DERIVATA DI LIE DI UN CAMPO VETTORIALE

$X, Y \in \mathfrak{X}(M)$ Sia $\{\vartheta_t\}$ = flusso di X

$$L_X Y := \frac{d}{dt}\Big|_{t=0} (\vartheta_{-t})_* Y$$

Vogliamo dimostrare che $L_X Y$ e' un campo liscio e vogliamo anche dimostrare che $L_X Y = [X, Y]$.

Sia $(U, \varphi = (x^1 \ldots x^n))$ una carta su M.

Fissato $p \in U$ possiamo trovare $\varepsilon > 0$ e V = intorno di p t.c.

$$\vartheta : (-\varepsilon, \varepsilon) \times V \longrightarrow U$$

sia $\bar{\vartheta} : (-\varepsilon, \varepsilon) \times \varphi(V) \longrightarrow \varphi(U)$ la rappresentazione locale.

e sia
$$X = \sum \xi^i \frac{\partial}{\partial x^i}$$
$$Y = \sum \eta^i \frac{\partial}{\partial x^i} \quad \text{su } U.$$

$\xi^i, \eta^i \in C^\infty(U)$.

Se $|t|<\varepsilon$

$$\left((\vartheta_{-t})_* Y\right)(p) = (d\vartheta_{-t})_{\vartheta_t(p)}\left(Y(\vartheta_t(p))\right) =$$

$$= (d\vartheta_{-t})_{\vartheta_t(p)}\left(\sum \eta^i(\vartheta_t(p))\ \frac{\partial}{\partial x^i}\Big|_{\vartheta_t(p)}\right)$$

$$= \sum \eta^i(\vartheta_t(p))\cdot(d\vartheta_{-t})_{\vartheta_t(p)}\left(\frac{\partial}{\partial x^i}\Big|_{\vartheta_t(p)}\right)$$

ora $\displaystyle (d\vartheta_{-t})_{\vartheta_t(p)}\ \frac{\partial}{\partial x^i}\Big|_{\vartheta_t(p)} =$

$$= \sum_{j=1}^{n} \frac{\partial \overline{\vartheta_{-t}}^{\,j}}{\partial x^i}\left(\varphi\vartheta_t(p)\right)\cdot\frac{\partial}{\partial x^j}\Big|_{\vartheta_{-t}\vartheta_t(p)}$$

perché il differenziale di ϑ_{-t} e' rappresentato nelle basi dalla matrice Jacobiana della sua rappresentazione locale. Quindi

$$(d\vartheta_{-t})_{\vartheta_t(p)}\left(\frac{\partial}{\partial x^i}\Big|_{\vartheta_t(p)}\right) = \sum_{j=1}^{n} \frac{\partial \overline{\vartheta}^{\,j}_{-t}}{\partial x^i}(\varphi\vartheta_t(p))\cdot\frac{\partial}{\partial x^j}\Big|_p$$

$$\left((\vartheta_{-t})_* Y\right)(p) = \sum_{i,j=1}^{n} \eta^i\circ\vartheta_t(p)\ \frac{\partial \overline{\vartheta}^{\,j}_{-t}}{\partial x^i}(\varphi\vartheta_t(p))\ \frac{\partial}{\partial x^j}\Big|_p.$$

Da questa formula vediamo che le componenti di $(\theta_{-t})_* Y$ sono le funzioni

$$\sum_{i=1}^{n} \eta^i \circ \theta_t \quad \frac{\partial \bar{\theta}^j_{-t}}{\partial x^i} \circ \varphi \circ \theta_t =: \xi^j(t,\cdot)$$

queste sono funzioni C^∞ su $(-\varepsilon, \varepsilon) \times V$. Quindi se le deriviamo rispetto a t e le calcoliamo per $t=0$ otteniamo delle funzioni lisce su V.

Questo dimostra intanto che $L_X Y$ e' ben definito ed e' un campo vettoriale C^∞.

Ora vogliamo calcolare effettivamente le componenti di $L_X Y$

$$(\theta_{-t})_* Y(p) = \sum_{j=1}^{n} \xi^j(t,p) \left.\frac{\partial}{\partial x^j}\right|_p$$

$$L_X Y(p) = \sum_{j=1}^{n} \frac{\partial \xi^j}{\partial t}(0,p) \left.\frac{\partial}{\partial x^j}\right|_p$$

$$\frac{\partial z^j}{\partial t}(0,p) = \sum_{i=1}^{n} \frac{\partial}{\partial t}\eta^i \circ \vartheta_t(0,p) \cdot \frac{\partial \overline{\vartheta}^j_{-t}}{\partial x^i} \circ \varphi \circ \vartheta_t(0,p) +$$

$$+ \sum_{i=1}^{n} \eta^i(\vartheta_0(p)) \cdot \frac{\partial}{\partial t}\left(\frac{\partial \overline{\vartheta}^j_{-t}}{\partial x^i} \circ \varphi \circ \vartheta_t\right)(0,p)$$

$$\vartheta_0 = id_M \implies \eta^i \circ \vartheta_t(0,p) = \eta^i(p).$$

$$\frac{\partial}{\partial t}(\eta^i \circ \vartheta_t)(0,p) = \left.\frac{d}{dt}\right|_{t=0} \eta^i(\vartheta_t(p)) =$$

$$= X\eta^i(p).$$

$$\frac{\partial \overline{\vartheta}^j_{-t}}{\partial x^i} \circ \varphi \circ \vartheta_t(0,p) = \frac{\partial \overline{\vartheta}^j_0}{\partial x^i} \circ \varphi \circ \vartheta_0(p) =$$

$$= \frac{\partial \overline{\vartheta}^j_0}{\partial x^i}(\varphi(p)).$$

$\overline{\vartheta} : (-\varepsilon, \varepsilon) \times \varphi(V) \longrightarrow \varphi(U)$ è il flusso locale di $\varphi_* X = \sum_{i=1}^{n} \xi^i \circ \varphi^{-1} \cdot e_i$

$$\implies \overline{\vartheta}_0(x) = x \implies \frac{\partial \overline{\vartheta}^j_0}{\partial x^i} = \delta_{ij}$$

(5)

$$\frac{\partial}{\partial t}\left(\frac{\partial \overline{\vartheta}_{-t}^{\,j}}{\partial x^i}(\varphi\vartheta_t(p))\right) = ? \qquad (*)$$

~~poniamo~~ $$= \frac{\partial}{\partial t}\left(\frac{\partial \overline{\vartheta}^{\,j}}{\partial x^i}(-t, \varphi\vartheta_t(p))\right) =$$

$$= \frac{\partial^2 \overline{\vartheta}^{\,j}}{\partial t\, \partial x^i}(0, \varphi(p))\,(-1) +$$

$$+ \sum_{k=1}^{n} \frac{\partial^2 \overline{\vartheta}^{\,j}}{\partial x^k \partial x^i}(0, \varphi(p)) \cdot \left.\frac{d}{dt}\right|_{t=0} x^k\vartheta_t(p)$$

$$\left.\frac{d}{dt}\right|_{t=0} x^k\vartheta_t(p) = X x^k(p) = \xi^k(p)$$

quindi

$$\underset{\substack{\parallel \\ (*)}}{\frac{\partial}{\partial t}(\ldots)} = -\frac{\partial^2 \overline{\vartheta}^{\,j}}{\partial t\, \partial x^i}(0, \varphi(p)) +$$

$$+ \sum \frac{\partial^2 \overline{\vartheta}^{\,j}}{\partial x^k \partial x^i}(0, \varphi(p)) \cdot \xi^k(p)$$

Ma $\quad \dfrac{\partial^2 \bar{\theta}^j}{\partial t\, \partial x^i}(0,x) =$ $\qquad x := \varphi(p)$ 6

$$= \frac{\partial}{\partial x^i} \frac{\partial \bar{\theta}^j}{\partial t}(0,x)$$

$\bar{\theta}$ e' il flusso di $\varphi_* X = \sum \xi^i \circ \varphi^{-1} \cdot e_i$

$$\Rightarrow \frac{d}{dt} \bar{\theta}(t,x) = \varphi_* X(x)$$

$$\Rightarrow \frac{\partial^2 \bar{\theta}^j}{\partial t \partial x^i}(0,x) = \frac{\partial}{\partial x^i} \xi^j(\varphi^{-1}(x)) =$$

$$= \frac{\partial\, \xi^j \circ \varphi}{\partial x^i}(x) = \frac{\partial \bar{\xi}^j}{\partial x^i}(\varphi(p)) = \left.\frac{\partial}{\partial x^i}\right|_p \cdot \xi^j$$

mentre

$$\frac{\partial^2 \bar{\theta}^j}{\partial x^k \partial x^i}(0,x) = \frac{\partial}{\partial x^k} \frac{\partial \bar{\theta}^j}{\partial x^i}(0,x) =$$

$$= \frac{\partial}{\partial x^k} \frac{\partial \bar{\theta}_0^j}{\partial x^i}(x) = \frac{\partial}{\partial x^k} \delta_{ij} \equiv 0.$$

Quindi

$$(*) = - \frac{\partial}{\partial x^i}\Big|_p \xi^j + 0$$

$$\frac{\partial z^j}{\partial t}(0,p) = \sum_{i=1}^{n} X\eta^i(p) \cdot \delta_{ij} +$$

$$+ \sum_{i=1}^{n} \eta^i(p) \cdot \left(- \frac{\partial}{\partial x^i}\Big|_p \xi^j \right) =$$

$$= X\eta^j(p) - \left(\eta^i(p) \frac{\partial}{\partial x^i}\Big|_p \right) \xi^j$$

$$= X\eta^j(p) - Y\xi^j(p).$$

Lemma

- Sia $f: M \to N$ diffeo, $X, Y \in \mathfrak{X}(M)$

$X = \{\vartheta_t\}$ $\qquad \frac{d}{dt}\Big|_{t=0} f_*(\vartheta_{-t})_* Y = f_* L_X Y$

dim $p \in M$ ~~$t \in \mathbb{R}$~~

$$f_*(\vartheta_{-t})_* Y(p) = df_{f^{-1}(p)}\left((\vartheta_{-t})_* Y(f^{-1}(p))\right)$$

$$= df_{f^{-1}(p)}\, (d\vartheta_{-t})_{(\vartheta_t f^{-1}(p))} \left(Y(\vartheta_t f^{-1}(p))\right)$$

derivo in t

$$\frac{d}{dt}\Big|_{t=0} f_* \vartheta_{-t*} Y(p) =$$

$$= df_{f^{-1}(p)}\left(\frac{d}{dt}\Big|_{t=0} (d\vartheta_{-t})_{\vartheta_t f^{-1}(p)}\left(Y(\vartheta_t f^{-1}(p))\right)\right)$$

$$= df_{f^{-1}(p)}\left(L_X Y(p)\right).$$ ///

Abbiamo usato questo fatto

$(-\varepsilon, \varepsilon) \xrightarrow{v} V$ $\qquad$ V = sp. vett.
$\qquad W$ "

$L: (-\varepsilon, \varepsilon) \to L(V, W)$

$\frac{d}{dt} L(t)v(t) = \dot{L}v + L\dot{v}$ $\quad$ perché $L(V,W) \times V \to W$ è bilineare

Esercizio: V, W, U = sp. vett su $\mathbb{R}$ dim < +∞

$B : U \times V \longrightarrow W$ bilineare

se $u = u(t)$ $v = v(t)$ sono curve in U e V

$$\frac{d}{dt} B(u(t), v(t)) = B(\dot{u}, v) + B(u, \dot{v})$$

dim

$$\cancel{B(u,v)} \; B(u(t), v(t)) - B(u(t_0), v(t_0)) =$$

$$= B(u(t), v(t)) - B(u(t_0), v(t)) + B(u(t_0), v(t))$$

$$- B(u(t_0), v(t_0)) =$$

$$= B(u(t) - u(t_0), v(t)) + B(u(t_0), v(t) - v(t_0))$$

$$\frac{B(u(t), v(t)) - B(u(t_0), v(t_0))}{t - t_0} = B\left(\frac{u(t) - u(t_0)}{t - t_0}, v(t)\right)$$

$$+ B\left(u(t_0), \frac{v(t) - v(t_0)}{t - t_0}\right)$$

$$\longrightarrow \underline{B(\dot{u}(t_0), v(t_0)) + B(u(t_0), \dot{v}(t_0))}$$

Teo Se ~~∃ ε>0 tc~~ $\varphi_s^X \varphi_t^Y = \varphi_t^Y \varphi_s^X$ per tempi piccoli

per $|t|, |s| < \varepsilon$ $\Longrightarrow$ $[X, Y] = 0$.

Viceversa, se $[X, Y] = 0$ $\Longrightarrow$ $\varphi_s^Y \varphi_t^X = \varphi_t^X \varphi_s^Y$

per tutti i tempi.

dim

~~Supp.~~ Fisso p supp. $\exists\ \varepsilon > 0$ e $U \ni p$

tc $\varphi^X : (-\varepsilon, \varepsilon) \times U \to M$ $\varphi^Y : (-\varepsilon, \varepsilon) \times U \to M$

e $\varphi_t^X \varphi_s^Y = \varphi_s^Y \varphi_t^X$ per $|t|, |s| < \delta$

e su $V \subset\subset U$

dove sono definite

le composiz

allora $\varphi_t^X \varphi_s^Y \varphi_{-t}^X = \varphi_s^Y$

flusso di $\varphi_{t*} Y$ $\Longrightarrow$ $\varphi_{t*} Y = Y$

per $|t| < \delta$ $\Longrightarrow$ $[X, Y] = L_X Y \equiv 0$.

Viceversa, supp. $[X, Y] \equiv 0$ $\Longrightarrow$ $L_X Y = 0$

$$\left.\frac{d}{dt}\right|_{t=t_0} (\varphi_{-t})_* Y = \left.\frac{d}{dt}\right|_{t=t_0} (\varphi_{-t_0})_* (\varphi_{t_0 - t})_* Y$$

$$= \left.\frac{d}{ds}\right|_{s=0} (\varphi_{-t_0})_* (\varphi_{-s})_* Y$$

$$= (\varphi_{-t_0})_* \left.\frac{d}{ds}\right|_{s=0} (\varphi_{-s})_* Y = (\varphi_{-t_0})_* \mathcal{L}_X Y \equiv 0$$

$$\Rightarrow \quad \frac{d}{dt}(\varphi_{-t})_* Y \equiv 0 \qquad \forall t$$

$$\Rightarrow \quad (\varphi_{-t})_* Y \equiv Y$$

$$\Rightarrow \quad \varphi^X_{-t}\, \varphi^Y_s\, \varphi^X_t = \varphi^Y_s \qquad \text{OK.}$$