Advanced Mathematical Methods for Engineers - February 23 2015

1. Determine the general solution of the linear homogeneous system

$$\underline{z}' = \mathbb{A}\underline{z}, \quad \text{where} \quad \mathbb{A} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix}.$$

2. Consider the Cauchy Problem

$$\begin{cases} y' = e^{\frac{1}{y}}(y+2) \\ y(x_o) = y_o, \end{cases} \qquad (x_o, y_o) \in D = \mathbb{R}^2 \backslash \{y = 0\}.$$

   Determine the main properties of the solution and draw its qualitative graph, as (x_o, y_o) ranges in D.

3. Compute the Fourier Transform of the tempered distribution $u = \operatorname{sign} x$, taking into account that in the sense of distributions $(\operatorname{sign} x)' = 2\delta$.

   Then, relying on the previous result, and on the fundamental properties of the Fourier transform, compute

$$\mathcal{F}(x|x|) = \mathcal{F}(x^2 \operatorname{sign} x).$$

4. Consider the Hilbert space $H = L^2(\mathbb{R})$ and its complete orthonormal system $\{u_n\}$, $n = 0, 1, \ldots$ where

$$u_n(x) = \frac{1}{\sqrt{2^n\, n! \sqrt{\pi}}}\, e^{-x^2/2}\, H_n(x), \qquad H_n(x) = (-)^n e^{x^2} \frac{d^n}{dx^n}\, e^{-x^2}.$$

   Given the function

$$f(x) = (7x - 4)e^{-x^2/2},$$

   compute its Fourier expansion $f = \sum_n c_n u_n$, $c_n = (f, u_n)$, and relying on the fundamental relation $\|f\|^2 = \sum_n |c_n|^2$, determine the value of a proper integral over the interval $(-\infty, \infty)$.