HARNACK INEQUALITIES for NON-NEGATIVE SOLUTIONS to DEGENERATE and SINGULAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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on

Qualitative Properties of Solutions to Elliptic and Parabolic Equations

PLAN

1. Introduction
   Harnack inequality for the heat equation : $\boxed{p=2}$

2. Parabolic p-laplacian : $\boxed{p > 2}$

3. Consequences of the Harnack inequality for the p-laplacian

4. Parabolic p-laplacian : $\boxed{1 < p < 2}$

REMARK

* Model problem vs equations with full quasi-linear structure
* Lack of homogeneity vs non-linearity
* Porous media equation

Model problem

* $u_t - \Delta u = 0$

* $u_t - \operatorname{div}(|Du|^{p-2} Du) = 0$

Full quasi-linear structure

* $u_t - \operatorname{div} \mathbb{A}(x,t,u,Du) = B(x,t,u,Du)$

where

$$\mathbb{A} : E_T \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^N$$
$$B : E_T \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$$

and

$$\mathbb{A}(x,t,u,Du) \cdot Du \geq C_0 |Du|^p - C^p$$

$$|\mathbb{A}(x,t,u,Du)| \leq C_1 |Du|^{p-1} + C^{p-1}$$

$$|B(x,t,u,Du)| \leq C |Du|^{p-1} + C^{p-1}$$

a.e. in E_T

A bit of history

Elliptic (original !) Harnack inequality (Harnack, 1887) Let $u \geq 0$ be a harmonic function in a domain $E \subset \mathbb{R}^N$. Then there exists a constant $c = c(N)$ such that for every ball $B_{4\rho}(x_0) \subset E$

$$\sup_{B_\rho(x_0)} u \leq c \inf_{B_\rho(x_0)} u$$

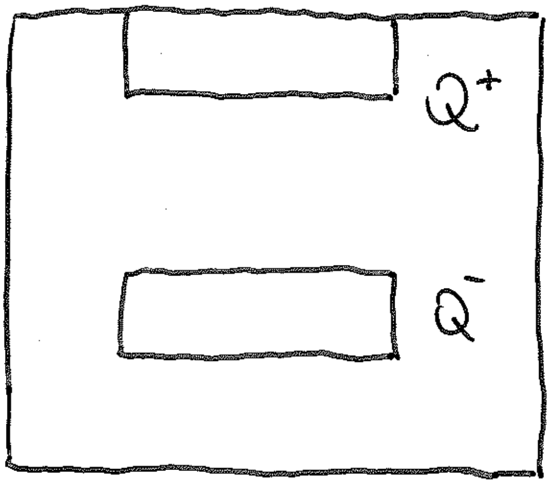
Remarks

* Proof based on explicit representation
* Extensions (Moser, Serrin, Trudinger, DiBenedetto, ...)

Parabolic Harnack inequality (Hadamard [13], Pini [21]). Let u be a non-negative solution of

$$u_t - \Delta u = 0$$

in $Q = \{|x| < \rho\} \times (0, T)$



Let

$$Q^+ = \{|x| < \rho'\} \times (\tau_1^+, \tau^+)$$

$$Q^- = \{|x| < \rho'\} \times (\tau_1^-, \tau_2^-)$$

Q $[0 < \rho' < \rho, \; 0 < \tau_1^- < \tau_2^- < \tau_1^+ < \tau]$

Then

$$\sup_{Q^-} u \leq \gamma \inf_{Q^+} u$$

where $\gamma = \gamma(N, \rho, \rho', \tau_1^+, \tau_1^-, \tau^+, \tau_2^-)$

<u>Equivalent statement</u>. u same as above

Let $(x_0, t_0) \in E_T$, $Q_\rho \equiv B_\rho(x_0) \times (t_0 - \rho^2, t_0 + \rho^2)$

There exist $c, \gamma = c, \gamma(N)$ such that for every $Q_{2\rho} \subset E_T$

$$u(x_0,t_0) \leq \gamma \inf_{B_\rho(x_0)} u(x, t_0 + C\rho^2)$$

$(\gamma, C > 0!)$

Remarks

* Since the two statements are equivalent, we will refer to the second one.

* Explicit representation of solutions

* Extensions (Moser, Aronson-Serrin, Trudinger ...)

* $B_\rho(x_0) \times (0, \rho^2)$ reflects the parabolic scaling

* Homogeneity with respect to u

* Physical meaning

* Mathematical meaning

<u>Idea</u> In Moser's proof the linearity is immaterial. Hence, with

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0$$

things should work with the scaling $B_\rho(x_0) \times (0, \rho^p)$

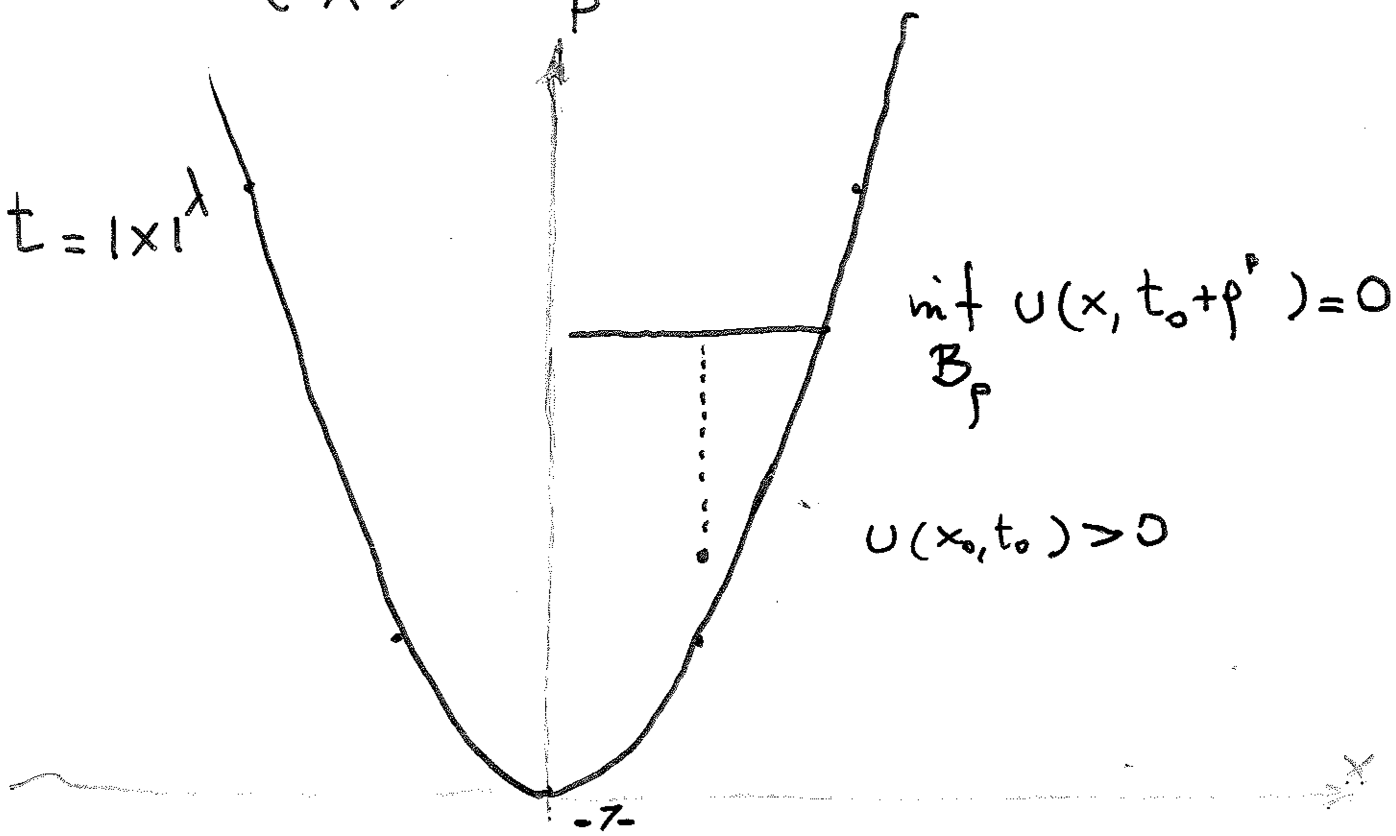
<u>Counterexample</u> The Barenblatt fundamental solution

$$\mathcal{B}(x,t) = t^{-N/\lambda}\left\{1 - \gamma_p\left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}}\right\}_+^{\frac{p-1}{p-2}}$$

$t > 0$

$\lambda = N(p-2) + p, \qquad p > 2$

$\gamma_p = \left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}} \frac{p-2}{p}$



Claim Lack of homogeneity

New Idea Look at

$$u_t - \operatorname{div}(|Du|^{p-2} Du) = 0$$

as if it were

$$\partial_t u^{p-1} - \operatorname{div}(|Du|^{p-2} Du) = 0$$

written in a time scale intrinsic to the solution itself, of the order of

$$t\,[u(x,t)]^{2-p}$$

Main fact INTRINSIC RESCALING

$[u(x_o,t_o)]^{2-p}$ is the intrinsic scaling factor.

Question How do I implement it?

Answer Back to the heat equation!

Remark : a proof based only on structural properties. Namely, take

$$u_t - \Delta u = 0,$$

multiply by

$$\pm (u-k)_\pm \zeta^2$$

with ζ proper cut-off function, integrate over $Q_\rho = B_\rho(x_0) \times (t_0, t_1)$ and get Energy estimates

$$\sup_{t_0 < t < t_1} \int_{B_\rho(x_0)} (u-k)_\pm^2 \zeta^2(x,t)\,dx + \iint_{Q_\rho} |D(u-k)_\pm|^2 \zeta^2\,dx\,d\tau \qquad (1.1)$$

$$\leq \gamma \left[\iint_{Q_\rho} (u-k)_\pm^2 \left(|D\zeta|^2 + \zeta_t \right) dx\,d\tau \right]$$

$$+ \int_{B_\rho(x_0)} (u-k)_\pm^2 \zeta^2(x, t_0)\,dx$$

The whole proof is based on (1.1)

Two technical tools

* Expansion of positivity
* Construction of a proper initial positivity cylinder or set

Remarks

* Properly adjusted, both for the degenerate and the singular p-laplacian we will use the same tools

* For the consthuction of the positivity set, we have an "easy" method, but also a different, more effective, more "complicated" one (stay tuned!)

Proposition 1.1 (Expansion of positivity)

Let u satisfy (1.1), $u \geq 0$. There exist constants ϑ, λ that depend only on the data, $\vartheta, \lambda \in (0,1)$ such that if

$$u(x, t_*) \geq h > 0 \quad \forall x \in B_\rho(x_*) \tag{1.2}$$

then

$$u(x, t_* + \vartheta\rho^2) \geq \lambda h \qquad \forall x \in B_{2\rho}(x_*) \tag{1.3}$$

provided $B_{4\rho}(x_*) \times [t_* - 4\vartheta\rho^2, t_* + 4\vartheta\rho^2] \subset E_T$

Remarks

* We use the same techniques as in [15], but we read them under an expanding point of view, rather than the usual shrinking one.
* We divide the proof in simpler steps

PROOF We can assume $(x_*, t_*) = (0,0)$ and we set

$$A_{h,R}(\cdot) = \{ x \in B_R : u(x,\cdot) < h \}$$

$$Q_R(\vartheta) = B_R \times (-\vartheta R^2, 0)$$

STEP I From (1.2)

$$|A_{h,4\rho}(t_*)| \leq (1 - 4^{-N}) |B_{4\rho}| \qquad (1.4)$$

STEP II Let (1.2) hold. $\exists\, \vartheta, \eta = \vartheta, \eta(\text{data})$ such that

$$|A_{\eta h, 4\rho}(t)| \leq (1 - 4^{-(N+1)}) |B_{4\rho}|$$
$$\forall t \in (t_*, t_* + \vartheta \rho^2) \qquad (1.5)$$

Proof of Step II

- Write (1.1) over $B_{4\rho} \times (0, \vartheta\rho^2)$
- Assume $\zeta = \zeta(x) \in W_0^{1,\infty}(B_{4\rho})$ such that

$$\zeta \equiv 1 \text{ on } B_{4\rho(1-\sigma)}, \quad |D\zeta| \leq \frac{1}{4\sigma\rho}$$

$$\int_{B_{4\rho(1-\sigma)}} (u-h)_-^2(x,\tau)\,dx \le \int_{B_{4\rho}} (u-h)_-^2(x,0)\,dx$$

$$+ \frac{\gamma}{(4\sigma\rho)^2} h^2 \theta \rho^{N+2} \qquad (\text{by } (1.4))$$

$$\le h^2 \left[(1-4^{-N}) |B_{4\rho}| + \frac{\gamma\theta}{(4\sigma)^2} \rho^N \right]$$

This holds $\forall \tau \in (0, \theta\rho^2)$

Notice that

$$\int_{B_{4\rho(1-\sigma)}} (u-h)_-^2\,dx \ge \int_{B_{4\rho(1-\sigma)} \cap [u<\eta h]} (u-h)_-^2\,dx$$

$$\ge h^2(1-\eta)^2 |A_{\eta h, 4\rho(1-\sigma)}|$$

Moreover

$$|A_{\eta h, 4\rho}| = |A_{\eta h, 4\rho(1-\sigma)}| + |A_{\eta h,} \cap [B_{4\rho} \setminus B_{4\rho(1-\sigma)}]|$$

$$\le |A_{\eta h, 4\rho(1-\sigma)}| + N\sigma |B_{4\rho}|$$

Hence, putting everything together

$$|\Delta_{\eta h, 4\rho}(\tau)| \leq \frac{1}{h^2(1-\eta)^2} h^2\left[(1-4^{-N}) + \frac{\tilde{\gamma}\vartheta}{(4\sigma)^2}\right]|B_{4\rho}|$$

$$+ N\sigma |B_{4\rho}|$$

$$= \left\{\frac{1}{(1-\eta)^2}\left[(1-4^{-N}) + \frac{\tilde{\gamma}\vartheta}{(4\sigma)^2}\right] + N\sigma\right\}|B_{4\rho}|$$

Now choose η, σ, ϑ sufficiently close to zero and we are done $\blacksquare$

From here on, till the end of the proof, θ is fixed.

Consider $(x_*, t_o) = (x_*, t_* + \vartheta\rho^2)$

With a second change of variables, w.l.o.g. we can assume $(x_*, t_o) \equiv (0,0)$

Hence $Q_{\alpha R}(\beta) = B_{\alpha R} \times (-\beta R^2, 0)$

Let us now recall De Giorgi's version of Poincaré inequality