Ex 1

$$\begin{cases} y' + 4y = y^3 \\ y(0) = \alpha \end{cases}$$

⊛ Risolviamo l'eq. diff. come eq. di Bernoulli, cercando soluzioni non nulle:

$$z = y^{1-\alpha} = y^{-2} \qquad z' = -2y^{-3}y'$$

$$= -2y^{-3}(y^3 - 4y)$$

$$= -2 + 8z$$

$$z' - 8z = -2$$

Int. generale $\quad z = ce^{+8t} + \frac{1}{4}$; $\quad z(0) = \alpha^{-2}$

$$\Updownarrow$$

Da cui $\quad y = \pm \frac{1}{z^{1/2}} = \pm \frac{1}{\left(\frac{1}{4} + ce^{8t}\right)^{1/2}} \qquad c = \frac{4-\alpha^2}{4\alpha^2}$

$$y(x) = \pm \frac{\sqrt{4\alpha^2}}{\sqrt{\alpha^2 + (4-\alpha^2)e^{8t}}}$$

⊛ L'eq. diff. ammette $y = 0$ come soluzione.
Ciò corrisponde ad $\alpha = 0$.

$$y^2 = \frac{4\alpha^2}{\alpha^2+(4-\alpha^2)e^{8t}} \longrightarrow y = \pm \frac{\sqrt{4\alpha^2}}{\sqrt{\alpha^2+(4-\alpha^2)e^{8t}}}$$

Se $\alpha>0$ si sceglie $+$: $y(x) = \dfrac{2\alpha}{\sqrt{\alpha^2+(4-\alpha^2)e^{8t}}}$

(la soluzione parte da un valore positivo all'istante $t=0$ e rimane positiva perché non può intersecare l'asse t per unicità)

Se $\alpha<0$ si sceglie $-$: $y(x) = -\dfrac{2(-\alpha)}{\sqrt{\alpha^2+(4-\alpha^2)e^{8t}}}$

(come sopra)

Se $\alpha=0$ $\quad y\equiv 0$.(vedi ⊛) sol. def. in $\mathbb{R}$

Se $\alpha\neq 0$ e $(4-\alpha^2)\geq 0$ allora $\alpha^2+(4-\alpha^2)e^{8t}>0$

e y risulta definita in tutto $\mathbb{R}$.

Quindi la sol. del PdC è globale per $-2\leq\alpha\leq+2$

Ex 2

$$Z(t) = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \quad ; \quad \det Z(t) = 1-t^2$$

$1-t^2\neq 0 \iff t\neq\pm 1$

$Z(t)$ è matrice fondamentale in $(-\infty,-1)$
$(-1,1)$
$(1,+\infty)$

~~Se~~ ~~t~~ In ciascuno dei suddetti intervalli

$$Z'(t) = A(t)\, Z(t)$$

da cui $\qquad A(t) = Z'(t)\, Z(t)^{-1}$

$$Z'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad Z(t)^{-1} = \frac{1}{1-t^2} \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}$$

$$A(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix} \frac{1}{1-t^2} = \begin{pmatrix} \dfrac{-t}{1-t^2} & \dfrac{1}{1-t^2} \\[2ex] \dfrac{1}{1-t^2} & \dfrac{-t}{1-t^2} \end{pmatrix}$$

$$\vec{Y}(t) = Z(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix}$$

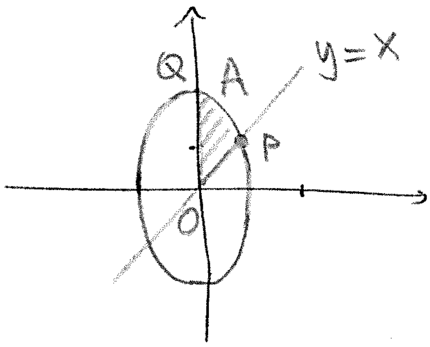
$$\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \iff c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \iff \begin{array}{l} c_1 = 1 \\ c_2 = 1 \end{array}$$

Int. particolare $\qquad \vec{Y}(t) = \begin{pmatrix} 1 + t \\ t + 1 \end{pmatrix}$

Ex 3

$f = 3x^2 - 2y^2 + 3y \qquad A = \{ x \geq 0,\ y \geq x,\ 4x^2 + y^2 \leq 4 \}$

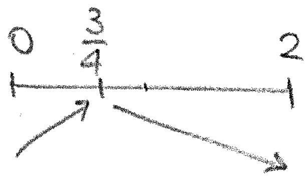
Vale il T. di Weierstrass $\Rightarrow$ esistono gli estremi assoluti di f in A



$\nabla f = (6x, -4y+3) = (0,0) \iff \begin{cases} x = 0 \\ y = \frac{3}{4} \end{cases}$ non è interno ad A

su $\overline{OQ}$: $f(0,y) = -2y^2 + 3y \qquad y \in [0,2]$

$\frac{d}{dy}$: $-4y + 3 \geq 0 \iff y \leq \frac{3}{4}$



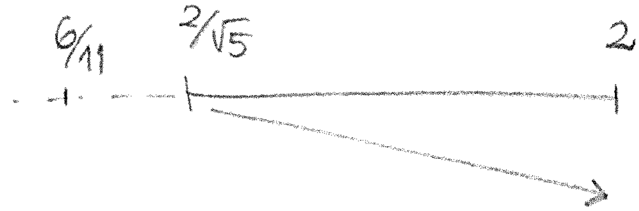
su $\overline{OP}$: $f(x,x) = x^2 + 3x \qquad x \in [0, \frac{2}{\sqrt{5}}]$

$\frac{d}{dx}$: $2x + 3 \geq 0 \iff x \geq -\frac{3}{2}$

su $\overset{\frown}{PQ}$: $x^2 = 1 - \frac{y^2}{4} \qquad y \in [\frac{2}{\sqrt{5}}, 2]$

$h(y) = 3 - \frac{3}{4}y^2 - 2y^2 + 3y = -\frac{11}{4}y^2 + 3y + 3$

$$h'(y) = -\frac{11}{2}y + 3 \geq 0 \iff y \leq \frac{6}{11}$$



$$f(0,0)=0$$

$$f\left(0, \frac{3}{4}\right) = \frac{9}{8}$$

$$f(0,2) = -2 \qquad \underline{\min}$$

$$f\left(\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{4}{5} + \frac{6}{\sqrt{5}} = \frac{4+6\sqrt{5}}{5} \qquad \underline{\max}$$

EX 4

$\vec{F}$ è di classe C^1 in $\mathbb{R}^4$

$$\vec{F}(1,1,1,1) = (0,0)$$

$$\mathrm{Jac}\,\vec{F} = \begin{pmatrix} 2x & -2t & 2z & -2y \\ 5x^4 & -2t & 5z^4 & -2y \end{pmatrix}$$

$$\mathrm{Jac}\,\vec{F}(1,1,1,1) = \begin{pmatrix} 2 & \boxed{\begin{matrix} -2 & 2 \\ -2 & 5 \end{matrix}} & -2 \\ 5 & & -2 \end{pmatrix}$$

Siccome $\det \begin{pmatrix} -2 & 2 \\ -2 & 5 \end{pmatrix} \neq 0$, vale il T. di Dini

e quindi esiste un'unica f. implicita come richiesto.

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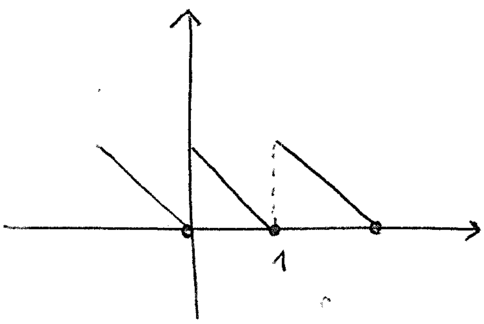
$$\mathrm{Jac}\,\vec{\phi}\,(1,1) = -\left[\mathrm{Jac}_{yz}\vec{F}\,(1,1,1,1)\right]^{-1}\mathrm{Jac}_{xt}\vec{F}\,(1,1,1,1).$$

$$= -\begin{pmatrix} -2 & 2 \\ -2 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 2 & -2 \\ 5 & -2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 0 & * \\ * & * \end{pmatrix} = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$$

$$\partial_x g_1(1,1) = 0$$

Ex 5



$$a_k = 2\int_0^1 (1-x)\cos(2\pi k x)\,dx =$$

$$= 2\left\{\left[(1-x)\frac{\operatorname{sen}(2\pi k x)}{2\pi k}\right]_0^1 + \int_0^1 \frac{\operatorname{sen}(2\pi k x)}{2\pi k}\,dx\right\}$$

$$= \frac{1}{\pi k}\left[-\frac{\cos 2\pi k x}{2\pi k}\right]_0^1 = 0$$

$$a_0 = 2\int_0^1 (1-x)\,dx = 1$$

$$b_k = 2\int_0^1 (1-x)\,\mathrm{sen}\,(2\pi k x)\,dx =$$

$$= 2\left\{\left[(1-x)\,\frac{-\cos(2\pi k x)}{2\pi k}\right]_0^1 - \int_0^1 \frac{\cos(2\pi k x)}{2\pi k}\,dx\right\}$$

$$= \frac{1}{\pi k}$$

$$f(x) \leadsto \frac{1}{2} + \sum_{k=1}^{+\infty} \frac{1}{\pi k}\,\mathrm{sen}(2\pi k x)$$

$$s(x) = \begin{cases} f(x) & x \neq k, \quad k \in \mathbb{Z} \\ \frac{1}{2} & x = k, \quad k \in \mathbb{Z} \end{cases}$$

$$\int_0^1 f(x)^2\,dx = \frac{1}{2}\left\{\frac{a_0^2}{2} + \sum_{k=1}^{+\infty} a_k^2 + b_k^2\right\}$$

$$\int_0^1 (1-x)^2\,dx = \frac{1}{2}\left\{\frac{1}{2} + \sum_{k=1}^{+\infty} \frac{1}{\pi^2 k^2}\right\}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{k=1}^{+\infty} \frac{1}{k^2}$$

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Ex 6

$$\left|f(\rho\cos\theta, \rho\,\text{sen}\,\theta)\right| = \left|\frac{\cancel{\rho}\cos\theta\sqrt{|\rho|}\sqrt{|\text{sen}\,\theta|}}{\cancel{\rho}}\right| \leq \sqrt{|\rho|} \xrightarrow[\rho\to 0]{} 0$$

$\Rightarrow$ f è continua in $(0,0)$

$\exists\, \nabla f(0,0) = (0,0)$

$$\frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{x\sqrt{|y|}}{x^2+y^2}$$

In coord. polari:
$$\frac{\rho\cos\theta\,\rho^{1/2}\sqrt{|\text{sen}\,\theta|}}{\rho^2} = \frac{\cos\theta\sqrt{|\text{sen}\,\theta|}}{\rho^{1/2}} \underset{\rho\to 0}{\not\to} 0$$

f non è diff.^{a} in $(0,0)$

$$\frac{\partial f}{\partial \vec{v}}(0,0) = \lim_{t\to 0} \frac{f(tv_1, tv_2)}{t}$$

$$\frac{f(tv_1, tv_2)}{t} = \frac{\cancel{t}v_1\sqrt{|t|}\sqrt{|v_2|}}{\cancel{t}\,|t|} = \frac{v_1\sqrt{|v_2|}}{\sqrt{|t|}}$$

Il limite per $t\to 0$ esiste finito solo se $v_1\sqrt{|v_2|} = 0$

Quindi esistono solo le derivate parziali in $(0,0)$.