Ex 1

L' eq. è lineare a coefficienti continui in $\mathbb{R} \setminus \{0\}$ $= (-\infty, 0) \cup (0, +\infty)$.

Dato che il tempo iniziale -1 appartiene a $(-\infty, 0)$ la soluzione è definita in $(-\infty, 0)$.

$$A(x) = -\int \frac{1}{x} dx = -\log|x| = \log|x|^{-1}$$

$$e^{A(x)} = |x|^{-1} = \frac{1}{|x|} = -\frac{1}{x} \quad \text{in quanto } x<0$$

$$y' - \frac{1}{x} y = x^2 \log(x^2) \qquad \cdot e^{A(x)}$$

$$\left[y\, e^{A(x)} \right]' = e^{A(x)} x^2 \log(x^2)$$

$$y(x)\, e^{A(x)} = c + \int e^{A(x)} x^2 \log(x^2)$$

$$-\frac{1}{x} y(x) = c - \int x \log(x^2) dx$$

$$= c - \left[\frac{x^2}{2} \log(x^2) - \int \frac{\cancel{x^2}}{\cancel{2}} \cdot \frac{1}{\cancel{x^2}} \cdot \cancel{2}x \, dx \right]$$

$$= c - \frac{x^2}{2} \log(x^2) + \frac{x^2}{2}$$

$$y(x) = -cx + \frac{x^3}{2} \log(x^2) - \frac{x^3}{2}$$

$$y(-1) = \frac{1}{2} \iff c + \frac{1}{2} = \frac{1}{2} \iff c = 0$$

La soluzione è $y(x) = \frac{x^3}{2}\log(x^2) - \frac{x^3}{2}$

$\lim_{x \to 0^-} y(x) = 0$

$\lim_{x \to -\infty} y(x) = -\infty \implies$ la soluz. non è limitata

Ex 2

Il sistema si scrive come
$$\begin{cases} x' = 4x - z \\ y' = x + 5y + z \\ z' = x + 2z \end{cases}$$

Accoppiando la prima e la terza equazione e risolvendo con il metodo di eliminazione si trova:

$$x'' = 4x' - x - 2(4x - x')$$

$$x'' - 6x' + 9x = 0 \longrightarrow \lambda^2 - 6\lambda + 9 = 0 \longrightarrow \lambda_{1,2} = 3$$

$$x(t) = c_1 e^{3t} + c_2 t e^{3t}$$

$$z(t) = 4x - x' = \cancel{4}\, c_1 e^{3t} - c_2 e^{3t} + c_2 t e^{3t}$$

Infine

$$y' - 5y = (2c_1 - c_2)e^{3t} + 2c_2 t e^{3t}$$

sol. omogenea $c_3 e^{5t}$

sol. particolare $y_p = (A + Bt)\, e^{3t}$

$$y_p' = (B + 3A + 3Bt)\, e^{3t}$$

$$\cancel{y_p'' = (9A + 9Bt + 2B)\, e^{3t}}$$

$$(B + 3A + 3Bt - 5A - 5Bt) = 2c_1 - c_2 + 2c_2 t$$

$$\begin{cases} B - 2A = 2c_1 - c_2 \\ -2B = 2c_2 \end{cases} \qquad \begin{cases} -2A = 2c_1 \cancel{-c_2} \cancel{+c_2} \\ B = -c_2 \end{cases} \qquad \begin{cases} A = -c_1 \\ B = -c_2 \end{cases}$$

$$y_p = (-c_1 - c_2 t)\, e^{3t}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ c_3 e^{5t} - (c_1 + c_2 t) e^{3t} \\ (c_1 - c_2) e^{3t} + c_2 t e^{3t} \end{pmatrix} =$$

$$= c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} t \\ -t \\ -1+t \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Ex 3

$$f = x^2 - y^2 - x^4 + y^4 = x^2(1-x^2) - y^2(1-y^2)$$

$$\nabla f = (2x - 4x^3, -2y + 4y^3)$$

$$\begin{cases} 2x(1-2x^2) = 0 \\ -2y(1-2y^2) = 0 \end{cases} \qquad \begin{cases} x = 0 \vee x = \pm \frac{\sqrt{2}}{2} \\ y = 0 \vee y = \pm \frac{\sqrt{2}}{2} \end{cases}$$

Ci sono 9 punti critici

$$(0,0) \quad \left(0, \pm \frac{\sqrt{2}}{2}\right) \quad \left(\frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right) \quad \left(\frac{\sqrt{2}}{2}, 0\right)$$

$$\left(-\frac{\sqrt{2}}{2}, 0\right) \quad \left(-\frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$$

Ma $\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$ non sono interni.

Quindi selezioniamo solo

$$(0,0), \left(0, \pm \frac{\sqrt{2}}{2}\right), \left(\pm \frac{\sqrt{2}}{2}, 0\right)$$

Su $x^2+y^2=1$ f è identicamente nulla.

Dato che

$$f(0,0) = 0, \quad f\left(0, \pm \frac{\sqrt{2}}{2}\right) = -\frac{1}{4}, \quad f\left(\pm \frac{\sqrt{2}}{2}, 0\right) = \frac{1}{4}$$

concludiamo che

$$\max f = \frac{1}{4} = f\left(\pm \frac{\sqrt{2}}{2}, 0\right) \quad ; \quad \min f = -\frac{1}{4} = f\left(0, \pm \frac{\sqrt{2}}{2}\right).$$

Ex 4

$$f = \log(x^3+y) - \frac{xy}{4}$$

$$f \in C^1(\Omega) \quad , \quad \Omega = \{ y > -x^3 \}$$

$(1,0) = P \in \Omega$

$$f(P) = 0$$

$$f_x = \frac{3x^2}{x^3+y} - \frac{y}{4} \Big|_{\substack{x=1 \\ y=0}} = 3 \neq 0$$

$\Rightarrow \exists ! \; x = g(y)$ in un intorno di P con

$$g'(0) = -\frac{\partial_y f(P)}{\partial_x f(P)} = -\frac{1}{4}$$

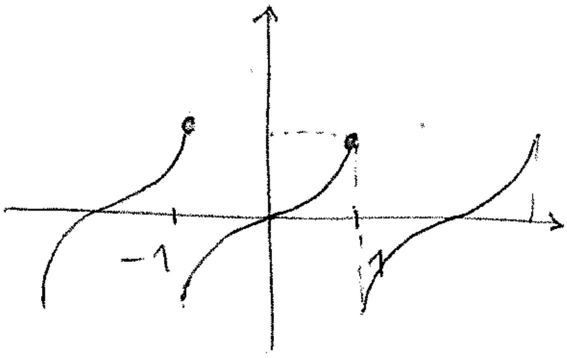
$$\partial_y f = \frac{1}{x^3+y} - \frac{x}{4} \Big|_{\substack{x=1 \\ y=0}} = \frac{3}{4}$$

$$\lim_{y\to 0} \frac{g(y)-1}{\sin y} = \frac{0}{0}$$

De L'Hopital: $\quad \dfrac{g'(y)}{\cos y} \to g'(0) = -\dfrac{1}{4}$

$\Rightarrow$ Il limite richiesto vale $-\frac{1}{4}$

Ex 5



f è dispari $\Rightarrow \quad a_k = 0$

$$b_k = 2\int_0^1 x^2 \sin(\pi k x)\,dx = 2\left[-x^2\frac{\cos(\pi k x)}{\pi k}\Big|_0^1 + \int_0^1 2x\frac{\cos(\pi k x)}{\pi k}\right]$$

$$= +\frac{4}{\pi k}\int_0^1 x\cos(\pi k x)\,dx - \frac{2}{\pi k}(-1)^k$$

$$= +\frac{4}{\pi k}\left[+x\,\frac{\sin(k\pi x)}{k\pi}\Big|_0^1 - \int_0^1 \frac{\sin(k\pi x)}{k\pi}\,dx\right] - \frac{2(-1)^k}{\pi k}$$

$0 \qquad 0$

$$= \frac{-4}{(k\pi)^2}\int_0^1 \sin(k\pi x)\,dx - \frac{2(-1)^k}{\pi k} = \frac{\pm 4}{k^3\pi^3}(\cos k\pi x)\Big|_0^1 \ldots$$

$$f(x) \rightsquigarrow \sum_{k=1}^{+\infty}\left[\frac{4[(-1)^k-1]}{\pi^3k^3} - \frac{2(-1)^k}{k\pi}\right]\operatorname{sen}(\pi k x) \qquad = \frac{4}{\pi^3k^3}\left[(-1)^k - 1\right] - \frac{2(-1)^k}{k\pi}$$

$$s(x) = \begin{cases} f(x) & x \in \mathbb{R}\setminus\{2k+1 : k\in\mathbb{Z}\} \\ 0 & x = 2k+1,\ k\in\mathbb{Z}\end{cases}$$

Ex 6

In coord. pol. $0 \leq \dfrac{x^2\left(|x|^{3/2}+y^2\right)}{x^2+y^2} = \cos^2\theta\left(\rho^{\frac{3}{2}}|\cos\theta|^{\frac{3}{2}}+\rho^2\operatorname{sen}^2\theta\right)$

$$\leq \rho^{3/2}+\rho^2 \longrightarrow 0$$

$\Rightarrow$ f è continua in $(0,0)$.

$f(0,y)=0 \Rightarrow \exists\, \partial_y f(0,0)=0$

$f(x,0) = |x|^{3/2} \underset{\frac{3}{2}>1}{\Longrightarrow} \exists\, \partial_x f(0,0)=0$

$$0 \leq \frac{f(x,y)}{\sqrt{x^2+y^2}} \underset{\text{in coord. polari}}{=} \cos^2\theta\left(\rho^{1/2}|\cos\theta|^{3/2}+\rho\operatorname{sen}^2\theta\right) \leq$$

$$\leq \rho^{1/2}+\rho \longrightarrow 0$$

$\Rightarrow$ f è diff.^{le} in $(0,0)$