1)

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3 - xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

In coord. polari:

$$\left| \frac{\rho^{\cancel{3}} (\cos^3\theta - \sin^3\theta - \cos\theta \sin^2\theta)}{\cancel{\rho^2}} \right| \leq 3\rho \longrightarrow 0$$

$\Rightarrow$ f è continua in $(0,0)$

$$f(x,0) = x \quad \Rightarrow \quad \frac{\partial f}{\partial x}(0,0) = 1$$

$$f(0,y) = -y \quad \Rightarrow \quad \frac{\partial f}{\partial y}(0,0) = -1$$

$$\frac{f(x,y) - f(0,0) - \nabla f(0,0) \cdot (x,y)}{\sqrt{x^2+y^2}} =$$

$$= \frac{\dfrac{x^3 - y^3 - xy^2}{x^2+y^2} - x + y}{\sqrt{x^2+y^2}} = \frac{\cancel{x^3} - \cancel{y^3} - xy^2 - \cancel{x^3} - xy^2 + \cancel{y^3} + x^2y}{(x^2+y^2)^{3/2}} =$$

$$= \frac{-2xy^2 + x^2y}{(x^2+y^2)^{3/2}} \underset{\substack{\uparrow \\ \text{coord.} \\ \text{polari}}}{=} -2\cos\theta \sin^2\theta + \cos^2\theta \sin\theta \not\to 0$$

f non è diff.^{le} in $(0,0)$

$$\frac{\partial f}{\partial v}(0,0) = \lim_{t\to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3(v_1^3 - v_2^3 - v_1 v_2^2)}{t^2(v_1^2+v_2^2)}}{t}$$

$$\underset{v_1^2+v_2^2=1}{=} \lim_{t\to 0} \left(v_1^3 - v_2^3 - v_1 v_2^2 \right) = v_1^3 - v_2^3 - v_1 v_2^2$$

$\boxed{2}$

$$\begin{cases} y'' - \lambda y' + \lambda^2 y = 1, & \lambda \in \mathbb{R} \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

$t^2 - \lambda t + \lambda^2 = 0$ eq. caratt.

$$t_{1/2} = \frac{\lambda \pm \sqrt{\lambda^2 - 4\lambda^2}}{2} = \frac{\lambda \pm \lambda\sqrt{3}\, i}{2}$$

integrale generale omogenea:

$$y(x) = \begin{cases} c_1 + c_2 x & \text{se } \lambda = 0 \\ c_1 e^{\frac{\lambda}{2}x} \cos\left(\frac{\sqrt{3}\lambda x}{2}\right) + \\ \quad + c_2 e^{\frac{\lambda}{2}x} \operatorname{sen}\left(\frac{\sqrt{3}\lambda}{2} x\right) & \text{se } \lambda \neq 0 \end{cases}$$

Soluz. particolare:

$\lambda = 0$ $\qquad y_p(x) = Ax^2.$

L'eq. diventa $y'' = 1 \iff 2A = 1 \iff A = \frac{1}{2}$

$\lambda \neq 0$: $\quad y_p(x) = A.$

$$\lambda^2 A = 1 \implies A = \frac{1}{\lambda^2}$$

Integrale eq. completa :

$$y(x) = \begin{cases} c_1 + c_2 x + \frac{1}{2}x^2 & \lambda = 0 \\ c_1 e^{\lambda/2\, x} \cos\left(\frac{\sqrt{3}\lambda x}{2}\right) + c_2 e^{\frac{\lambda}{2} x} \operatorname{sen}\left(\frac{\sqrt{3}\lambda x}{2}\right) + \frac{1}{\lambda^2} & \lambda \neq 0 \end{cases}$$

cond. ai limiti :

$\lambda = 0$
$$\begin{cases} c_1 = 0 \\ c_1 + c_2 \pi + \frac{1}{2}\pi^2 = 0 \end{cases} \qquad \begin{matrix} c_1 = 0 \\ c_2 = -\frac{1}{2}\pi \end{matrix}$$

esiste un' unica soluzione $\quad -\frac{1}{2}\pi x + \frac{1}{2} x^2$

$\lambda \neq 0$
$$\begin{cases} c_1 + \frac{1}{\lambda^2} = 0 \\ c_1 e^{\frac{\lambda}{2}\pi} \cos\left(\frac{\sqrt{3}\lambda\pi}{2}\right) + c_2 e^{\frac{\lambda}{2}\pi} \operatorname{sen}\left(\frac{\sqrt{3}\lambda\pi}{2}\right) + \frac{1}{\lambda^2} = 0 \end{cases}$$

Matrice del sistema
$$\begin{pmatrix} 1 & 0 \\ e^{\frac{\lambda}{2}\pi} \cos\left(\frac{\sqrt{3}\lambda\pi}{2}\right) & e^{\lambda/2\,\pi} \operatorname{sen}\left(\frac{\sqrt{3}\lambda\pi}{2}\right) \end{pmatrix}$$

$\det = 0 \iff \operatorname{sen}\left(\frac{\sqrt{3}\lambda\pi}{2}\right)=0 \iff \frac{\sqrt{3}\lambda\pi}{2} = k\pi$ $k \in \mathbb{Z} \setminus \{0\}$

$\iff \lambda = \frac{2k}{\sqrt{3}} \quad k \in \mathbb{Z} \setminus \{0\}$

Se $\lambda \neq \frac{2k}{\sqrt{3}}$ allora $\exists!$ soluzione

se $\lambda = \frac{2k}{\sqrt{3}}$ allora il sistema diventa

$$\begin{cases} c_1 + \frac{1}{\lambda^2} = 0 \\ c_1 e^{\frac{\lambda}{2}\pi}(-1)^k + \frac{1}{\lambda^2} = 0 \end{cases}$$

$$\begin{cases} c_1 = -\frac{1}{\lambda^2} \\ -\frac{1}{\lambda^2} e^{\frac{\lambda}{2}\pi}(-1)^k + \frac{1}{\lambda^2} = 0 \end{cases}$$

$$-e^{\frac{\lambda}{2}\pi}(-1)^k + 1 = 0$$

$$e^{\frac{\lambda}{2}\pi}(-1)^k = 1$$

se k è dispari non ci sono soluzioni

se k è pari : $e^{\frac{\lambda}{2}\pi} = 1 \iff \frac{\lambda}{2}\pi = 0$ impossibile dato che $\lambda \neq 0$

Quindi se $\lambda = \frac{2k}{\sqrt{3}}$, con $k \in \mathbb{Z} \setminus \{0\}$ non esistono soluzioni

$\boxed{3}$

$$\vec{F}(x,y,z) = \left(xy + \log(x+z),\ 2e^{xz} - y\right)$$

$$P = (0,2,1)$$

$\vec{F} \in C^1(A) \qquad A = \{(x,y,z) \in \mathbb{R}^3 \mid x+z>0\}$

$\vec{F}(P) = (0,0)$

$$\operatorname{Jac}\vec{F} = \begin{pmatrix} y + \frac{1}{x+z} & x & \frac{1}{x+z} \\ 2ze^{xz} & -1 & 2xe^{xz} \end{pmatrix}$$

$$\operatorname{Jac}\vec{F}(P) = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\operatorname{Jac}_{xz}\vec{F}(P) = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{ha det} \neq 0$$

$\Rightarrow$ Per il T. di Dini esiste un'unica funzione

$\vec{g}(y) = (x(y), z(y))$ definita implicitamente

dal sistema $\vec{F} = \vec{0}$ e t.c. $\vec{g}(2) = (0,1)$

$$\vec{F}(x(y), y, z(y)) = (0,0)$$

$$\nabla g(2) = -\left[\operatorname{Jac}_{xz}\vec{F}(P)\right]^{-1} \operatorname{Jac}_{y}\vec{F}(P)$$

$$= -\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -3/2 \end{pmatrix} = \begin{pmatrix} x'(2) \\ z'(2) \end{pmatrix}$$

Eq. retta tang. in P a $\vec{\gamma}(y) = (x(y), y, z(y))$

è la retta passante per P e parallela a

$$\vec{\gamma}'(2) = (x'(2), 1, z'(2))$$

Dunque:

$$\begin{cases} x = 0 + t \cdot \frac{1}{2} \\ y = 2 + t \cdot 1 \\ z = 1 + t\left(-\frac{3}{2}\right) \end{cases} \qquad \begin{cases} y = 2 + 2x \\ z = 1 - 3x \end{cases}$$

$\boxed{4}$ ✓

$$f = x^2 - z$$

$$E = \{x^2 + 4y^2 + 9z^2 \leq 1\}$$

OSS: Valgono le ipotesi del T. di Weierstrass. La funzione ammette max e min assoluti in E

Weierstrass o.k. $\Rightarrow$ $\exists$ max e min assoluti in E di f

$\nabla f = (2x, 0, -1)$ non si annulla mai

$$\mathcal{L} = x^2 - z - \lambda(x^2 + 4y^2 + 9z^2 - 1)$$

$$\begin{cases} 2x - 2\lambda x = 0 \\ -8\lambda y = 0 \\ -1 - 18\lambda z = 0 \\ x^2 + 4y^2 + 9z^2 = 1 \end{cases}$$

$\longrightarrow \lambda = 0 \vee y = 0$

$\lambda = 0 \downarrow$ la III eq. è imposs.

$$\begin{cases} \lambda = 1 \vee x = 0 \\ y = 0 \\ z = -\frac{1}{18\lambda} \\ x^2 + 4y^2 + 9z^2 = 1 \end{cases} \qquad \begin{cases} x = 0 \\ y = 0 \\ \lambda = -\frac{1}{18z} \\ z = \pm\frac{1}{3} \end{cases} \vee \begin{cases} \lambda = 1 \\ y = 0 \\ z = -\frac{1}{18} \\ x^2 + 9 \cdot \frac{1}{18^2} = 1 \end{cases}$$

$x^2 = 1 - \frac{1}{36}$

$x = \frac{\pm\sqrt{35}}{6}$

I pti candidati sono

$\left(0, 0, \pm\frac{1}{3}\right)$

$\left(\pm\frac{\sqrt{35}}{6}, 0, -\frac{1}{18}\right)$

$f\left(0, 0, \pm\frac{1}{3}\right) = \mp\frac{1}{3}$

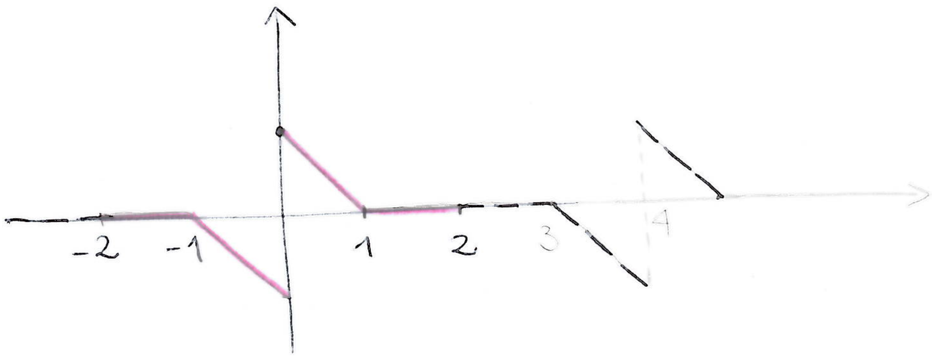
$f\left(\frac{\pm\sqrt{35}}{6}, 0, -\frac{1}{18}\right) = \frac{35}{36} + \frac{1}{18} = \frac{37}{36}$

$\max = \frac{37}{36} \qquad \min = -\frac{1}{3}$

$\boxed{5}$

$$f(x) = \begin{cases} 1-x & x \in [0,1] \\ 0 & x \in]1,2[\end{cases}$$

prolungata dispari in $(-2,0)$



La serie di Fourier avrà soli seni

$$a_k = 0 \qquad \forall k = 0,1,2,\dots$$

$$b_k = \frac{2}{4}\int_{-2}^{2} f(x)\,\mathrm{sen}\left(\frac{\pi k x}{2}\right)dx =$$

$$= \frac{4}{4}\int_{0}^{2} f(x)\,\mathrm{sen}\left(\frac{\pi k x}{2}\right)dx$$

$$= \int_{0}^{1} (1-x)\,\mathrm{sen}\left(\frac{\pi k x}{2}\right)dx =$$

$$= \left[-(1-x)\cos\left(\frac{k\pi x}{2}\right)\frac{2}{k\pi}\Big|_0^1 - \int_0^1 (-1)\left(-\cos\left(\frac{\pi k x}{2}\right)\right)\frac{2}{k\pi}dx \right]$$

$$= \left\{ 0 + \frac{2}{k\pi} - \frac{2}{k\pi}\left[\mathrm{sen}\left(\frac{k\pi x}{2}\right) \cancel{k\pi}\,\frac{2}{k\pi}\right]_0^1 \right\}$$

$$= \frac{2}{k\pi} - \frac{2}{k\pi}\,\mathrm{sen}\left(\frac{k\pi}{2}\right)\cdot\frac{2}{k\pi} = \frac{2}{k\pi} - \frac{4}{k^2\pi^2}\,\mathrm{sen}\left(\frac{k\pi}{2}\right)$$

$$\sum_{k=1}^{+\infty}\left[\frac{2}{k\pi}-\frac{4}{k^2\pi^2}\operatorname{sen}\left(\frac{k\pi}{2}\right)\right]\operatorname{sen}\left(\frac{k\pi}{2}x\right)$$

$$s(x)=\begin{cases} f(x) & x\neq 4k \quad k\in\mathbb{Z} \\ 0 & x=4k \quad k\in\mathbb{Z}\end{cases}$$

$\boxed{x=1}$

$$\operatorname{sen}\left(\frac{k\pi}{2}\right)=\begin{cases} 0 & k=2n \\ (-1)^n & k=2n+1\end{cases}$$

$$\sum_{n=0}^{+\infty}\left[\frac{2}{\pi(2n+1)}-\frac{4}{(2n+1)^2\pi^2}(-1)^n\right](-1)^n=0$$

$$\frac{1}{\cancel{\pi}}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}=\frac{2}{\pi^{\cancel{2}}}\underbrace{\sum_{n=1}^{+\infty}\frac{1}{(2n+1)^2}}_{=\frac{\pi^2}{8}}=\frac{\pi}{4}$$

$\boxed{6}$ $\checkmark$ $\begin{cases} y'=y+e^xy^3 \\ y(0)=1\end{cases}$

$y=0$ sol. eq. diff. ma non soddisfa la cond. iniz.

$z=y^{-2} \qquad z'=-2y^{-3}y'$

$$=-\frac{2}{y^3}\left(y+e^xy^3\right)=-2z-2e^x \qquad z'=-2z-2e^x$$

$z(x) = c_1 e^{-2x}$ sol. omog.

$z_p = A e^x$

$A \cancel{e^x} = -2A\cancel{e^x} - 2\cancel{e^x} \qquad A = -\frac{2}{3}$

$z(x) = c_1 e^{-2x} - \frac{2}{3} e^x$ int. gen.

$z(0) = 1 = (y(0))^{-2} \iff c_1 = \frac{5}{3}$

$z(x) = \frac{5}{3} e^{-2x} - \frac{2}{3} e^x$

$y^2(x) = (z(x))^{-1} = \left(\frac{5}{3} e^{-2x} - \frac{2}{3} e^x\right)^{-1}$

deve essere $\frac{5}{\cancel{3}} e^{-2x} - \frac{2}{\cancel{3}} e^x > 0 \iff 5 e^{-3x} > 2$

$\iff -3x > \log \frac{2}{5}$

$\iff x < -\frac{1}{3} \log \frac{2}{5} = \frac{1}{3} \log \frac{5}{2}$

La sol. del PdC è

$$y(x) = \frac{1}{\sqrt{\frac{5}{3} e^{-2x} - \frac{2}{3} e^x}} \qquad x \in \left]-\infty, \frac{1}{3} \log \frac{5}{2}\right[$$

si sceglie + per il dato iniziale