THIRD PART

or

How it happens that you get hungrier and hungrier as soon as you start eating

SOME QUESTIONS

In the following we are interested in local properties of solutions to

$$u_t - \operatorname{div} \mathbb{A}(x,t,u,Du) = 0 \qquad (3.1)$$

where

a) $\mathbb{A}(x,t,u,Du)\cdot Du \geq C_0 |Du|^p$ (3.2)

b) $|\mathbb{A}(x,t,u,Du)| \leq C_1 |Du|^{p-1}$ (3.3)

<u>First question</u> From Moser's Harnack inequality for non-negative solutions to

$$u_t - \operatorname{div}(a_{ij}(x,t) D_i u) = 0$$

with

$$a_{ij} \in L^\infty(E_T), \qquad a_{ij}\xi_i\xi_j \geq C|\xi|^2$$

it follows that

$$u(t,y) \geq u(s,x)\left(\frac{s}{t}\right)^{N/2} \exp\left[-A\left(1 + \frac{|x-y|^2}{t-s}\right)\right]$$

where A depends on the data and $0 < s < t < T$

See [2], [18], [22]

The lower bound has the same structure of the fundamental solution of the heat equation.

What can we say in our context?
Is there any hope to recover a similar result?

<u>Second question</u> Let us consider the p-laplacian

$$u_t - \operatorname{div}(|Du|^{p-2} Du) = 0$$

and the following family of Barenblatt-like functions

$$\Gamma_{\nu,p}(x,t;\bar{x},\bar{t}) = \frac{k\rho^{\nu}}{S^{\nu/\lambda}(t)} \left[1 - b(\nu,p) \left(\frac{|x-\bar{x}|}{S^{1/\lambda}(t)} \right)^{\frac{p}{p-1}} \right]_{+}^{\frac{p-1}{p-2}}$$

where $\nu \geq N$, k, ρ are parameters

$$S(t) = k^{p-2} \rho^{\nu(p-2)} (t-\bar{t}) + \rho^{\lambda}, \quad t \geq \bar{t}$$

$$\lambda = \nu(p-2) + p, \quad b(\nu,p) = \frac{1}{\lambda^{\frac{1}{p-1}}} \frac{p-2}{p}$$

* If $\nu = N$, $\Gamma_{\nu,p}$ are exact solutions for all $\kappa, \rho > 0$.

* If $\nu > N$, $\Gamma_{\nu,p}$ are subsolutions for all $\kappa, \rho > 0$

Main properties

* $\Gamma_{\nu,p} \leq \kappa$
* $\Gamma_{\nu,p}$ are supported in $\{|x-\bar{x}| < S^{\frac{1}{\lambda}}(s)\} \times (\bar{t}, t]$
* $\Gamma_{\nu,p}$ can be used to expand the positivity of any weak solution u (see [8])
* As $p \to 2$

$$\Gamma_{\nu,p} \longrightarrow \Gamma_{\nu}(x,t;\bar{x},\bar{t}) = \frac{\kappa\rho^{\nu}}{[(t-\bar{t})+\rho^2]^{\nu/2}} \exp\left\{-\frac{|x-\bar{x}|^2}{4[(t-\bar{t})+\rho^2]}\right\}$$

and Γ_ν have similar properties for the heat equation.

Hence $\Gamma_{\nu,p}$ for $\nu \geq N$ are sub-potentials

When dealing with (3.1) – (3.3) there is no explicit analog of $\Gamma_{\nu, p}$, since we have only a structural knowledge of the equation.
However, is there any similar behavior for solutions to (3.1) – (3.3)?

Third (related) question

By $\Gamma_{\nu, p}$ we know that solutions to the parabolic p-laplacian do not decay faster than $t^{-\nu/\lambda}$, $t \gg 1$
On the other hand, in the proof of the expansion of positivity, we have seen that for $t \gg 1$

$$u \simeq \frac{C}{t^{\frac{1}{p-2}}}$$

which amounts to taking $\nu \longrightarrow \infty$
In a bounded domain, such a decay is sharp.

Is there a way to explain this kind of behavior ?

Fourth question In [8] the following result is proved

Proposition Let $u \geq 0$ be a weak solution of the p-laplacian. There exists a constant $B > 1$ depending only upon the data, such that $\forall (x_0, t_0) \in E_T$ $\forall \rho, \vartheta > 0$, such that $B_{4\rho}(x_0) \times (t_0 - 4^p\vartheta, t_0 + 4^p\vartheta) \subset \Omega_T$

$$u(x_0,t_0) \leq B\left\{\left(\frac{\rho^p}{\vartheta}\right)^{\frac{1}{p-2}} + \left(\frac{\vartheta}{\rho^p}\right)^{N/p}\left[\inf_{B_\rho(x_0)} u(x, t_0+\vartheta)\right]^{\lambda/p}\right\}$$

Remarks

* It is an alternative form of the Harnack inequality where the geometry can be a priori prescribed, independent of the solution

* The two forms are **equivalent**

* It is instrumental in determining the optimal growth of the initial data as $|x| \to \infty$ for the solvability of the Cauchy Problem (see [DiBenedetto - Herrero T.A.M.S. (1989)).
* It implies the optimal decay

$$u(t) \geq \frac{c}{t^{N/\lambda}}, \qquad t \gg 1, \quad \lambda = N(p-2)+p$$

Can we have something similar in the quasi-linear setting?

Fifth (stupid) question

Can we get a simpler and more straightforward proof of the expansion of positivity?

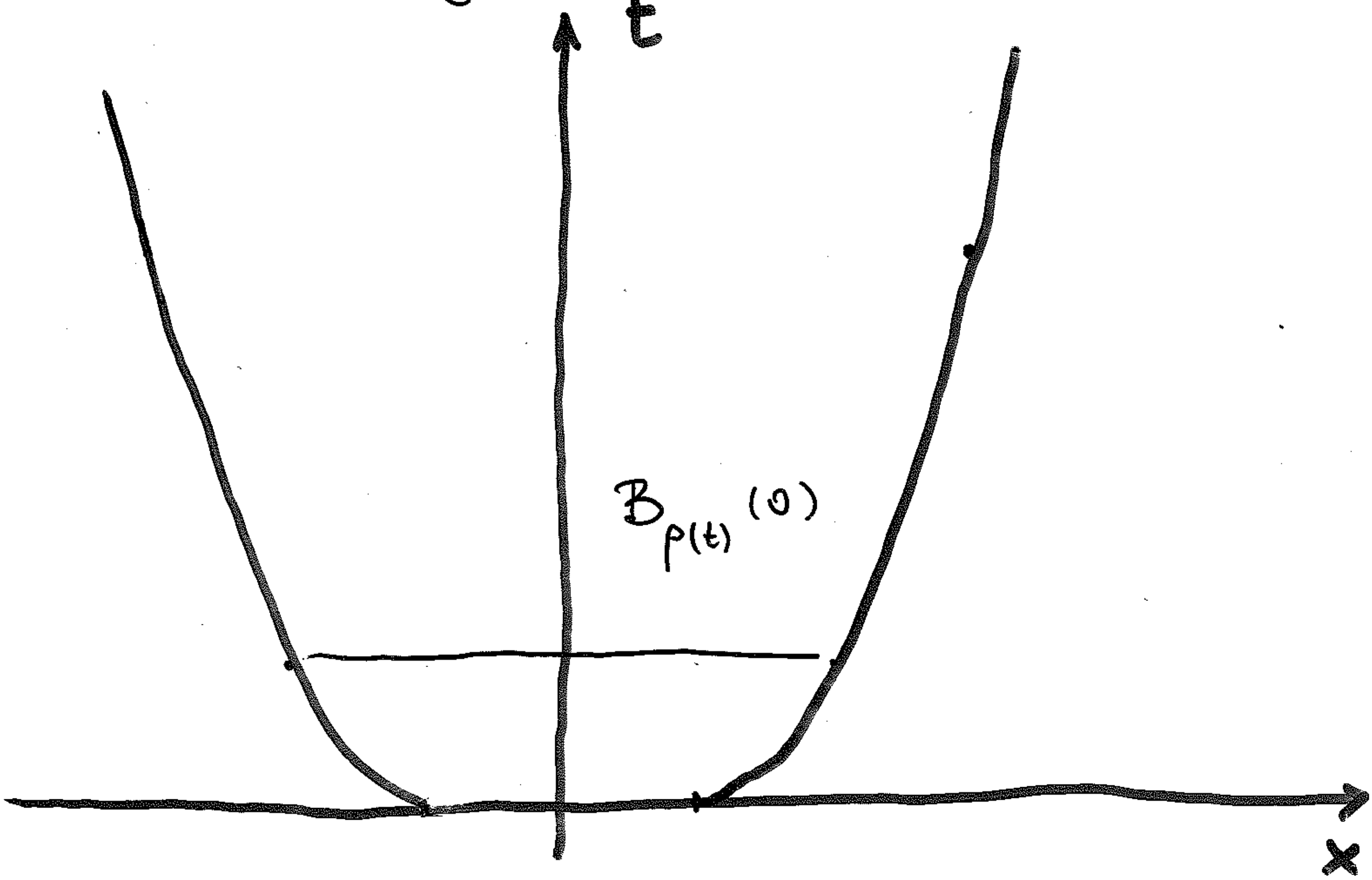
Remarks

* All the previous questions ask if the Barenblatt-like functions drive the behavior in the quasi-linear setting
* Ill-posed question?
* We cannot work in a fixed cylinder because of the time decay,

$$t^{-\frac{1}{p-2}} \quad \text{vs} \quad t^{-\frac{N}{N(p-2)+p}}$$

First idea

If we want to mimic the behavior of Barenblatt-like functions choose a Barenblatt-like cylinder, instead of a straight one



Unfortunately, it doesn't work

Second idea

In the proof of the expansion of positivity, we need w, which is obtained from u, by time-rescaling

Hence

* Barenblatt-like space-time domain

+

* time-varying cuts $(u-k(t))_-$

As before, let

$$\lambda = \nu(p-2) + p \qquad \nu \text{ positive}$$

to be chosen

$$S(t) = \lambda k^{p-2} \rho^{\nu(p-2)} (t-\bar{t}) + \rho^{\lambda}$$

with

ρ, k positive, $\quad t > \bar{t}$

$$\mathcal{D}(t) = \{ |x-\bar{x}| < S^{1/\lambda}(s) \} \times (\bar{t}, t]$$

These are Barenblatt-like space-time domains.

Then

<u>Proposition 3.1</u>

Let $\quad u(\cdot, \bar{t}) \geq k \quad \text{in } B_\rho(\bar{x})$

and $\quad \mathcal{D}(t) \subseteq E_T$

Then the constant λ (and correspondingly ν) can be chosen depending only upon the data and independent of k such that

$$u(x,t) \geqslant \frac{k\rho^{\nu}}{S^{\nu/\lambda}(t)} \qquad \forall\, t > \bar{t},\ |x-\bar{x}| < \frac{1}{2} S^{1/\lambda}(t) \tag{3.4}$$

Remarks

* We have a different (but equivalent) way of stating the expansion of positivity of Proposition 2.1
* It allows a different, simpler and more direct proof of the intrinsic Harnack inequality
* The Proposition holds also for supersolutions
* Once λ is identified, the proof shows that the Proposition continues to hold

for any larger λ (and ν). Hence (3.4) gives a family of lower bounds, parametrized by ν.

* These estimates are forward in time and for them to hold the solution u is not required to exist for times $t < \bar{t}$!

* We have answered some (but not all yet) of the previous questions,