Ex 1

Eq. caratteristica $\alpha^2 + 2\lambda\alpha + 3 = 0$

Soluzioni $\alpha_{1/2} = -\lambda \pm \sqrt{\lambda^2 - 3}$

$\lambda^2 - 3 = 0 \iff \lambda = \pm\sqrt{3}$

$\alpha_1 = \alpha_2 = -\lambda$ $\quad y(x) = c_1 e^{-\lambda x} + c_2 x e^{-\lambda x}$

integrale generale

$\lambda^2 - 3 > 0 \iff \lambda < -\sqrt{3} \vee \lambda > +\sqrt{3}$

$$y(x) = c_1 e^{+\alpha_1 x} + c_2 e^{\alpha_2 x}$$

α_1 e α_2 radici reali e distinte

$\lambda^2 - 3 < 0 \iff -\sqrt{3} < \lambda < \sqrt{3}$

α_1 e α_2 radici complesse coniugate

$$y(x) = e^{-\lambda x}\left(c_1 \cos(\sqrt{3-\lambda^2}\,x) + c_2 \operatorname{sen}(\sqrt{3-\lambda^2}\,x)\right)$$

Imponiamo le cond. ai limiti:

$\boxed{\lambda^2 - 3 = 0}$

$$\begin{cases} c_1 = 0 \\ c_1 e^{-\lambda} + c_2 e^{-\lambda} = 0 \end{cases} \qquad \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

Il pb ammette solo la soluzione nulla $y(x) \equiv 0$

$\lambda^2 - 3 > 0$

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\alpha_1} + c_2 e^{\alpha_2} = 0 \end{cases}$$

matrice associata: $\begin{pmatrix} 1 & 1 \\ e^{\alpha_1} & e^{\alpha_2} \end{pmatrix}$

$\det = e^{\alpha_2} - e^{\alpha_1} \neq 0$ in quanto $\alpha_1 \neq \alpha_2$

Dunque anche in questo caso $c_1 = c_2 = 0$

e $y(x) \equiv 0$

$\lambda^2 - 3 < 0$

$$\begin{cases} c_1 = 0 \\ c_2 \operatorname{sen}\left(\sqrt{3-\lambda^2}\right) = 0 \end{cases}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & \operatorname{sen}\sqrt{3-\lambda^2} \end{vmatrix} = \operatorname{sen}\sqrt{3-\lambda^2} = 0$$

$\Leftrightarrow \sqrt{3-\lambda^2} = k\pi \Leftrightarrow 3-\lambda^2 = k^2\pi^2$

$\Leftrightarrow \lambda^2 = 3 - k^2\pi^2$

Imponendo $3 - k^2\pi^2 \geq 0$ si vede che è possibile solo $k=0$. Tuttavia per $k=0$ $\lambda^2 = 3$ non è accettabile.

Pertanto $c_1 = c_2 = 0$ e $y(x) \equiv 0$

Ex 2

$xy' = 3(1-y^2)$ Eq. a variabili separabili

$1-y^2=0 \iff y=\pm 1$ <u>soluzioni costanti</u>

$$\int \frac{dy}{1-y^2} = \int \frac{3\,dx}{x}$$

$$\frac{1}{1-y^2} = \frac{1}{2}\,\frac{1}{1-y} + \frac{1}{2}\,\frac{1}{1+y}$$

$$-\frac{1}{2}\log|1-y| + \frac{1}{2}\log|1+y| = 3\log|x| + c$$

$$\log\left|\frac{1+y}{1-y}\right|^{1/2} = \log|x|^3 + c$$

$$\left|\frac{1+y}{1-y}\right|^{\frac{1}{2}} = |x|^3 e^c$$

$$\left|\frac{1+y}{1-y}\right| = x^6 e^{2c} \longrightarrow \frac{1+y}{1-y} = kx^6 \qquad k\in\mathbb{R}$$

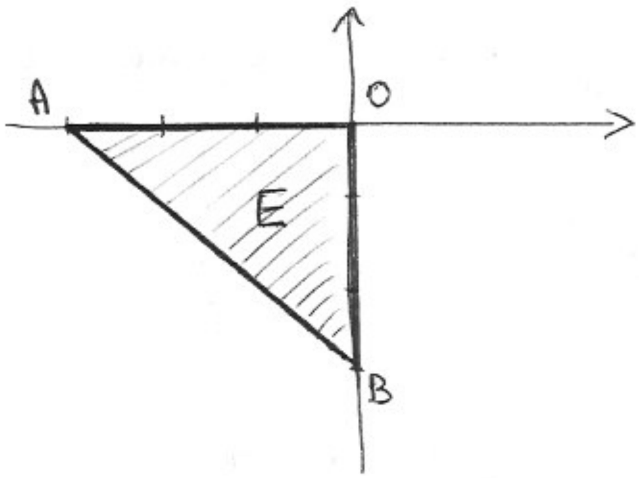
$$\longrightarrow y = \frac{kx^6-1}{1+kx^6}$$

$y(1)=1 \longrightarrow$ la Soluzione è $y(x)\equiv 1$

$y(1)=0 \Rightarrow \frac{k-1}{1+k}=0 \rightarrow k=1 \Rightarrow y(x)=\frac{x^6-1}{1+x^6}$

Ex 3

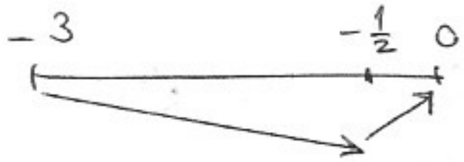
$$f(x,y) = x^2 + y^2 - xy + x + y$$



$P(-1,-1)$ pto stazionario interno

su $\overline{OA}$: $f(x,0) = x^2 + x$ $\quad x \in [-3,0]$

$\frac{d}{dx}$: $2x + 1 \geq 0 \iff x \geq -\frac{1}{2}$

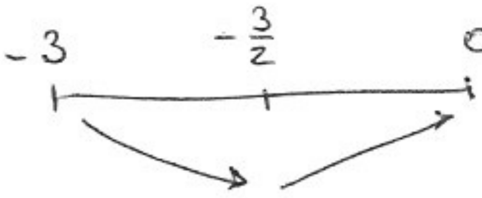


su $\overline{OB}$ $f(0,y) = y^2 + y$ $\quad y \in [-3,0]$

$\frac{d}{dy}$: $2y + 1$ come sopra

su $\overline{AB}$: $f(x, -x-3) = 3x^2 + 9x + 6$ $\quad x \in [-3,0]$

$\frac{d}{dx}$: $6x + 9 \geq 0 \iff x \geq -\frac{3}{2}$



$A(-3,0)$ $B(0,-3)$ $O(0,0)$

$D(-\frac{1}{2},0)$ $F(0,-\frac{1}{2})$ $G(-\frac{3}{2},-\frac{3}{2})$

$f(P) = -1$

$f(A) = f(B) = 6$

$f(O) = 0$

$f(D) = f(F) = -\frac{1}{4}$

$f(G) = -\frac{3}{4}$

$\min_E f = -1$

$\max_E f = 6$

Ex 4

$\vec{F} = (x - u^3 - v^3, \; y - uv + v^2)$

$\vec{F}(2,0,1,1) = (0,0)$

$$\operatorname{Jac}\vec{F} = \begin{pmatrix} 1 & 0 & -3u^2 & -3v^2 \\ 0 & 1 & -v & -u+2v \end{pmatrix}$$

$$\operatorname{Jac}\vec{F}(2,0,1,1) = \begin{pmatrix} 1 & 0 & -3 & -3 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} -3 & -3 \\ -1 & 1 \end{pmatrix} = -6 \neq 0$$

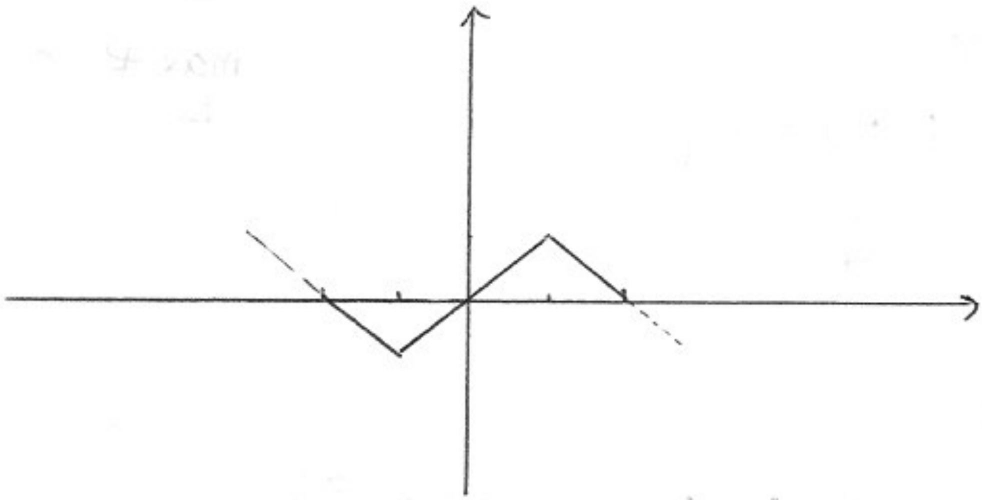
$\underset{\text{Dini}}{\Longrightarrow}$ $\exists!$ $\vec{g}(x,y) = (u(x,y), v(x,y))$ in un intorno di $(2,0)$ t.c. $\vec{g}(2,0) = (1,1)$

$$\operatorname{Jac}\vec{g}(2,0) = -\left[\operatorname{Jac}_{uv}\vec{F}\right]^{-1} \operatorname{Jac}_{xy}\vec{F} \; (2,0,1,1) =$$

$$= -\begin{pmatrix} -3 & -3 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 3 \\ 1 & -3 \end{pmatrix}$$

Ex 5

Occorre estendere f in modo dispari su $[-2,0]$:



$a_k = 0 \qquad b_k = \int_0^1 x \,\text{sen}\left(\frac{\pi k x}{2}\right) dx + \int_1^2 (2-x)\, \text{sen}\left(\frac{\pi k x}{2}\right) dx$

$$= \dots = \frac{8}{k^2 \pi^2}\, \text{sen}\left(\frac{\pi k}{2}\right)$$

$$\sum_{k=1}^{+\infty} \left[\frac{8}{k^2\pi^2}\, \text{sen}\left(\frac{k\pi}{2}\right)\right] \text{sen}\left(\frac{k\pi x}{2}\right)$$

Essendo f regolare a tratti e continua, la serie di Fourier converge puntualmente a $f(x)$, $\forall x \in \mathbb{R}$.

Poi, essendo $|b_k| \leq \frac{8}{k^2\pi^2}$ con $\sum \frac{1}{k^2}$ convergenti, si ha anche conv. totale in $\mathbb{R}$.

Infine, scegliendo $x = 1$:

$$1 = \sum_{k=1}^{+\infty} \frac{8}{k^2\pi^2} \sin^2\left(\frac{k\pi}{2}\right)$$

$$\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 0 & k = 2n \\ (-1)^n & k = 2n+1 \end{cases}$$

$$1 = \sum_{=0}^{+\infty} \frac{8}{\pi^2 (2n+1)^2} \Rightarrow \sum_{=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Ex 6

f è continua in $\mathbb{R}^2$ in quanto composizione di funz. continue.

$$f(x,0) = 0 \Rightarrow \frac{\partial f}{\partial x}(0,0) = 0$$

$$f(0,y) = |y| e^{y^2}$$

$$\frac{f(0,y) - f(0,0)}{y} = \frac{|y| e^{y^2}}{y} = \begin{cases} e^{y^2} & y > 0 \\ -e^{y^2} & y < 0 \end{cases}$$

$$\xrightarrow{y \to 0^+} +1$$

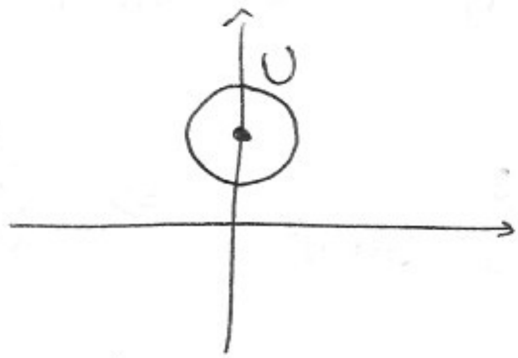
$$\nexists \text{ limite} \Rightarrow \nexists \frac{\partial f}{\partial y}(0,0)$$

$$\xrightarrow{y \to 0^-} -1$$

$\Rightarrow$ f non è diff.^{le} in $(0,0)$

$$f(x,y) = y e^{x^2+y^2} \quad \underline{\text{se } y>0}$$

Scelgo U intorno di $(0,1)$ dove $y>0$:



Pertanto $f \in C^1(U)$ (non essendoci il valore assoluto)

e questo assicura la differenziabilità in U, in particolare in $(0,1)$.