PARTE A

(1) $$I = vp \int_{\mathbb{R}} \frac{t}{(t-2)(t^2+4)}\, dt$$

$$I = 2\pi i \,\mathrm{Res}\left(\frac{z}{(z-2)(z^2+4)}, 2i\right) + \pi i \,\mathrm{Res}\left(\frac{z}{(z-2)(z^2+4)}, 2\right)$$

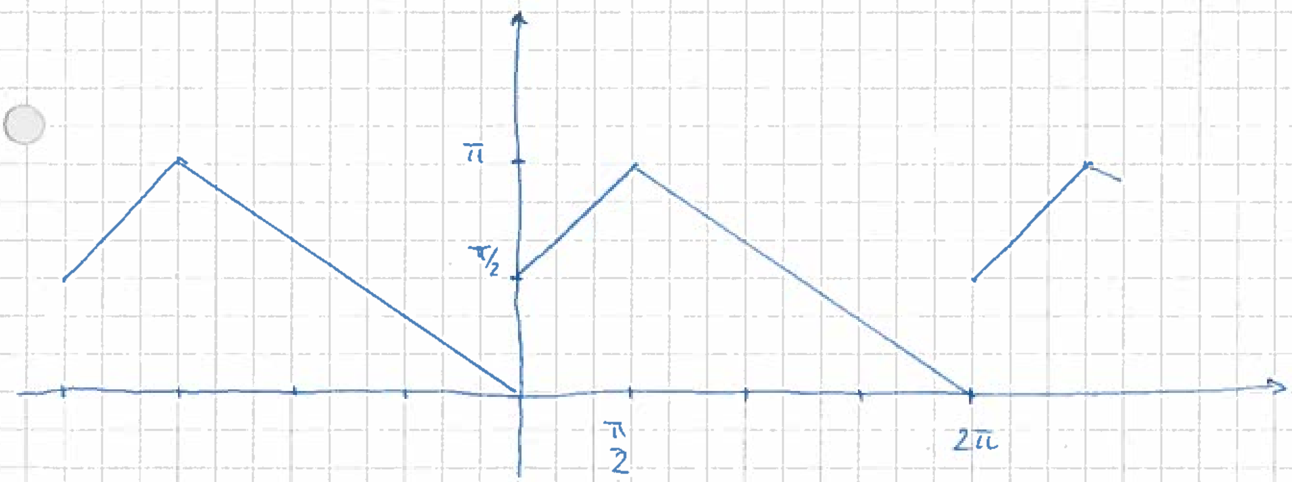
$$= \cancel{2}\pi i \frac{\cancel{2i}}{(2i-2)\cdot\cancel{2}\cdot\cancel{2i}} + \pi i \frac{2}{4+4} =$$

$$= \pi i \left[\frac{1}{4} - \frac{1}{2-2i}\right] = \pi i \left[\frac{1}{4} - \frac{2+2i}{4+4}\right]$$

$$= \pi i \left[\cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{i}{4}\right] = \frac{\pi}{4}$$

---

(2) $$f(t) = \begin{cases} \frac{\pi}{2} + t & \text{se } 0 < t \le \frac{\pi}{2} \\ \frac{2}{3}(2\pi - t) & \text{se } \frac{\pi}{2} < t \le 2\pi \end{cases}$$



f non è né pari, né dispari

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\left[\left(\frac{\pi}{2} + \pi\right)\cdot\frac{\pi}{2}\cdot\frac{1}{2} + \pi\cdot\frac{3\pi}{2}\cdot\frac{1}{2}\right]$$

$$\frac{1}{2\pi}\left[\frac{3\pi}{2}\cdot\frac{\pi}{2} + \frac{3\pi}{2}\cdot\pi\right] = \frac{1}{2\cancel{\pi}}\,\frac{3\pi^2}{2}\,\frac{3}{2} = \frac{9\pi}{8}$$

Quindi $\quad \frac{a_0}{2} = \frac{9}{16}\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\left(\frac{\pi}{2}+t\right)\cos nt\,dt + \int_{\pi/2}^{2\pi}\frac{2}{3}(2\pi-t)\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+t\right)\frac{\sin nt}{n}\Big|_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin nt}{n}\,dt + \right.$$

$$\left. + \frac{2}{3}(2\pi-t)\frac{\sin nt}{n}\Big|_{\pi/2}^{2\pi} + \frac{2}{3}\int_{\pi/2}^{2\pi}\frac{\sin nt}{n}\,dt\right]$$

$$= \frac{1}{\pi}\left[\cancel{\pi\frac{\sin n\pi/2}{n}} + \frac{\cos nt}{n^2}\Big|_0^{\pi/2}\right.$$

$$\left. \cancel{-\frac{2}{3}\frac{3}{2}\pi\frac{\sin n\pi/2}{n}} - \frac{2}{3}\frac{\cos nt}{n^2}\Big|_{\pi/2}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{2}{3}\frac{1}{n^2} + \frac{2}{3}\frac{\cos n\pi/2}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{5}{3}\frac{\cos n\pi/2}{n^2} - \frac{5}{3}\frac{1}{n^2}\right]$$

$$= \frac{5}{3\pi}\;\frac{\cos n\pi/2 - 1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\left(\frac{\pi}{2}+t\right)\sin nt\,dt + \int_{\pi/2}^{2\pi}\frac{2}{3}(2\pi-t)\sin nt\,dt\right]$$

$$= \frac{1}{\pi}\left[-\left(\frac{\pi}{2}+t\right)\frac{\cos nt}{n}\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{\cos nt}{n}\,dt + \right.$$

$$\left. -\frac{2}{3}(2\pi-t)\frac{\cos nt}{n}\Big|_{\pi/2}^{2\pi} - \frac{2}{3}\int_{\pi/2}^{2\pi}\frac{\cos nt}{n}\,dt\right]$$

$$= \frac{1}{\pi}\left[-\cancel{\pi\frac{\cos n\pi/2}{n}} + \frac{\pi}{2}\frac{1}{n} + \frac{\sin nt}{n^2}\Big|_0^{\pi/2}\right.$$

$$+ \frac{\cancel{2}}{\cancel{3}} \frac{3\pi}{\cancel{2}} \cancel{\frac{\cos n\pi/2}{n}} - \frac{2}{3} \frac{\sin nt}{n^2} \Big|_{\pi/2}^{2\pi} \Big]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2n} + \frac{\sin n\pi/2}{n^2} + \frac{2}{3} \frac{\sin n\pi/2}{n^2} \right]$$

$$= \frac{1}{2n} + \frac{5}{3\pi} \frac{\sin n\pi/2}{n^2}$$

Quindi, la serie di Fourier cercata è

$$S(t) = \frac{9}{16}\pi + \sum_{n=1}^{\infty} \frac{5}{3\pi} \frac{\cos n\pi/2 - 1}{n^2} \cos nt +$$

$$+ \left(\frac{1}{2n} + \frac{5}{3\pi} \frac{\sin n\pi/2}{n^2} \right) \sin nt \Big\}$$

Osserviamo che

a) Se $t \neq 2k\pi$ con $k \in \mathbb{Z}$ e $t \neq \frac{\pi}{2} + 2m\pi$ con $m \in \mathbb{Z}$, f è continua e derivabile in t. Pertanto in tal caso

$$S(t) = f(t)$$

b) Se $t_k = 2k\pi$ con $k \in \mathbb{Z}$, f ha in t_k un salto di ampiezza finita e le pendenze ai due lati del salto sono finite. Quindi

$$S(t_k) = \frac{1}{2}\left[f(t_k^+) + f(t_k^-) \right] = \frac{1}{2}\left(\frac{\pi}{2} + 0 \right) = \frac{\pi}{4}$$

c) Se $t_m = \frac{\pi}{2} + 2m\pi$ con $m \in \mathbb{Z}$, f in t_m è continua e le derivate destra e sinistra sono diverse, ma finite. Pertanto

$$S(t_m) = f(t_m)$$

(3) $\hat{f} = \int_{\mathbb{R}} \frac{1}{(t+3i)^2} e^{-i\omega t} dt$

$\omega < 0 \qquad \hat{f}(\omega) = 0$

$\omega > 0 \qquad \hat{f}(\omega) = -2\pi i \, \mathrm{Res}\left(\frac{e^{-i\omega t}}{(t+3i)^2}, -3i \right)$

$$= -2\pi i \lim_{\omega \to -3i} \frac{d}{dt} e^{-i\omega t}$$

$$= -2\pi i (-i\omega) \, e^{-i\omega(-3i)}$$

$$= -2\pi \omega \, e^{-3\omega}$$

Quindi, possiamo concludere che

$$\hat{f}(\omega) = -2\pi\omega H(\omega) e^{-3\omega}$$

Inoltre, per il Teorema di Plancherel

$$\int_{\mathbb{R}} \frac{1}{(t^2+9)^2} dt = \int_{\mathbb{R}} |f(t)|^2 dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} 4\pi^2 \int_0^{+\infty} \omega^2 e^{-6\omega} d\omega$$

$$= 2\pi \left[\cancel{\left. \omega^2 \frac{e^{-6\omega}}{-6} \right|_0^{+\infty}} + \frac{\cancel{2}^1}{\cancel{6}_3} \int_0^{+\infty} \omega e^{-6\omega} d\omega \right]$$

$$= \frac{2\pi}{3} \left[\cancel{\left. \omega \frac{e^{-6\omega}}{-6} \right|_0^{+\infty}} + \frac{1}{6} \int_0^{+\infty} e^{-6\omega} d\omega \right]$$

$$= \frac{2}{3}\pi \quad \frac{e^{-6\omega}}{-36}\Big|_0^{+\infty} = \frac{\cancel{2}}{3}\pi \frac{1}{\cancel{36}_{18}} = \frac{\pi}{54}.$$

---

(4) $$h(t) = \frac{1}{t+4i} * \frac{1}{(t+3i)^2}$$

Poiché

$$f(t) = \frac{1}{(t+3i)^2} \in L^1(\mathbb{R})$$

$$g(t) = \frac{1}{t+4i} \in L^2(\mathbb{R})$$

per il Teorema della convoluzione, abbiamo

$$\hat{h} = \hat{f} \cdot \hat{g}$$

$$= -2\pi\omega H(\omega) e^{-3\omega} \cdot (-2\pi i) e^{-4\omega} H(\omega)$$

dal momento che facilmente si ottiene

$$\hat{g} = -2\pi i \, H(\omega) e^{-4\omega}$$

Pertanto

$$\hat{h}(\omega) = -2\pi i \left[-2\pi\omega H(\omega) e^{-7\omega} \right]$$

da cui

$$h(t) = -2\pi i \frac{1}{(t+7i)^2}$$

(per analogia con quanto ottenuto nell'esercizio (3)).

BRUNNEN

(5) Sappiamo da sviluppi noti che

$$\forall \alpha \in \mathbb{R} \qquad (1+t)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n$$

con $|t| < 1$

Pertanto, se $\alpha = 1/2$

$$(1+t)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} t^n \qquad \forall |t|<1$$

da cui per $|z|>1$

$$\left(1+\frac{1}{z}\right)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} z^{-n}$$

Concludiamo, quindi, che

$$f_n = \binom{1/2}{n}.$$

PARTE B

(1) Dato $\Omega \subseteq \mathbb{C}$ un insieme aperto e $f: \Omega \subseteq \mathbb{C} \to \mathbb{C}$ una funzione olomorfa in Ω, preso $z_0 \in \Omega$ e $R>0$ t.c. $\overline{B_R(z_0)} \subset \Omega$ abbiamo che $\forall z \in B_R(z_0)$

$$f(z) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi$$

dove $C_R(z_0)$ è la circonferenza di centro z_0 e raggio R orientata positivamente.

---

② $f \in L^1 \implies \hat{f} \in C_0^0(\mathbb{R}),\ \hat{f}$ limitata

$f \in L^2 \iff \hat{f} \in L^2(\mathbb{R})$

$tf \in L^2 \implies \frac{d}{d\omega}\hat{f} \in L^2(\mathbb{R})$

$\forall n \in \mathbb{N}\quad \frac{d^n}{dt^n} f \in L^1 \implies \forall n \in \mathbb{N}\quad \omega^n \hat{f} \in C_0^0(\mathbb{R}),\ \omega^n \hat{f}$ limitata

$\forall n \in \mathbb{N}\quad \frac{d^n}{dt^n} f \in L^2 \implies \forall n \in \mathbb{N},\ \omega^n \hat{f} \in L^2$

---

③ Sia $f: \mathbb{R} \to \mathbb{R}$, 2π-periodica, tale che

$$\int_{-\pi}^{\pi} |f(t)|^2 dt < +\infty.$$ Allora

$$\int_{-\pi}^{\pi} |f(t)|^2 dt = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \right]$$

Se esprimiamo f con la serie di Fourier in forma esponenziale, abbiamo

$$\int_{-\pi}^{\pi} |f(t)|^2 dt = 2\pi \sum_{n=-\infty}^{+\infty} |c_n|^2.$$

4. Osserviamo che, posto

$$f = \frac{1}{(t+3i)^2}, \qquad g = \frac{1}{t+4i},$$

abbiamo

$$f \in L^1, \quad g \in L^2 \implies h \in L^2$$

$$f \in L^2, \quad g \in L^2 \implies h \in C^0(\mathbb{R}) \text{ e } \lim_{|t|\to\infty} h(t) = 0$$

---

5. È immediato osservare che

$$\int_0^{+\infty} \frac{1-\cos t}{t^2} e^{-4t^2} e^{-\lambda t}\, dt$$

converge per ogni $\lambda \in \mathbb{R}$, a causa della rapida decrescenza a zero quando $t \to +\infty$ dell'esponenziale e^{-4t^2}.

Poiché λ_F è l'estremo inferiore dei λ per cui l'integrale converge, necessariamente $\lambda_F = -\infty$.