Ex 1

$$f(x) = \left| x - \frac{\pi}{2} \right| \qquad x \in [0, 2\pi)$$

$$= \begin{cases} \frac{\pi}{2} - x & x \in \left[0, \frac{\pi}{2}\right] \\ x - \frac{\pi}{2} & x \in \left[\frac{\pi}{2}, 2\pi\right) \end{cases}$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx = \frac{1}{\pi} \left[\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \cos kx \, dx + \right.$$

$$\left. + \int_{\frac{\pi}{2}}^{2\pi} \left(x - \frac{\pi}{2}\right) \cos kx \, dx \right] =$$

$$= \frac{1}{\pi} \left[\left[\left(\frac{\pi}{2} - x\right) \cdot \frac{\operatorname{sen} kx}{k}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\operatorname{sen} kx}{k} \, dx \right] +$$

$$+ \frac{1}{\pi} \left[\left(x - \frac{\pi}{2}\right) \frac{\operatorname{sen} kx}{k} \Bigg|_{\frac{\pi}{2}}^{2\pi} - \int_{\frac{\pi}{2}}^{2\pi} \frac{\operatorname{sen} kx}{k} \, dx \right]$$

$$= \frac{1}{\pi k} \left[\frac{-\cos kx}{k} \right]_0^{\frac{\pi}{2}} + \frac{1}{\pi} \left[\qquad + \frac{\cos kx}{k^2} \Bigg|_{\frac{\pi}{2}}^{2\pi} \right]$$

$$= \frac{1}{\pi k^2} \left[-\cos\left(\frac{k\pi}{2}\right) + 1 \right] + \qquad + \frac{1}{\pi k^2} - \frac{1}{\pi k^2} \cos\left(\frac{k\pi}{2}\right)$$

$$= -\frac{2}{\pi k^2} \cos\left(\frac{k\pi}{2}\right) + \frac{2}{\pi k^2}$$

$$a_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{2\pi}\left(x-\frac{\pi}{2}\right)dx$$

$$= \frac{1}{\cancel{\pi}}\cdot\frac{1}{2}\,\frac{\pi^{\cancel{2}}}{4} + \frac{1}{\pi}\cdot\left(\frac{3}{2}\pi\right)^2\cdot\frac{1}{2} = \frac{10}{8}\pi = \frac{5}{4}\pi$$

$$b_k = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\operatorname{sen} kx\,dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{2\pi}\left(x-\frac{\pi}{2}\right)\operatorname{sen} kx\,dx$$

$$= \frac{1}{\pi}\left\{\left[-\left(\frac{\pi}{2}-x\right)\frac{\cos kx}{k}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\frac{\cos kx}{k}\,dx\right\} +$$

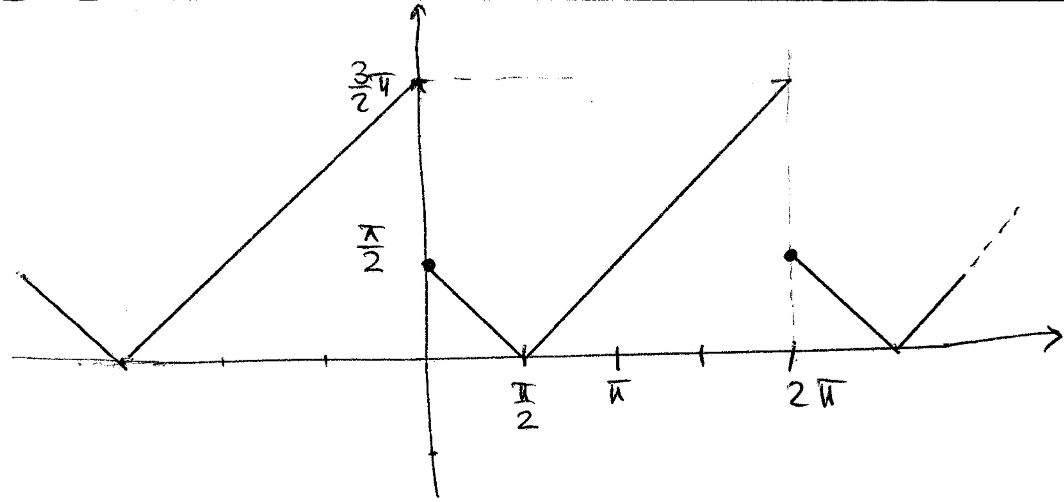
$$+ \frac{1}{\pi}\left\{\left[-\left(x-\frac{\pi}{2}\right)\frac{\cos kx}{k}\right]_{\frac{\pi}{2}}^{2\pi} + \int_{\frac{\pi}{2}}^{2\pi}\frac{\cos kx}{k}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{+\frac{\pi}{2}\cdot\frac{1}{k} - \left[\frac{\operatorname{sen} kx}{k^2}\right]_0^{\frac{\pi}{2}}\right\} +$$

$$+ \frac{1}{\pi}\left\{-\frac{3}{2}\pi\cdot\frac{1}{k} + \left[\frac{\operatorname{sen} kx}{k^2}\right]_{\frac{\pi}{2}}^{2\pi}\right\}$$

$$= \frac{1}{2k} - \frac{1}{\pi k^2}\operatorname{sen}\left(k\frac{\pi}{2}\right) - \frac{3}{2k} - \frac{1}{\pi k^2}\operatorname{sen}\left(k\frac{\pi}{2}\right)$$

$$= -\frac{1}{k} - \frac{2}{\pi k^2}\operatorname{sen}\left(k\frac{\pi}{2}\right)$$



$$s(x) = \begin{cases} f(x) & x \neq 2k\pi \,, \quad k \in \mathbb{Z} \\ \dfrac{\frac{\pi}{2} + \frac{3}{2}\pi}{2} = \pi & x = 2k\pi \,, \quad k \in \mathbb{Z} \end{cases}$$

$$f(x) \sim \frac{5}{8}\pi + \sum_{k=1}^{+\infty} \left(\frac{2}{\pi k^2} - \frac{2}{\pi k^2} \cos k\frac{\pi}{2} \right) \cos kx \; +$$

$$+ \sum_{k=1}^{+\infty} \left(-\frac{1}{k} - \frac{2}{\pi k^2} \operatorname{sen} k\frac{\pi}{2} \right) \operatorname{sen} kx$$

$x = 0$

$$\pi = \frac{5}{8}\pi + \sum_{k=1}^{+\infty} \frac{2}{\pi k^2} \left(1 - \cos k\frac{\pi}{2} \right)$$

$$\cos\left(k\frac{\pi}{2} \right) = \begin{cases} (-1)^n & k = 2n \\ 0 & k = 2n+1 \end{cases}$$

$$\frac{3\pi}{8} = -\frac{2}{\pi} \sum_{n=1}^{+\infty} \frac{1}{4n^2} \left[(-1)^n \right] + \frac{2}{\pi} \sum_{k=1}^{+\infty} \frac{1}{k^2}$$

$$\frac{3}{8}\pi = -\frac{1}{2\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} + \frac{\pi}{3} \Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} = -\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$$

Ex 2

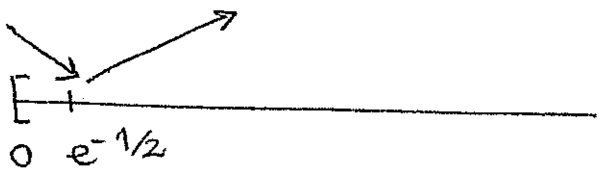
$$f(x,y) = (x^2+y^2)\log(x^2+y^2) \qquad (x,y)\neq(0,0)$$

- $\lim_{(x,y)\to(0,0)} f(x,y) = 0 \implies$ f si prolunga con continuità ponendo $f(0,0)=0$

- f è radiale : $f(x,y) = g(\sqrt{x^2+y^2})$ dove

$$\begin{cases} g(\rho) = \rho^2 \log \rho^2 & \rho > 0 \\ g(0) = 0 \end{cases}$$

$$g'(\rho) = 2\rho\left[\log\rho^2 + 1\right] \geq 0 \iff \rho^2 \geq e^{-1} \iff \rho \geq e^{-1/2}$$



$\rho = e^{-1/2}$ pto di min assoluto per g

$\rho = 0$ pto di max relativo

$\implies$ I punti della circonferenza $x^2+y^2 = e^{-1}$ sono di min assoluto per f ; l'origine $(0,0)$ è un punto di max locale.

EX 3

$$(P)\quad \begin{cases} y'' + 2y' + 2\lambda y = 0 \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

$$t^2 + 2t + 2\lambda = 0 \qquad t_{1/2} = -1 \pm \sqrt{1-2\lambda}$$

$\boxed{1-2\lambda > 0}$

$$y(x) = c_1 e^{(-1+\sqrt{1-2\lambda})x} + c_2 e^{(-1-\sqrt{1-2\lambda})x}$$

$$\begin{cases} y(0) = c_1 + c_2 = 0 \\ y(\pi) = c_1 e^{t_1\pi} + c_2 e^{t_2\pi} = 0 \end{cases}$$

$$\begin{vmatrix} 1 & 1 \\ e^{t_1\pi} & e^{t_2\pi} \end{vmatrix} = e^{t_2\pi} - e^{t_1\pi} \neq 0 \quad \text{in quanto } t_1 \neq t_2$$

$\Rightarrow$ L'unica soluzione di (P) è $y \equiv 0$

$\boxed{1-2\lambda = 0}$

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

$$\begin{cases} y(0) = c_1 = 0 \\ y(\pi) = c_2 \pi e^{-\pi} = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

$\Rightarrow$ L'unica sol. di (P) è $y \equiv 0$

$\boxed{1-2\lambda<0}$ $\qquad y(x) = e^{-x}\left(c_1 \cos\left(\sqrt{2\lambda-1}\,x\right) + c_2 \operatorname{sen}\left(\sqrt{2\lambda-1}\,x\right)\right)$

$$\begin{cases} y(0) = c_1 = 0 \\ y(\pi) = e^{-\pi}\left(c_2 \operatorname{sen}\left(\sqrt{2\lambda-1}\,\pi\right)\right) = 0 \end{cases}$$

Se $\operatorname{sen}\left[\sqrt{(2\lambda-1)}\,\pi\right] = 0$ si ottengono gli autovalori.

$$\sqrt{(2\lambda-1)}\,\pi = k\pi \quad \Rightarrow \quad 2\lambda - 1 = k^2$$

$$\Rightarrow \quad \lambda_k = \frac{k^2+1}{2} \qquad k \in \mathbb{N}$$

Le corrisp. autofunzioni sono

$$y_k(x) = c_2 e^{-x} \operatorname{sen}(kx); \qquad k \in \mathbb{N},\ c_2 \in \mathbb{R}$$

Ex 4

$$\begin{cases} y'' + 4y' + 13y = e^{-2t}\cos 3t \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

$$\lambda^2 + 4\lambda + 13 = 0 \qquad \lambda_{1/2} = -2 \pm \sqrt{4-13} = -2 \pm 3i$$

$$y_{om} = c_1 e^{-2t}\cos 3t + c_2 e^{-2t} \operatorname{sen} 3t$$

$$e^{-2t}\cos 3t = \operatorname{Re}\left[e^{(-2+3i)t}\right]$$

EX 5

$f(x,y) = e^{xy} - \sin(xy) - y = 0 \qquad P(0,1)$

$f(0,1) = 1 - 1 = 0$

$f_y = x e^{xy} - x\cos(xy) - 1 \Big|_P = -1 \neq 0$

$\Rightarrow \exists!\ y(x)$ definita impl. dall' eq. $f = 0$
in un intorno di P con $y(0) = 1$
$f(x, y(x)) = 0$

$$y'(x) = -\frac{\partial_x f}{\partial_y f}(x, y(x))$$

$\partial_x f = y e^{xy} - y\cos(xy) \Big|_P = 0 \qquad \Rightarrow y'(0) = 0$

$$e^{xy(x)} - \operatorname{sen}(xy(x)) - y(x) = 0$$

$$e^{xy(x)}\left[y(x) + xy'(x)\right] - \cos[xy(x)]\left(y(x) + xy'(x)\right) - y'(x) = 0$$

$$e^{xy(x)}\left[y(x) + xy'(x)\right]^2 + e^{xy(x)}\left[2y'(x) + xy''(x)\right] +$$

$$+ \sin(xy(x))\left[y(x) + xy'(x)\right]^2 - \cos(xy(x))\left[2y'(x) + xy''(x)\right]$$

$$- y''(x) = 0$$

$x = 0$
$y(0) = 1$
$y'(0) = 0$
$\Bigg| \Rightarrow \quad 1 - y''(0) = 0 \quad \Rightarrow \quad y''(0) = 1$

$$z_p(t) = A t e^{(-2+3i)t}$$

$$z_p' = A e^{(-2+3i)t} + (-2+3i) A t e^{(-2+3i)t}$$

$$z_p'' = 2A(-2+3i) e^{(-2+3i)t} + At(-2+3i)^2 e^{(-2+3i)t}$$

$$z_p'' + 4z_p' + 13 z_p = e^{(-2+3i)t}\Big[\cancel{-4A} + 6iA + At(4-9-12i)$$

$$+ \cancel{4A} + 4(-2+3i)At + 13At\Big] \overset{\uparrow}{=} e^{(-2+3i)t}$$

$$6iA \cancel{-5At} \cancel{-12Ait} \cancel{-8At} \cancel{+12Ait} \cancel{+13At} = 1$$

$$6iA = 1 \quad \Longrightarrow A = \frac{1}{6i} = -\frac{1}{6}i$$

$$z_p = \left(-\frac{1}{6}i\right) t e^{(-2+3i)t}$$

$$y_p = \operatorname{Re} z_p = e^{-2t} t \left[+\frac{1}{6} \operatorname{sen} 3t\right]$$

L'int. gen è: $y = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \operatorname{sen} 3t + \frac{1}{6} t e^{-2t} \operatorname{sen} 3t$

$$y(0) = c_1 = 0$$

$$\cancel{y(\pi) = c_2 e^{-2\pi} \underbrace{\operatorname{sen}(3\pi)}_{0} + \frac{1}{6}\pi e^{-2\pi} \operatorname{sen} 3\pi = 0}$$

$$y'(0) = c_2 \cdot 3 \quad \cancel{\ast \tfrac{3}{6}} = 1 \quad \Longrightarrow \quad c_2 = \frac{1}{3}$$

$$y = \left(\frac{1}{3} + \frac{1}{6}t\right) e^{-2t} \operatorname{sen} 3t$$

$$y(x) = 1 + \frac{1}{2}x^2 + o(x^2)$$

$x = 0$ pto di min

Ex 6

$$J(x) = \int_0^1 (\dot{x}^2 + 10tx)\, dt$$

$x(0) = 1$

$x(1) = 2$

$f_{\dot{x}} = 2\dot{x}$

$f_x = 10t$

$$2\ddot{x} - 10t = 0 \Rightarrow \ddot{x} = 5t$$

$$\Rightarrow \dot{x} = 5\frac{t^2}{2} + c_1$$

$$\Rightarrow x = \frac{5}{2}\cdot\frac{t^3}{3} + c_1 t + c_2$$

$x(0) = c_2 = 1$

$x(1) = \frac{5}{6} + c_1 + 1 = 2 \Rightarrow c_1 = \frac{1}{6}$

$$\hat{x} = \frac{5}{6}t^3 + \frac{1}{6}t + 1$$

$$Hf_{x\dot{x}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$ semid. pos $\Rightarrow$ $\hat{x}$ è minimante