Esercizio 1

Il sistema differenziale è
$$\begin{cases} x' = x - y + e^t \\ y' = 5x + 2y \\ z' = 3z - t \end{cases}$$

La III eq. può essere risolta indipendentemente dal sistema:
$$z' - 3z = -t$$
ha come int. generale $\qquad z = \frac{1}{3}t + \frac{1}{9} + c_1 e^{3t}$

$$\begin{cases} x' = x - y + e^t \\ y' = 5x + 2y \end{cases}$$

$$y'' = 5x' + 2y'$$
$$= 5x - 5y + 5e^t + 2y' = y' - 2y - 5y + 5e^t + 2y'$$

$$y'' - 3y' + 7y = 5e^t$$

$$\lambda^2 - 3\lambda + 7 = 0 \qquad \lambda_{1/2} = \frac{3 \pm \sqrt{19}\, i}{2}$$

$$y_p(t) = Ae^t \quad : \quad (A - 3A + 7A)e^t = 5e^t \qquad A = 1$$

$$y(t) = c_2 e^{\frac{3}{2}t} \cos\left(\frac{\sqrt{19}}{2}t\right) + c_3 e^{3/2 t} \operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right) + e^t$$

$$5x(t) = y' - 2y$$
$$= \frac{3}{2} c_2 e^{3/2 t} \cos\left(\frac{\sqrt{19}}{2}t\right) - c_2 \frac{\sqrt{19}}{2} e^{\frac{3}{2}t} \operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right) + \frac{3}{2} c_3 e^{\frac{3}{2}t} \operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right)$$
$$+ \frac{\sqrt{19}}{2} c_3 e^{3/2 t} \cos\left(\frac{\sqrt{19}}{2}t\right) + e^t - 2c_2 e^{\frac{3}{2}t} \cos\left(\frac{\sqrt{19}}{2}t\right)$$
$$- 2c_3 e^{3/2 t} \operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right) - 2e^t$$

$$x(t) = \frac{1}{5}c_2\left(-\frac{1}{2}e^{\frac{3}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) - \frac{\sqrt{19}}{2}e^{\frac{3}{2}t}\operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right)\right) +$$

$$+\frac{1}{5}c_3\left(-\frac{1}{2}e^{\frac{3}{2}t}\operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right) + \frac{\sqrt{19}}{2}e^{\frac{3}{2}t}\cos\frac{\sqrt{19}}{2}t\right) - \frac{1}{5}e^{t}$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1\begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix} + c_2\begin{pmatrix} -\frac{1}{10}e^{\frac{3}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) - \frac{\sqrt{19}}{10}e^{\frac{3}{2}t}\operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right) \\ e^{\frac{3}{2}t}\cos\frac{\sqrt{19}}{2}t \\ 0 \end{pmatrix}$$

$$+ c_3\begin{pmatrix} -\frac{1}{10}e^{\frac{3}{2}t}\operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right) + \frac{\sqrt{19}}{10}e^{\frac{3}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) \\ e^{3/2\,t}\operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right) \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{1}{5}e^{t} \\ e^{t} \\ \frac{1}{3}t + \frac{1}{9} \end{pmatrix}$$

EX 2

L'eq. è omogenea : $$y' = \frac{x^2-y^2}{xy+y^2} = \frac{1-\left(\frac{y}{x}\right)^2}{\frac{y}{x}+\frac{y^2}{x^2}}$$

$$z = \frac{y}{x} \Rightarrow z' = \frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x}\left(y' - z\right)$$

$$= \frac{1}{x}\left(\frac{1-z^2}{z+z^2} - z\right) = \frac{1}{x}\left(\frac{1-z}{z} - z\right)$$

$$= \frac{1}{x}\left(\frac{1-z-z^2}{z}\right)$$

$$\frac{z' \cdot z}{1-z-z^2} = \frac{1}{x}$$

$$\int \frac{z\,dz}{1-z-z^2} = \underbrace{\int \frac{dx}{x}}_{=\log|x|}$$

$$\parallel$$

$$-\frac{1}{2}\int \frac{-2z-1+1}{1-z-z^2}\,dz = -\frac{1}{2}\int \frac{-2z-1}{1-z-z^2}\,dz - \frac{1}{2}\int \frac{dz}{1-z-z^2}$$

$$= -\frac{1}{2}\log|1-z-z^2| + \frac{1}{2}\int \frac{dz}{(z-z_1)(z-z_2)}$$

$$z_{1/2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$= -\frac{1}{2}\log|1-z-z^2| + \frac{1}{2}\int \frac{\sqrt{5}/5}{z-z_1}\,dz + \frac{1}{2}\int \frac{-\sqrt{5}/5}{z-z_2}\,dz$$

$$= -\frac{1}{2}\log|1-z-z^2| + \frac{\sqrt{5}}{10}\log|z-z_1| - \frac{\sqrt{5}}{10}\log|z-z_2| + c$$

La soluzione rimane in forma implicita.

Ex 3

Gli estremi assoluti esistono per il Teorema di Weierstrass.

$$\mathcal{L} = 4 - z - \lambda(x^2+y^2-8) - \mu(x+y+z-1)$$

$$\begin{cases} -2\lambda x - \mu = 0 \\ -2\lambda y - \mu = 0 \\ -1 - \mu = 0 \longrightarrow \mu = -1 \\ x^2+y^2 = 8 \\ x+y+z = 1 \end{cases}$$

$2\lambda x = 1 \qquad x = \frac{1}{2\lambda}$

$2\lambda y = 1 \qquad y = \frac{1}{2\lambda} \qquad x = y$

$$\begin{cases} x = \frac{1}{2\lambda} \\ x = y \\ \mu = -1 \\ 2x^2 = 8 \longrightarrow x = \pm 2 \\ z = 1 - 2x \longrightarrow z = 1 \mp 4 \end{cases}$$

Le soluzioni del sistema sono

$$\left(2, 2, -3, \frac{1}{4}, -1\right)$$

$$\left(-2, -2, 5, -\frac{1}{4}, -1\right)$$

$f(2,2,-3) = 7 \qquad$ max

$f(-2,-2,5) = -1 \qquad$ min

Ex 4

$f = e^{xy} - xz^2 \log y - \pi$ $\qquad$ $\operatorname{dom} f = \{y > 0\}$

$\partial_x f = y e^{xy} - z^2 \log y$ $\qquad$ $P(\log\pi, 1, \sqrt{\pi})$

$\partial_x f(\log\pi, 1, \sqrt{\pi}) = \pi \neq 0$

$f(\log\pi, 1, \sqrt{\pi}) = \pi - \pi = 0$

Per il th. di Dini esiste un'unica funzione implicita $g(y,z)$ di classe C^1 in un intorno di $(1, \sqrt{\pi})$ t.c. $g(1, \sqrt{\pi}) = \log\pi$ e

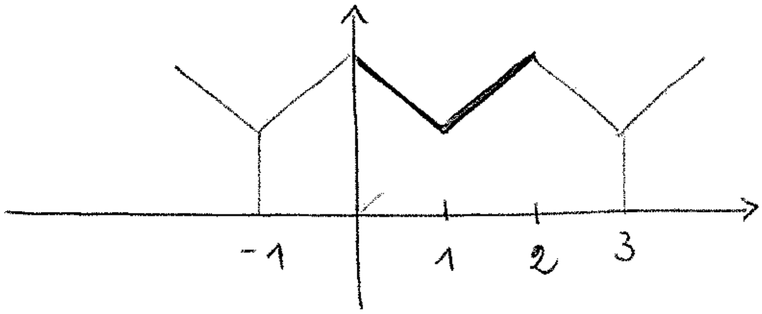
$$\nabla g(1, \sqrt{\pi}) = \frac{\left(-\partial_y f(P), -\partial_z f(P)\right)}{\pi}$$

$$= (0, 0)$$

g di classe $C^1 \Rightarrow g$ differenziabile

Ex 5

$$f(x) = |x-1| + 1 = \begin{cases} x & x \geq 1 \\ 2-x & x < 1 \end{cases}$$



f è pari $\Rightarrow b_k = 0$

$$a_0 = 3$$

$$a_k = \frac{2}{k^2\pi^2}\left[1-(-1)^k\right] = \begin{cases} \dfrac{4}{(2n+1)^2\pi^2} & k = 2n+1 \\ 0 & k \text{ pari} \end{cases}$$

serie di Fourier $\quad \displaystyle \frac{3}{2} + \sum_{n=0}^{+\infty} \frac{4}{(2n+1)^2\pi^2} \cos[(2n+1)\pi x]$

conv. puntualmente e totalmente a f in $\mathbb{R}$

ID. di Parseval: $\quad \displaystyle \frac{a_0^2}{2} + \sum_{k=1}^{+\infty} (a_k^2 + b_k^2) = \frac{2}{T}\int_0^T f^2(x)\,dx$

$$\int_0^2 f^2(x)\,dx = \int_{-1}^1 f^2(x)\,dx = 2\int_0^1 (2-x)^2\,dx = -2\,\frac{(2-x)^3}{3}\Big|_0^1$$

$$= -\frac{2}{3}(1-8) = \frac{14}{3}$$

$$\frac{9}{2} + \sum_{n=0}^{+\infty} \frac{16}{(2n+1)^4\pi^4} = \frac{14}{3}$$

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \left(\frac{14}{3} - \frac{9}{2}\right)\frac{\pi^4}{16} = \frac{\pi^4}{96}$$

Ex 6

(i) In coord. polari

$$\frac{\rho^2|\cos\theta \operatorname{sen}\theta|}{\rho\sqrt{1+\cos\theta \operatorname{sen}\theta}} = \frac{\rho|\overbrace{\cos\theta}^{\leq 1} \overbrace{\operatorname{sen}\theta}^{\leq 1}|}{\sqrt{1+\frac{1}{2}\underbrace{\operatorname{sen}2\theta}_{\geq -1}}} \leq \frac{\rho}{\sqrt{1-\frac{1}{2}}} = \sqrt{2}\rho \longrightarrow 0$$

Quindi f è continua in $(0,0)$.

(ii) $$\frac{f(tv_1,tv_2)-f(0,0)}{t} = \frac{t^2 v_1v_2}{t|t|\sqrt{v_1^2+v_1v_2+v_2^2}} = \frac{t\, v_1v_2}{|t|\sqrt{\ldots}}$$

Il limite per $t\to 0$ esiste se e solo se $v_1v_2=0$

ossia $v_1=0$ oppure $v_2=0$

Pertanto esistono solo le derivate parziali nulle in $(0,0)$.

(iii) f non è diff.^{le} in $(0,0)$ in quanto non ammette tutte le derivate direzionali in tale punto