EX 1

Dividendo per x si ottiene un'equazione di Eulero:

$$xy' + 2y = \frac{1}{x}$$

Eq. omogenea $\quad xy' + 2y = 0$

$$\alpha + 2 = 0 \quad \rightarrow \alpha = -2$$

$$y(x) = \frac{c_1}{x^2} \quad \text{int. gen. omogenea}$$

Soluz. particolare con il metodo di somiglianza

$$y_p(x) = \frac{A}{x}$$

$$y_p' = -\frac{A}{x^2} \qquad x y_p' + 2y_p = \frac{1}{x}$$

$$-\frac{A}{x} + \frac{2A}{x} = \frac{1}{x} \quad \rightarrow A = 1$$

Int. gen. eq. completa: $y(x) = \frac{c_1}{x^2} + \frac{1}{x}, \quad c_1 \in \mathbb{R}$

È chiaro che $\lim_{x \to +\infty} \frac{c_1}{x^2} + \frac{1}{x} = 0$, per ogni $c_1 \in \mathbb{R}$.

$$y(2) = 2\,y(1) \iff \frac{c_1}{4} + \frac{1}{2} = 2(c_1 + 1)$$

$$\iff c_1 = -\frac{6}{7}$$

$$y(x) = -\frac{6}{7x^2} + \frac{1}{x}$$

Ex 2

$$\begin{cases} x' = -x + 4y + 2 + e^{3t} \\ y' = -x + 3y - 1 \end{cases}$$

Metodo di riduzione (o sostituzione)

$$x'' = -x' + 4(-x + 3y - 1) + 3e^{3t}$$
$$= -x' - 4x + 12y - 4 + 3e^{3t}$$
$$= -x' - 4x + 3(x' + x - 2 - e^{3t}) - 4 + 3e^{3t}$$
$$= -x' - 4x + 3x' + 3x - 6 - \cancel{3e^{3t}} - 4 + \cancel{3e^{3t}}$$

$$x'' - 2x' + x = -10$$

$$\lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda - 1)^2 = 0$$

$$x_0(t) = c_1 e^t + c_2 t e^t$$

$$x_p(t) = A \quad \rightsquigarrow \quad A = -10$$

$$x(t) = c_1 e^t + c_2 t e^t - 10$$

$$\cancel{y = -c_1 e^t - c_2 t}$$

$$4y = x' + x - 2 - e^{3t}$$
$$= c_1 e^t + c_2 e^t + c_2 t e^t + c_1 e^t + c_2 t e^t - 10 - 2 - e^{3t}$$
$$= (2c_1 + c_2) e^t + 2c_2 t e^t - 12 - e^{3t}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 t e^t - 10 \\ \frac{2c_1 + c_2}{4} e^t + \frac{c_2}{2} t e^t - 3 - \frac{1}{4} e^{3t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} e^t \\ \frac{e^t}{2} \end{pmatrix} + c_2 \begin{pmatrix} t e^t \\ \frac{1}{4} e^t + \frac{1}{2} t e^t \end{pmatrix} - \begin{pmatrix} 10 \\ 3 + \frac{1}{4} e^{3t} \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} - \begin{pmatrix} 10 \\ \frac{13}{4} \end{pmatrix}$$

$$\begin{cases} c_1 - 10 = 0 \\ \frac{c_1}{2} + \frac{c_2}{4} - \frac{13}{4} = 0 \end{cases} \qquad \begin{cases} c_1 = 10 \\ c_2 = -7 \end{cases}$$

Ex 3

$$f(x,y,z) = x^2 - x + y^2 + e^x - z + y^2 e^z \qquad P = (0,0,1)$$

$f \in C^1(\mathbb{R}^3)$

$f(0,0,1) = 0$

$f_z = -1 + y^2 e^z \qquad f_z(P) = -1 \neq 0$

$\Rightarrow \exists!\ g(x,y)$ definita implicitamente in un intorno di $(0,0)$ t.c. $g(0,0) = 1$

$$g_x(0,0) = -\frac{f_x(0,0,1)}{f_z(0,0,1)} \qquad g_y(0,0) = -\frac{f_y(0,0,1)}{f_z(0,0,1)}$$

$f_x = 2x - 1 + e^x \qquad f_x(P) = 0$

$f_y = 2y + 2y e^z \qquad f_y(P) = 0 \qquad \Rightarrow \nabla g(0,0) = (0,0)$

$\Rightarrow (0,0)$ è punto critico per g.

$$f(x,y,g(x,y)) = 0 \iff x^2 - x + y^2 + e^x - g + y^2 e^g = 0$$

∂_x: $\quad 2x - 1 + e^x - g_x + y^2 e^g g_x = 0$

∂_{xx}: $\quad 2 + e^x - g_{xx} + y^2 e^g (g_x)^2 + y^2 e^g g_{xx} = 0$

in P: $\quad 2+1-g_{xx}(P)+0=0 \quad \Rightarrow \quad g_{xx}(P)=3$

∂_{xy}: $\quad -g_{xy}+2ye^{g}g_x+y^2e^{g}g_xg_y+y^2e^{g}g_{xy}=0$

in P: $\quad -g_{xy}(P)+0+0+0=0 \quad \Rightarrow g_{xy}(P)=0$

∂_y: $\quad 2y-g_y+2ye^{g}+y^2e^{g}g_y=0$

∂_{yy}: $\quad 2-g_{yy}+2e^{g}+2ye^{g}g_y+2ye^{g}g_y+y^2e^{g}g_y^2$

$+y^2e^{g}g_{yy}=0$

in P: $\quad +2-g_{yy}(P)+2e+0+0+0+0=0$

$g_{yy}(P)=2e+2$

$$\mathcal{H}g(0,0)=\begin{pmatrix}3 & 0\\ 0 & 2e+2\end{pmatrix} \quad \text{def. positiva}$$

$\Rightarrow (0,0)$ è punto di min rel. stretto per g

EX 4

$\mathcal{L}= xy+2z-\lambda(x+y+z)-\mu(x^2+y^2+z^2-24)=0$

$$\begin{cases} y-\lambda-2\mu x=0 \\ x-\lambda-2\mu y=0 \\ 2-\lambda-2\mu z=0 \\ x+y+z=0 \\ x^2+y^2+z^2=24 \end{cases}$$

} differenza m.a m.

$$\begin{cases} y-x+2\mu(y-x)=0 \\ x-\lambda-2\mu y=0 \\ \text{---} \\ \text{---} \\ \text{---} \end{cases} \qquad \begin{cases} (y-x)(1+2\mu)=0 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{cases}$$

$$\begin{cases} y=x \\ \text{---} \\ \text{---} \\ z=-2x \\ x^2+x^2+4x^2=24 \to x=\pm 2 \end{cases} \vee \begin{cases} \mu=-\frac{1}{2} \\ x-\lambda+y=0 \\ 2-\lambda+z=0 \\ \text{---} \\ \text{---} \end{cases}$$

$$\begin{cases} x=2 \\ y=2 \\ z=-4 \\ \mu=0 \\ \lambda=2 \end{cases} \vee \begin{cases} x=-2 \\ y=-2 \\ z=4 \\ \mu=\frac{1}{3} \\ \lambda=-\frac{2}{3} \end{cases}$$

$$\begin{cases} \mu=-\frac{1}{2} \\ y=\lambda-x \\ z=\lambda-2 \\ \lambda+\lambda-2=0 \to \lambda=1 \\ \text{---} \end{cases}$$

$$\begin{cases} \mu=-\frac{1}{2} \\ y=1-x \\ z=-1 \\ \lambda=1 \\ x^2+(1-x)^2+1=24 \to \end{cases}$$

$$\to x_{1/2}=\frac{1\pm 3\sqrt{5}}{2}$$

$$\begin{cases} x=\frac{1+3\sqrt{5}}{2} \\ y=\frac{1-3\sqrt{5}}{2} \\ z=-1 \\ \mu=-\frac{1}{2} \\ \lambda=1 \end{cases} \vee \begin{cases} x=\frac{1-3\sqrt{5}}{2} \\ y=\frac{1+3\sqrt{5}}{2} \\ z=-1 \\ \mu=-\frac{1}{2} \\ \lambda=1 \end{cases}$$

$$f(2,2,-4) = -4$$

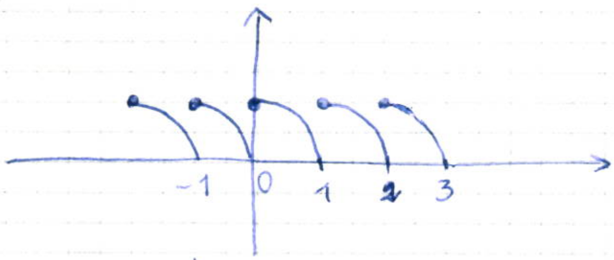
$$f(-2,-2,4) = 12 \qquad \text{Max}$$

$$f\left(\frac{1+3\sqrt{5}}{2}, \frac{1-3\sqrt{5}}{2}, -1\right) = \frac{1-45}{4} - 2 = -13$$

$$f\left(\frac{1-3\sqrt{5}}{2}, \frac{1+3\sqrt{5}}{2}, -1\right) = -13 \qquad \text{Min}$$

Ex 5

f :



$$a_0 = 2\int_0^1 (1-x^2)\,dx = \frac{4}{3}$$

$$a_k = 2\int_0^1 (1-x^2)\cos(2\pi k x)\,dx =$$

$$= 2\left\{(1-x^2)\frac{\operatorname{sen}(2\pi k x)}{2\pi k}\Big|_0^1 - \int_0^1 -2x \cdot \frac{\operatorname{sen}(2\pi k x)}{2\pi k}\,dx\right\}$$

$$= 2\cdot\frac{1}{\pi k}\left\{\left[-\frac{x\cos(2\pi k x)}{2\pi k}\right]_0^1 + \int_0^1 \frac{\cos(2\pi k x)}{2\pi k}\,dx\right\}$$

$$= -\frac{1}{\pi^2 k^2}$$

$$b_k = 2\int_0^1 (1-x^2)\operatorname{sen}(2\pi k x)\,dx =$$

$$= 2\left\{-(1-x^2)\frac{\cos(2\pi k x)}{2\pi k}\Big|_0^1 - \int_0^1 \cancel{2}x\,\frac{\cos(2\pi k x)}{\cancel{4}\pi k}\,dx\right\}$$

$$= 2\cdot\frac{1}{2\pi k} - \frac{2}{\pi k}\left\{\underbrace{\cancel{x\,\frac{\operatorname{sen}(2\pi k x)}{2\pi k}\Big|_0^1}}_{0} - \int_0^1 \frac{\operatorname{sen}(2\pi k x)}{2\pi k}\,dx\right\}$$

$$= \frac{1}{\pi k}$$

$$f(x) \rightsquigarrow \frac{2}{3} + \sum_{k=1}^{+\infty} \frac{-1}{\pi^2 k^2}\cos(2\pi k x) + \frac{1}{\pi k}\,\mathrm{sen}(2\pi k x)$$

$f(x)$ è regolare a tratti $\Rightarrow$ la serie di Fourier converge puntualmente a

$$s(x) = \begin{cases} f(x) & x \notin \mathbb{Z} \\ \frac{1}{2} & x \in \mathbb{Z} \end{cases}$$

$x = 0$:

$$\frac{1}{2} = \frac{2}{3} + \sum_{k=1}^{+\infty} -\frac{1}{\pi^2 k^2} \Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Id. di Parseval :

$$\int_0^1 (1-x^2)^2\,dx = \frac{1}{2}\left\{ \frac{16}{9}\cdot\frac{1}{2} + \sum_{k=1}^{+\infty}\left(\frac{1}{k^4\pi^4} + \frac{1}{k^2\pi^2}\right)\right\}$$

$$2\left(1 + \frac{1}{5} - \frac{2}{3}\right) = \frac{8}{9} + \frac{1}{\pi^4}\sum_{k=1}^{+\infty}\frac{1}{k^4} + \frac{1}{6}$$

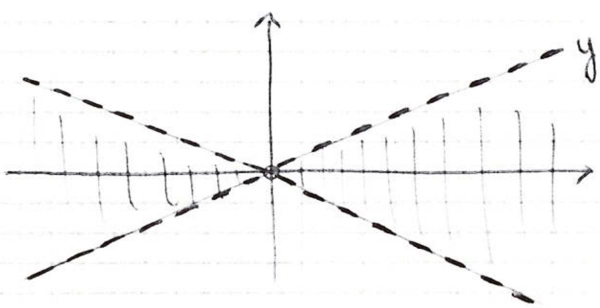
$$\Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

Ex 6

$$f(x,y) = \log(2x^2 - 3y^2)$$

$$\mathrm{dom} f = \{(x,y) \in \mathbb{R}^2 \mid 2x^2 - 3y^2 > 0\}$$

$$2x^2 - 3y^2 > 0 \iff y^2 < \frac{2}{3}x^2 \iff -\sqrt{\frac{2}{3}}|x| < y < \sqrt{\frac{2}{3}}|x|$$



$\partial_x f = \dfrac{4x}{2x^2-3y^2}$ $\partial_y f = \dfrac{-6y}{2x^2-3y^2}$

Sono ben definite e continue in $\mathrm{dom} f$; dunque f è di classe C^1 in $\mathrm{dom} f$.

Se f è C^1 allora è differenziabile

$$\nabla f(1,0) = \left(\frac{4}{2}, 0\right) = (2,0)$$

π: $$z = \log 2 + (2,0)\cdot(x-1, y)$$
$$z = 2x - 2 + \log 2$$