(1) $y' = \dfrac{9-y^2}{1-x^2}$

Si tratta di una equazione a variabili separabili. Osserviamo che

$$9-y^2=0 \qquad\qquad y=\pm 3$$

Le due rette sono integrali particolari. Dunque la soluzione del Pb di Cauchy

$$\begin{cases} y' = \dfrac{9-y^2}{1-x^2} \\ \\ y(0)=3 \end{cases}$$

è proprio la retta $y=3$.

Per risolvere l'altro problema, separiamo le variabili. Abbiamo

$$\int_0^y \frac{ds}{9-s^2} = \int_0^x \frac{dp}{1-p^2}$$

Inoltre

$$\int_0^y \frac{ds}{9-s^2} = \frac{1}{6}\int_0^y \left(\frac{1}{3-s}+\frac{1}{3+s}\right) ds$$

$$= \frac{1}{6} \ln\left|\frac{3+s}{3-s}\right| \Bigg|_0^y = \frac{1}{6}\ln\left|\frac{3+y}{3-y}\right| = \frac{1}{6}\ln\left(\frac{3+y}{3-y}\right);$$

$$\int_0^x \frac{dp}{1-p^2} = \frac{1}{2}\int_0^x \left(\frac{1}{1-p}+\frac{1}{1+p}\right) dp$$

$$= \frac{1}{2}\ln\left|\frac{1+p}{1-p}\right| \Bigg|_0^x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Perciò

$$\frac{1}{6}\ln\left(\frac{3+y}{3-y}\right)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\frac{3+y}{3-y}=\left(\frac{1+x}{1-x}\right)^3$$

$$(y+3)(1-x)^3=(3-y)(1+x)^3$$

$$y\left[(x+1)^3-(x-1)^3\right]=3\left[(x+1)^3+(x-1)^3\right]$$

$$y=\frac{3\,(x^3\cancel{+3x^2}+3x\cancel{+1}+x^3\cancel{-3x^2}+3x\cancel{-1})}{\cancel{x^3}+3x^2+\cancel{3x}+1\cancel{-x^3}+3x^2-\cancel{3x}+1}$$

$$y=\frac{3\,(2x^3+6x)}{6x^2+2}$$

$$y=\frac{6x\,(x^2+3)}{2\,(3x^2+1)} \qquad y=3x\,\frac{(x^2+3)}{3x^2+1}$$

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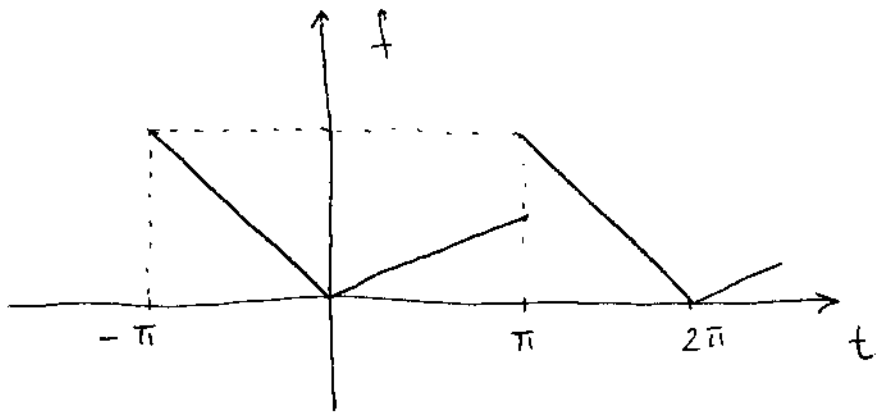

Poiché f è limitata e discontinua in numero finito di punti nel periodo, certamente $\int_{-\pi}^{\pi}|f(t)|^2dt<+\infty$.
Possiamo, dunque, concludere che f è sviluppabile in serie di Fourier.

Abbiamo

$$a_0 = \frac{1}{\pi}\left[\frac{\pi\cdot\pi}{2} + \frac{\pi}{2}\cdot\frac{\pi}{2}\right] = \frac{3}{4}\pi \qquad \frac{a_0}{2} = \frac{3}{8}\pi$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\cos nt\,dt + \int_0^{\pi}\frac{t}{2}\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\cancel{-t\frac{\sin nt}{n}\Big|_{-\pi}^{0}} + \int_{-\pi}^{0}\frac{\sin nt}{n}\,dt + \cancel{\frac{t}{2}\frac{\sin nt}{n}\Big|_0^{\pi}} + \right.$$

$$\left. - \int_0^{\pi}\frac{1}{2}\frac{\sin nt}{n}\,dt\right] = \frac{1}{\pi}\left[-\frac{\cos nt}{n^2}\Big|_{-\pi}^{0} + \frac{1}{2}\frac{\cos nt}{n^2}\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{-1+(-)^n}{n^2} + \frac{1}{2}\frac{(-)^n-1}{n^2}\right) = \frac{3}{2\pi}\left[\frac{(-)^n-1}{n^2}\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\sin nt\,dt + \int_0^{\pi}\frac{t}{2}\sin nt\,dt\right]$$

$$= \frac{1}{\pi}\left[t\frac{\cos nt}{n}\Big|_{-\pi}^{0} \cancel{-\int_{-\pi}^{0}\frac{\cos nt}{n}\,dt} - \frac{t}{2}\frac{\cos nt}{n}\Big|_0^{\pi} + \right.$$

$$\left. \cancel{+\int_0^{\pi}\frac{1}{2}\frac{\cos nt}{n}\,dt}\right] = \frac{1}{\pi}\left[-\frac{(-\pi)\cos n\pi}{n} - \frac{\pi}{2}\frac{\cos n\pi}{n}\right]$$

$$= \frac{1}{\pi}\frac{\pi}{n}\left[(-)^n - \frac{1}{2}(-)^n\right] = \frac{1}{\pi}\frac{\pi(-)^n}{2n} = \frac{(-)^n}{2n}$$

Dunque

$$S(t) = \frac{3}{8}\pi + \sum_{n=1}^{\infty} \frac{3((-)^n-1)}{2\pi n^2}\cos nt + \frac{(-)^n}{2n}\sin nt$$

Per quanto riguarda la convergenza, certamente

$$f(t) \overset{E}{=} S(t)$$

Inoltre,

$\forall t \neq k\pi$ $\quad f$ è continua e derivabile e dunque

$$S(t) = f(t)$$

$\forall t_n = 2n\pi$ $\quad f$ è continua ed esistono finite $f'(t_n^+)$ e $f'(t_n^-)$. Dunque $S(t_n) = f(t_n)$

$\forall t_k = (2k+1)\pi$ $\quad f$ ha un salto di ampiezza finita e le pendenze da entrambi i lati del salto sono finite. Dunque

$$S(t_k) = \frac{f(t_k^+) + f(t_k^-)}{2} = \frac{\pi + \frac{\pi}{2}}{2} = \frac{3\pi}{4}$$

Scegliendo $t = 0$, abbiamo

$$f(0) = S(0)$$

$$0 = \frac{3}{8}\pi + \frac{3}{2\pi} \sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

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③ $\quad f = y\, e^{9x^2 - 8y^2}$

$$\nabla f = \left(18xy\, e^{9x^2-8y^2},\ (1-16y^2)\, e^{9x^2-8y^2}\right)$$

Quindi,

$$\nabla f = 0 \qquad \begin{cases} 18xy = 0 \\ 1 - 16y^2 = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = \pm\frac{1}{4} \end{cases}$$

Passiamo alle derivate seconde

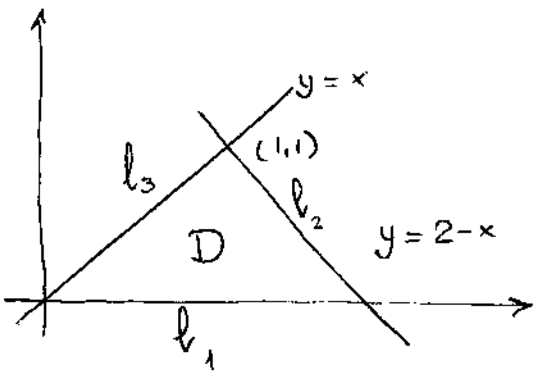
$$f_{xx} = (18y + 324x^2y)\exp(9x^2 - 8y^2)$$

$$f_{xy} = 18x(1 - 16y^2)\exp(9x^2 - 8y^2)$$

$$f_{yy} = (256y^3 - 48y)\exp(9x^2 - 8y^2)$$

$$\mathbb{H}_f(0, \tfrac{1}{4}) = \begin{bmatrix} \frac{9}{2}\exp(-\frac{1}{2}) & 0 \\ 0 & -8\exp(-\frac{1}{2}) \end{bmatrix} \Longrightarrow \text{punto di colle}$$

$$\mathbb{H}_f(0, -\tfrac{1}{4}) = \begin{bmatrix} -\frac{9}{2}\exp(-\frac{1}{2}) & 0 \\ 0 & 8\exp(-\frac{1}{2}) \end{bmatrix} \Longrightarrow \text{punto di colle}$$



Poiché $f \geq 0$ in D e $f\big|_{l_1} \equiv 0$, è chiaro che $m_D = 0$

Il massimo è assunto su $l_2 \cup l_3$

$$f\big|_{l_3} = x\, e^{x^2} = g(x) \qquad x \in [0, 1]$$

$$g'(x) = e^{x^2}(1 + 2x^2) \geq 0 \qquad \text{Dunque}$$

$$m_3 = 0, \qquad M_3 = g(1) = e$$

$$f\big|_{l_2} = (2-x)\, e^{9x^2 - 8(2-x)^2} = (2-x)\, e^{x^2 + 32x - 32} = h(x)$$

$$x \in [1, 2]$$

$$h'(x) = -e^{x^2+32x-32} + (2-x)(2x+32)\,e^{x^2+32x-32}$$

$$= e^{x^2+32x-32}\left(-2x^2-28x+63\right)$$

$h'(x) \geq 0$ $\qquad$ $-2x^2-28x+63 \geq 0$

$$2x^2+28x-63 \leq 0$$

$$x = \frac{-14 \pm \sqrt{196+126}}{2}$$

$$x = \frac{-14 \pm \sqrt{322}}{2} \quad \begin{cases} \dfrac{-14+\sqrt{322}}{2} = x_1 \\[2mm] \dfrac{-14-\sqrt{322}}{2} = x_2 \end{cases}$$

$- \quad x_2 \quad + \quad 1 \quad + \quad x_1 \quad 2 \quad -$

Quindi
x_1 è punto di massimo lungo ℓ_2. Inoltre

$$h(x_1) > h(1) = g(1) = M_3$$

Pertanto il massimo assoluto è $M_D = h(x_1)$.

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(4) $\quad f \in C^\infty(\mathbb{R}^2)$

Inoltre

$f(0,0) = 0$ $\qquad$ $f_y = \cos(x+y+\pi) - \cos(x+y)$
$\qquad\qquad\qquad\qquad + y\sin(x+y)$

$f_y(0,0) = -2$

Tutte le ipotesi del Teorema di Dini sono verificate e dunque esistono $I_{x=0}$, $J_{y=0}$ e $g: I \to J$ con $g \in C^\infty(I)$ t.c. $g(0)=0$ e $\forall x \in I$
$f(x, g(x)) \equiv 0$.

Inoltre

$$f_x(x,y) = \cos(x+y+\pi) + y\sin(x+y) + 2x$$

$$f_x(0,0) = -1$$

$$\Longrightarrow \quad g'(0) = -\frac{f_x(0,0)}{f_y(0,0)} = -\frac{-1}{-2} = -\frac{1}{2}$$

Infine

$$f_{xx} = \sin(x+y+\pi) + y\cos(x+y) + 2$$

$$f_{xy} = \sin(x+y+\pi) + \sin(x+y) + y\cos(x+y)$$

$$f_{yy} = \sin(x+y+\pi) + 2\sin(x+y) + y\cos(x+y)$$

$$\Longrightarrow \quad g''(0) = -\frac{f_{xx}(0,0) + 2f_{xy}\,g'(0) + f_y(0,0)(g'(0))^2}{f_y(0,0)}$$

$$= -\frac{2}{-2} = 1$$