\pm i\sqrt{24}$$

Le radici interne a $C_1(0)$ sono dunque $z_0=0$, $z_1=(-5+\sqrt{24})i$. Pertanto

$$I=\frac{1}{i}2\pi i\left[\operatorname{Res}(f,0)+\operatorname{Res}(f,(-5+\sqrt{24})i)\right]$$

$$=2\pi\left[\frac{-1}{-1}+\frac{[(-5+\sqrt{24})i]^2-1}{(-5+\sqrt{24})i\,(\cancel{-10i}+2\sqrt{24}\,i\cancel{+10i})}\right]$$

$$=2\pi\left[+1+\frac{-1-(25+24-10\sqrt{24})}{(+5-\sqrt{24})\,2\sqrt{24}}\right]$$

$$= 2\pi \left[+1 + \frac{-50 + 10\sqrt{24}}{(5-\sqrt{24})\, 2\sqrt{24}} \right]$$

$$= 2\pi \left[+1 + \frac{-\cancel{10}^{\,5}\,\cancel{(5-\sqrt{24})}}{\cancel{(5-\sqrt{24})}\,\cancel{2}\sqrt{24}} \right]$$

$$= 2\pi \left[+1 - \frac{5}{\sqrt{24}} \right] = 2\pi \left[\frac{2\sqrt{6} - 5}{2\sqrt{6}} \right]$$

$$= \frac{\pi}{\sqrt{6}} \left(2\sqrt{6} - 5 \right)$$

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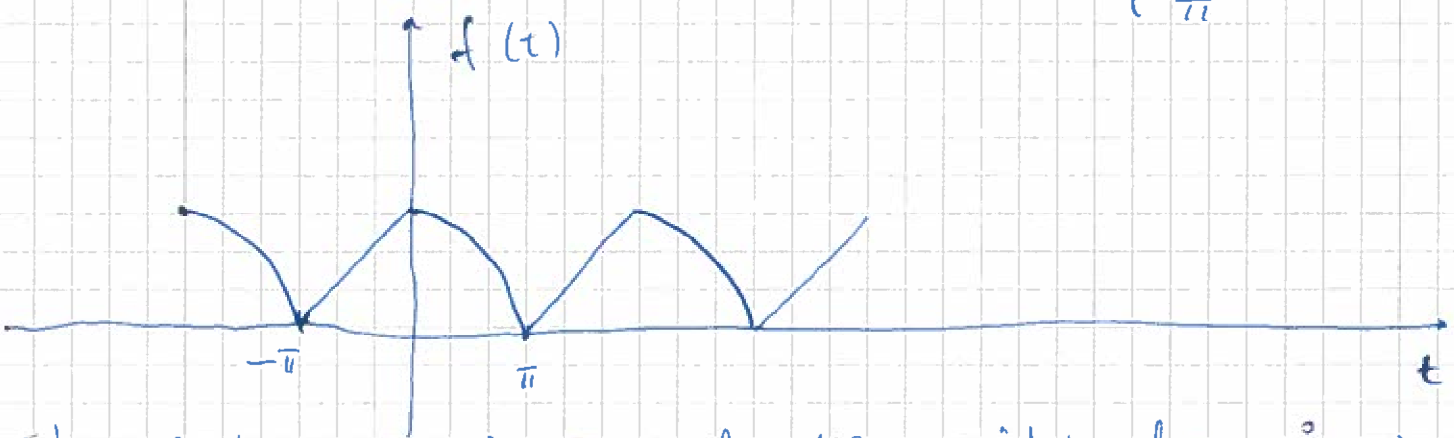
(2) $f : \mathbb{R} \to \mathbb{R}$, 2π periodica, $f(t) = \begin{cases} t+\pi, & -\pi < t < 0 \\ \dfrac{\pi^2 - t^2}{\pi} & 0 \le t \le \pi \end{cases}$



E' evidente che f è sviluppabile, poiché $f \in C^0(\mathbb{R})$ ed è ovunque limitata. Osserviamo che f non è né pari, né dispari. Abbiamo

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi} \left[\int_{-\pi}^{0} (\pi + t)\,dt + \int_0^{\pi} \frac{\pi^2 - t^2}{\pi}\,dt \right]$$

$$= \frac{1}{\pi} \left[\frac{(\pi + t)^2}{2} \Big|_{-\pi}^{0} + \frac{1}{\pi} \left(\pi^2 t - \frac{t^3}{3} \right) \Big|_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{1}{\pi} \left(\pi^3 - \frac{\pi^3}{3} \right) \right] = \frac{\pi^2}{\pi} \left[\frac{1}{2} + 1 - \frac{1}{3} \right]$$

$$= \pi \, \frac{3 + 6 - 2}{6} = \frac{7}{6} \pi .$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+t)\cos nt\,dt\, +\right.$$

$$\left. + \frac{1}{\pi}\int_{0}^{\pi}(\pi^2-t^2)\cos nt\,dt\right] =$$

$$= \frac{1}{\pi}\left[\cancel{(\pi+t)\frac{\sin nt}{n}\Big|_{-\pi}^{0}} - \int_{-\pi}^{0}\frac{\sin nt}{n}\,dt\, +\right.$$

$$\left. + \frac{1}{\pi}\cancel{(\pi^2-t^2)\frac{\sin nt}{n}\Big|_{0}^{\pi}} + \frac{2}{\pi}\int_{0}^{\pi} t\,\frac{\sin nt}{n}\,dt\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos nt}{n^2}\Big|_{-\pi}^{0} - \frac{2}{\pi}\,t\,\frac{\cos nt}{n^2}\Big|_{0}^{\pi} + \cancel{\frac{2}{\pi}\int_{0}^{\pi}\frac{\cos nt}{n^2}\,dt}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-)^n}{n^2} - \frac{2}{\cancel{\pi}}\,\cancel{\pi}\,\frac{(-)^n}{n^2}\right]$$

$$= \frac{1-3(-)^n}{\pi n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+t)\sin nt\,dt\, +\right.$$

$$\left. + \frac{1}{\pi}\int_{0}^{\pi}(\pi^2-t^2)\sin nt\,dt\right]$$

$$= \frac{1}{\pi}\left[-(\pi+t)\frac{\cos nt}{n}\Big|_{-\pi}^{0} + \cancel{\int_{-\pi}^{0}\frac{\cos nt}{n}\,dt}\right.$$

$$\left. - \frac{1}{\pi}(\pi^2-t^2)\frac{\cos nt}{n}\Big|_{0}^{\pi} - \frac{2}{\pi}\int_{0}^{\pi} t\,\frac{\cos nt}{n}\,dt\right]$$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\pi}{n}} + \cancel{\frac{\pi^2}{\pi}}\,\frac{1}{n} - \cancel{\frac{2}{\pi}\,t\,\frac{\sin nt}{n^2}\Big|_{0}^{\pi}}\right.$$

$$\left. + \frac{2}{\pi}\int_{0}^{\pi}\frac{\sin nt}{n^2}\,dt\right] = -\frac{2}{\pi^2}\,\frac{\cos nt}{n^3}\Big|_{0}^{\pi}$$

$$= \frac{2}{\pi^3}\,\frac{1-(-)^n}{n^3}.$$

Quindi

$$S(t) = \frac{7}{12}\pi + \sum_{n=1}^{\infty} \frac{1-3(-)^n}{\pi n^2} \cos nt + \frac{2}{\pi^3} \frac{1-(-)^n}{n^3} \sin nt$$

Osserviamo che $\forall\, t \neq k\pi$ con $k \in \mathbb{Z}$, f è continua e derivabile in t; pertanto, in tali t abbiamo

$$S(t) = f(t).$$

In $t_k = k\pi$, $k \in \mathbb{Z}$, f è continua e la derivate dx e sx sono finite. Pertanto, abbiamo ancora

$$S(t_k) = f(t_k)$$

Dunque $\forall t \in \mathbb{R}$ è $S(t) = f(t)$. Questo non sorprende, perché f è di classe C^1 a tratti e dunque la serie S converge uniformemente alla funzione f su tutto $\mathbb{R}$.

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③ $$\hat{f}(\omega) = \int_{\mathbb{R}} \frac{1}{(t-4i)(t+7i)} e^{-i\omega t}\, dt.$$

Pertanto

$\omega < 0$ $$\hat{f}(\omega) = 2\pi i \operatorname{Res}\left(\frac{e^{-i\omega t}}{(t-4i)(t+7i)}, 4i\right)$$

$$= 2\pi i \frac{e^{-i\omega(4i)}}{4i+7i} = \frac{2\pi}{11} e^{4\omega}.$$

$\omega > 0$ $$\hat{f}(\omega) = -2\pi i \operatorname{Res}\left(\frac{e^{-i\omega t}}{(t-4i)(t+7i)}, -7i\right)$$

$$= -2\pi i \frac{e^{-i\omega(-7i)}}{-7i-4i} = \frac{2\pi}{11} e^{-7\omega}.$$

In definitiva, dunque

$$\hat{f}(\omega) = \frac{2\pi}{11} \begin{cases} e^{4\omega} & \omega < 0 \\ 1 & \omega = 0 \\ e^{-7\omega} & \omega > 0 \end{cases}$$

Per il Teorema di Plancherel

$$\int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega = 2\pi \int_{\mathbb{R}} |f(t)|^2 dt.$$

Pertanto

$$\int_{\mathbb{R}} \frac{1}{(t^2+16)(t^2+49)} dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega$$

$$= \frac{1}{\cancel{2\pi}} \left[\int_{-\infty}^{0} \frac{\cancel{4\pi^2}^{\,2}}{121} e^{8\omega} d\omega + \right.$$

$$\left. + \int_{0}^{+\infty} \frac{\cancel{4\pi^2}^{\,2}}{121} e^{-14\omega} d\omega \right] = \frac{2\pi}{121} \left[\frac{e^{8\omega}}{8} \Big|_{-\infty}^{0} + \right.$$

$$\left. + \frac{e^{-14\omega}}{-14} \Big|_{0}^{+\infty} \right] = \frac{\cancel{2}\pi}{121} \cdot \left(\frac{1}{\cancel{8}_4} + \frac{1}{\cancel{14}_7} \right)$$

$$= \frac{\pi}{\cancel{121}_{11}} \cdot \frac{\cancel{11}^{\,1}}{28} = \frac{\pi}{308}.$$

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④ $$\begin{cases} X'' - 5X' + 6X = H(t)e^{4t} \\ X(0) = 0 \qquad X'(0) = 1 \end{cases}$$ Applicando la $\mathcal{L}$-trasf. abbiamo

$$(s^2 x - s\cancel{0} - 1) - 5(sx \cancel{-0}) + 6x = \frac{1}{s-4}$$

$$(s^2 - 5s + 6)x = \frac{1}{s-4} + 1$$

$$x = \frac{1}{(s-2)\cancel{(s-3)}} \cdot \frac{\cancel{s-4}+1}{s-4}$$

$$x = \frac{1}{(s-2)(s-4)} = \frac{\operatorname{Res}(x(s), 2)}{s-2} + \frac{\operatorname{Res}(x(s), 4)}{s-4}$$

$$= \frac{1}{2} \frac{1}{s-4} - \frac{1}{2} \frac{1}{s-2} .$$

Quindi

$$x(t) = \frac{H(t)}{2} \left[e^{4t} - e^{2t} \right],$$

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(5) $$F^*(z) = \frac{1}{2} \log\left(1 + \frac{1}{z}\right) - \frac{1}{2} \log\left(1 - \frac{1}{z}\right)$$

Ricordiamo che

$$\ln(1+t) = \sum_{n=0}^{\infty} (-)^n \frac{t^{n+1}}{n+1} \quad \text{in } |t| < 1$$

ed anche $\ln(1-t) = -\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}$ in $|t|<1$.

Pertanto

$$\log\left(1 + \frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-)^n}{n+1} \left(\frac{1}{z}\right)^{n+1}$$

$$+ \log\left(1 - \frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-)^n}{n+1} \left(-\frac{1}{z}\right)^{n+1} =$$

$$= + \sum_{n=0}^{\infty} \frac{(-)^{2n+1}}{n+1} \left(\frac{1}{z}\right)^{n+1} =$$

$$= + \sum_{n=0}^{\infty} \frac{(-1)}{n+} \left(\frac{1}{z}\right)^{n+1}$$

Pertanto, sottraendo i due termini

$$F^*(z) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-)^n + 1}{n+1} \left(\frac{1}{z}\right)^{n+1}$$

Osserviamo che se $n = 2k$ abbiamo

$$\frac{1}{2}\,\frac{(-)^{n}+1}{n+1}\left(\frac{1}{z}\right)^{n+1} = \frac{1}{2}\,\frac{(-)^{2k}+1}{2k+1}\left(\frac{1}{z}\right)^{2k+1}$$

$$= \frac{1}{2k+1}\left(\frac{1}{z}\right)^{2k+1}$$

Se, invece, $n = 2k+1$, abbiamo

$$\frac{1}{2}\,\frac{(-)^{n}+1}{n+1}\left(\frac{1}{z}\right)^{n+1} = \frac{1}{2}\,\frac{(-)^{2k+1}+1}{2k+2}\left(\frac{1}{z}\right)^{2k+2} = 0.$$

Pertanto,

$$F^{*}(z) = \sum_{k=0}^{\infty} \frac{1}{2k+1}\left(\frac{1}{z}\right)^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2k+1}\, z^{-(2k+1)}$$

e, per definizione di Z^{*} trasformata, possiamo concludere che

$$f_{2k} = 0$$

$$\Longrightarrow \quad f = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right\}.$$

$$f_{2k+1} = \frac{1}{2k+1}$$

PARTE B

(1) Sia $\Omega \subseteq \mathbb{C}$ un insieme aperto, $z_0 \in \Omega$ e f olomorfa in $\Omega \setminus \{z_0\}$. Diciamo che f ha in z_0 un polo di ordine 4 se $\lim_{z \to z_0} f(z) = z_{\infty}$ e se $\lim_{z \to z_0} f(z)\,(z-z_0)^{4} = L$ <u>finito e diverso da zero</u>.

La funzione $f(z) = \frac{1}{(z-3)^4}$, olomorfa in $\mathbb{C} \setminus \{3\}$ ha in $z_0 = 3$ un polo di ordine 4.

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(2) $f \in L^1 \Longrightarrow \hat{f} \in C_0^0 \cap L^\infty$

$f \in L^2 \Longleftrightarrow \hat{f} \in L^2$

$tf \in L^2 \Longrightarrow \hat{f}' \in L^2$

$\forall n \in \mathbb{N} \quad f^{(n)} \in L^1 \Longrightarrow \forall n \in \mathbb{N} \quad \omega^n \hat{f} \in C_0^0 \cap L^\infty$

$\forall n \in \mathbb{N} \quad f^{(n)} \in L^2 \Longrightarrow \forall n \in \mathbb{N} \quad \omega^n \hat{f} \in L^2$

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(3) Siano $f, g, h : \mathbb{R} \to \mathbb{R}$.

Se

1) $f, g, h \in L^1$

oppure

2) $f, g, h \in E$

allora vale la proprietà associativa, ossia

$$(f * g) * h = f * (g * h)$$

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(4) Siano $f, g : \mathbb{R} \to \mathbb{R}$. Se

a) $f \in L^1, \ g \in L^1$

b) $f \in L^1, \ g \in L^2$ $\Longrightarrow \widehat{f * g} = \hat{f} \cdot \hat{g}$

c) $f \in L^2, \ g \in L^2$

Analogamente, se

$\hat{f}, \hat{g} \in L^1$

$\hat{f}, \hat{g} \in L^2$ $\quad \Longrightarrow \quad \widehat{fg} = \frac{1}{2\pi} \hat{f} * \hat{g}.$

$\hat{f} \in L^1, \hat{g} \in L^2$

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(5) Sia f una funzione di variabile complessa e sia f olomorfa in $\operatorname{Re} s > \lambda$ per un qualche $\lambda \in \mathbb{R}$ <u>finito</u>.

Se $\exists\, \mu \geq \lambda$, $\exists\, \alpha \geq 1$, $\exists\, M > 0$ t.c.

$$|f(s)| \leq \frac{M}{|s|^\alpha} \quad \text{in } \operatorname{Re} s > \mu$$

allora f è la $\mathcal{L}$-trasformata di una funzione di E e tale funzione F è data da

$$F(t) = \frac{1}{2\pi i}\, \mathrm{vp} \int_{x-i\infty}^{x+i\infty} f(s)\, e^{st}\, ds,$$

ove x è arbitrario, purché $x > \mu$.