LEMMA Suppose that $u \in W^{1,1}(B_\rho)$, $u \geq 0$, and $u = 0$ on some set $\mathcal{E}_0$ of positive measure. Then for any measurable set $\mathcal{E}$ from K_ρ we have

$$\int_{\mathcal{E}} u(x)\,\varphi(x)\,dx \leq C(N)\,\frac{\rho^N}{|\mathcal{E}_0|}\,|\mathcal{E}|^{1/N} \int_{K_\rho} |Du(x)|\,\varphi(x)\,dx$$

where $\varphi(x) = \varphi(|x|)$ is an arbitrary non-increasing function of $|x|$ with values from $[0,1]$ that is equal to unity on $\mathcal{E}_0$.

For the proof, see [15], page 89, or [8], page 5.

In the following we will frequently use this Corollary

Corollary Let $v \in W^{1,1}(B_\rho) \cap C(B_\rho)$, and let $k, l \in \mathbb{R}$ such that $k < l$.

Then

$$(l-k)\,|[v > l]| \le \gamma \frac{\rho^{N+1}}{|[v<k]|} \int_{[k<v<l]} |Dv|\,dx$$

STEP $\overline{III}$ $\forall\, \varepsilon \in (0,1)$ $\exists$ $\eta_1 = \eta_1(\varepsilon, \text{data})$ such that

$$|[u < \eta_1 h] \cap Q_{4\rho}(\theta)| < \varepsilon\, |Q_{4\rho}(\theta)|$$

Proof of Step $\overline{III}$

* Write (1.1) over $Q_{5\rho}(2\theta)$
* Assume $\zeta(x,t) \equiv 1$ on $Q_{4\rho}(\theta)$, $|D\zeta| < \frac{1}{\rho}$, $0 \leq \zeta_t \leq \frac{2}{\theta\rho^2}$
* Set $k = \eta h 2^{-s}$, η as in Step $\overline{II}$, $s \in \mathbb{N}$ to be determined

By (1.1) we get

$$\iint_{Q_{4\rho}(\theta)} |D(u - \tfrac{\eta h}{2^s})_-|^2 \leq \gamma_1 \frac{\eta^2 h^2}{2^{2s}} \rho^N,$$

$\gamma_1 = \gamma_1(\gamma, \theta, N)$

By Step $\overline{II}$ and DeGiorgi's Lemma

$$\int_{B_{4\rho}} (u - \tfrac{\eta h}{2^s})_-(x,\tau)\,dx \leq \gamma_2 \rho \int_{B_{4\rho} \cap [\frac{\eta h}{2^{s+1}} < u < \frac{\eta h}{2^s}]} |Du|(x,\tau)\,dx$$

If we integrate in $d\tau$ over $t \in (-\vartheta\rho^2, 0)$ and set

$$A_s(\tau) = \{ x \in B_{4\rho} : u(x,\tau) \leq \eta \frac{h}{2^s} \}$$

$$|A_s| = \int_{-\vartheta\rho^2}^{0} |A_s(\tau)|\, d\tau$$

We obtain

$$\eta \frac{h}{2^{s+1}} |A_{s+1}| \leq \gamma_2 \rho \left(\iint_{Q_{4\rho}(\theta)} |D(u - \eta \tfrac{h}{2^s})_-|^2 dx\, d\tau \right)^{1/2} \cdot |A_s \setminus A_{s+1}|^{1/2}$$

$$\leq \gamma_3 \eta \frac{h}{2^s} \rho^{N/2 + 1} |A_s \setminus A_{s+1}|^{1/2}$$

and also

$$|A_{s+1}|^2 \leq \gamma_4 |A_s \setminus A_{s+1}|\, |Q_{4\rho}(\theta)|$$

Adding for $s = 0, 1, \text{—}, s^* - 1$

$$|A_{s^*}| \leq \left(\frac{\gamma_4}{s_* - 1} \right)^{1/2} |Q_{4\rho}(\theta)|$$

and we are finished once we choose s^* such that

$$\left(\frac{\gamma_4}{s^* - 1} \right)^{1/2} \leq \varepsilon \quad \Longrightarrow \quad \eta_1 = \frac{\eta}{2^{s_*}}$$

<u>Step IV</u> Let u satisfy (1.1). $\forall a \in (0,1)$ $\exists \nu = \nu(a, \text{data})$ such that if

$$|[u < k] \cap Q_{4\rho}(\theta)| < \nu |Q_{4\rho}(\theta)|$$

then

$$u > ak \quad \text{in } B_{2\rho} \times \left(-\frac{\theta}{2}\rho^2, 0\right)$$

We skip the proof of this. It is the core of De Giorgi's iteration method [4] as revisited in the parabolic context in [15].

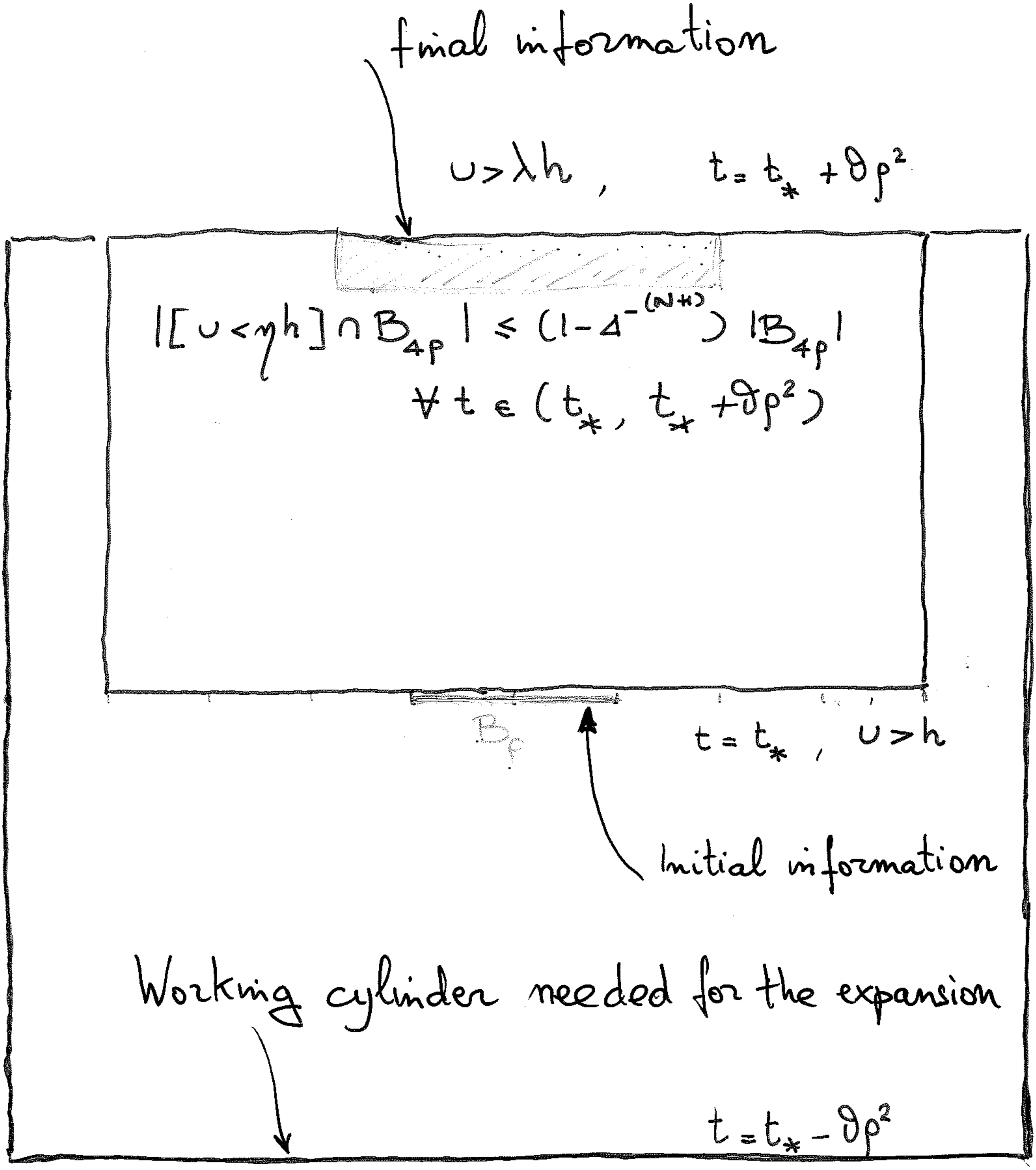
<u>Remark</u> ν does not depend on k.

End of the Proof of Proposition 1.1.

Choose $a = \frac{1}{2}$, ε of Step III as ν and by Step IV conclude that

$$u(x,0) > \frac{1}{2}\eta_1 h \qquad \forall x \in B_{2\rho}$$

$[\Longrightarrow \lambda = \frac{1}{2}\eta_1$; recall that $t=0 \Longrightarrow t = \theta\rho^2]$


final information
$u>\lambda h$, $t=t_*+\vartheta\rho^2$
$|[u<\eta h]\cap B_{4\rho}| \le (1-4^{-(N+1)})|B_{4\rho}|$
$\forall t\in(t_*, t_*+\vartheta\rho^2)$
B_ρ
$t=t_*$, $u>h$
Initial information
Working cylinder needed for the expansion
$t=t_*-\vartheta\rho^2$

Remarks

* In order to achieve the expansion of positivity, we need a large working cylinder (larger than the final set)

* The initial information allow us to control the measure of the level sets in a proper time range, whose width depends only on the data

PROOF OF THE HARNACK INEQUALITY

* Fix $(x_0,t_0)\in E_T$ and $\rho>0$ s.t.
$B_{8\rho}(x_0)\times\left(t_0-(8\rho)^2,\, t_0+(8\rho)^2\right)\subseteq E_T$
$u(x_0,t_0)>0$

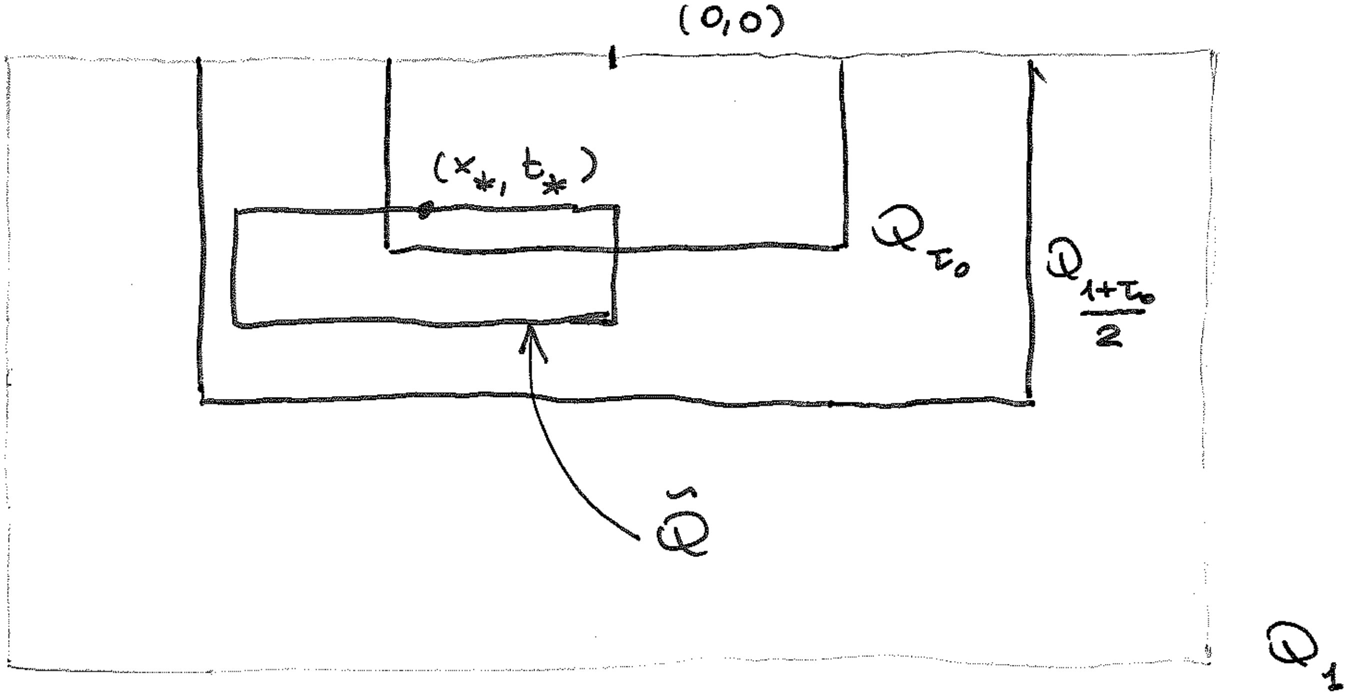
* We make a change of variables
$$y=\frac{x-x_0}{\rho}, \qquad \tau=\frac{t-t_0}{\rho^2}$$
$$v(y,\tau)=\frac{u(\cdot,\cdot)}{u(x_0,t_0)}$$
and redefine u, x, t the new variables. We assume u continuous

* We prove the Harnack inequality if we show that there exist two positive constants $C_1, C_2 = C_1, C_2$ (data) such that
$$\inf_{B_1(0)} u(x,C_1)\geq C_2$$

* For $\tau\in(0,1)$ define
$Q_\tau = B_\tau(0)\times(-\tau^2,0)$, $M_\tau=\sup_{Q_\tau} u$,
$N_\tau=(1-\tau)^{-\beta}$ $\beta>1$ to be determined

* Let τ_0 be the largest root of $M_\tau = N_\tau$
* Let $(x_*, t_*) \in Q_{\tau_0}$ such that

$$u(x_*, t_*) = (1-\tau_0)^{-\beta}$$



* The box

$$\tilde{Q} = B_{\frac{1-\tau_0}{2}}(x_*) \times \left(t_* - \left(\frac{1-\tau_0}{2}\right)^2, t_*\right) \subset Q_{\frac{1+\tau_0}{2}}$$

$$\Longrightarrow \quad \sup_{\tilde{Q}} u \leq N_{\frac{1+\tau_0}{2}} = 2^\beta (1-\tau_0)^{-\beta}$$

* Bounded functions that satisfy (1.1) are Hölder continuous, namely

$$\operatorname*{osc}_{Q_\rho} u \leq C \|u\|_{\infty, Q_R} \left(\frac{\rho}{R}\right)^\alpha$$

Hence $\forall \varepsilon \in (0,1)$, $\forall x \in B_{\varepsilon \frac{1-\tau_0}{2}}(x_*)$

$$u(x,t_*) \geq u(x_*,t_*) - C\,2^{\beta}(1-\tau_0)^{-\beta}\varepsilon^{\alpha}$$

$$\geq (1-\tau_0)^{-\beta}\left[1 - C\,2^{\beta}\varepsilon^{\alpha}\right]$$

Choose ε so small that $[\quad] = \frac{1}{2}$ and we have obtained

$$u(x,t_*) \geq \frac{1}{2}(1-\tau_0)^{-\beta} \quad \forall x \in B_{\varepsilon \frac{1-\tau_0}{2}}(x_*)$$

Remarks

* We have obtained the initial positivity block, relying on the Hölder continuity of u
* Since β is not determined yet, ε is only qualitatively known. In a moment it will be quantitatively estimated.
* Now we can apply Proposition 1.1 with $h = \frac{1}{2}(1-\tau_0)^{-\beta}$. Up to time-rescaling, we can assume $\theta \simeq 4$.

* By Proposition 1.1 we have

$$u(x,t_1) \geq \frac{1}{2}\lambda(1-\tau_0)^{-\beta}$$

$\forall x \in B_{r_1}(x_*)$, $\quad r_1 = 2\,\varepsilon\,\frac{1-\tau_0}{2}$,

$t_1 = t_* + 2^2\left(\varepsilon\,\frac{1-\tau_0}{2}\right)^2$

$$u(x,t_2) \geq \frac{1}{2}\lambda^2(1-\tau_0)^{-\beta}$$

$\forall x \in B_{r_2}(x_*)$, $\quad r_2 = 2^2\,\varepsilon\,\frac{1-\tau_0}{2}$

$t_2 = t_1 + 2^4\left(\varepsilon\,\frac{1-\tau_0}{2}\right)^2$

$\vdots$

$$u(x,t_n) \geq \frac{1}{2}\lambda^n(1-\tau_0)^{-\beta}$$

$\forall x \in B_{r_n}(x_*)$, $\quad r_n = 2^n\,\varepsilon\,\frac{1-\tau_0}{2}$

$t_n = t_{n-1} + 2^{2n}\left(\varepsilon\,\frac{1-\tau_0}{2}\right)^2$

Now choose $n \in \mathbb{N}$ such that

$1 \leq 2^n\,\frac{1-\tau_0}{2} \leq 2$ i.e. $\varepsilon \leq r_n \leq 2\varepsilon$

Correspondingly

$$\frac{1}{2}\lambda^n(1-\tau_0)^{-\beta} \geq \frac{1}{2}\lambda^n\,2^{(n-2)\beta}$$

$$= (\lambda 2^\beta)^n\,2^{-2\beta-1}$$

If we now choose $\beta = -\log_2 \lambda$, we have found a point $(x_*, \bar{t})$ and a quantitative ε such that

$$u(x, \bar{t}) \geq 2^{-(2\beta+1)} \quad \forall x \in B_\varepsilon(x_*)$$

with $\bar{t} \in (-1,1)$

* Now we apply Proposition 1.1

$$\ell = [|\log_2 \varepsilon|] + 1$$

times, that is as many times ℓ as needed to cover $B_1(0)$.

We find a level $\check{t} \in (0,2)$

$$u(x, \check{t}) \geq 2^{-(2\beta+1)} \lambda^\ell \quad \forall x \in B_1$$

A further application of Proposition 1.1 gives

$$u(x, 4) \geq 2^{-(2\beta+1)} \lambda^{\ell+1} \quad \forall x \in B_1$$

$\blacksquare$