Esercizio 1

$$\begin{cases} y'' + \lambda y = \lambda \sin x \\ y(0) = 0 \\ y(2\pi) = 0 \end{cases} \qquad \lambda > 0$$

$$\mu^2 + \lambda = 0 \Rightarrow \mu^2 = -\lambda \underset{\lambda>0}{\Rightarrow} \mu_{1/2} = \pm i\sqrt{\lambda}$$

$\lambda \neq 0 \Rightarrow$ le due radici sono distinte

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \operatorname{sen}\sqrt{\lambda}x$$

int. gen. omogenea

$$f(x) = \lambda \sin x = \lambda \operatorname{Im}(e^{ix})$$

caso I : $\lambda \neq 1$
caso II : $\lambda = 1$

Se $\lambda \neq 1$, allora $z_p(x) = Ae^{ix}$ risolve l'eq. completa

se e solo se $-Ae^{ix} + \lambda A e^{ix} = \lambda e^{ix}$

$$-A + \lambda A = \lambda \longrightarrow A = \frac{\lambda}{\lambda - 1}$$

Int. gen. eq. completa:

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x + \frac{\lambda}{\lambda-1} \operatorname{sen} x$$

Se $\lambda = 1$, allora $z_p(x) = Axe^{ix}$ risolve l'eq. comp.

se e solo se $\cancel{-Axe^{ix}} + \cancel{Axe^{ix}} + 2Aie^{ix} = e^{ix}$

$$2Ai = 1 \Rightarrow A = \frac{1}{2i} = \frac{-i}{2}$$

$$z_p = -\frac{i}{2} x e^{ix}$$

$$\operatorname{Im} z_p = -\frac{x}{2}\cos x$$

Sol. completa: $\quad y(x) = c_1 \cos x + c_2 \operatorname{sen} x - \frac{x}{2}\cos x$

Imponiamo le cond. ai limiti:

<u>caso I</u>:
$$\begin{cases} c_1 = 0 \\ c_1 \cos\sqrt{\lambda}\,2\pi + c_2 \operatorname{sen}(\sqrt{\lambda}\,2\pi) = 0 \end{cases}$$

$$\begin{vmatrix} 1 & 0 \\ \cos\sqrt{\lambda}\,2\pi & \operatorname{sen}\sqrt{\lambda}\,2\pi \end{vmatrix} = \operatorname{sen}(\sqrt{\lambda}\,2\pi) = 0 \iff \sqrt{\lambda}\,2\cancel{\pi} = k\cancel{\pi} \quad k \in \mathbb{N}$$

$$\iff \lambda = \left(\frac{k}{2}\right)^2 \quad k \in \mathbb{N}\setminus\{2\}$$

Se $\lambda = \lambda_k = \left(\frac{k}{2}\right)^2$ per qualche $k \in \mathbb{N}$, $k \neq 2$

allora il sistema omogeneo ha ∞ soluzioni $(0, c_2)$

e il pb ai limiti ha ∞ soluzioni

$$y_k(x) = c_2 \operatorname{sen}\left(\frac{k}{2}x\right) + \frac{\lambda_k}{\lambda_k - 1}\operatorname{sen} x, \qquad c_2 \in \mathbb{R}$$

Se $\lambda \neq \lambda_k \quad \forall k \in \mathbb{N}, k \neq 2$ esiste un'unica soluzione

$(c_1, c_2) = (0,0)$ e il pb. ai limiti ha l'unica

soluzione

$$y(x) = \frac{\lambda}{\lambda - 1}\operatorname{sen} x$$

<u>caso II</u> $\quad \lambda = 1$

$$\begin{matrix} y(0) = 0 \iff \\ y(2\pi) = 0 \iff \end{matrix} \begin{cases} c_1 = 0 \\ c_1 - \pi = 0 \end{cases} \qquad \text{impossibile}$$

Esercizio 2

$$\begin{cases} x' = 2x + y + e^{2t} \\ y' = -x + 2y + e^t \end{cases} \qquad y = x' - 2x - e^{2t}$$

I metodo: Derivo la I eq:

$$x'' = 2x' + y' + 2e^{2t}$$

sost. la II eq:

$$x'' = 2x' - x + 2y + e^t + 2e^{2t}$$

Sost. a y la I eq:

$$x'' = 2x' - x + 2x' - 4x - \cancel{2e^{2t}} + e^t + \cancel{2e^{2t}}$$

Risolvo $\quad x'' - 4x' + 5x = e^t$

$$\lambda^2 - 4\lambda + 5 = 0 \qquad \lambda_{1/2} = 2 \pm i$$

Sol. part. $x_p(t) = A e^t$

$$A - 4A + 5A = 1 \Rightarrow A = \frac{1}{2}$$

$$x(t) = c_1 e^{2t} \cos t + c_2 e^{2t} \operatorname{sen} t + \frac{1}{2} e^t$$

$$y(t) = x' - 2x - e^{2t}$$

$$= \cancel{2c_1 e^{2t} \cos t} - c_1 e^{2t} \operatorname{sen} t + \cancel{2c_2 e^{2t} \operatorname{sen} t} +$$

$$+ c_2 e^{2t} \cos t + \frac{1}{2} e^t - \cancel{2c_1 e^{2t} \cos t} - \cancel{2c_2 e^{2t} \operatorname{sen} t}$$

$$- e^t - e^{2t}$$

$$= -c_1 e^{2t} \operatorname{sen} t + c_2 e^{2t} \cos t - \frac{1}{2} e^t - e^{2t}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} e^{2t}\cos t \\ -e^{2t}\operatorname{sen} t \end{pmatrix} + c_2 \begin{pmatrix} e^{2t}\operatorname{sen} t \\ e^{2t}\cos t \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^t \\ -\frac{1}{2}e^t - e^{2t} \end{pmatrix}$$

<u>oppure</u>

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{ha autovalori } \lambda_{1/2} = 2 \pm i$$

$\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ è un autovettore relativo a $2+i$

$$\mathcal{B} = \left\{ \operatorname{Re}\left(e^{(2+i)t}\vec{v}\right), \operatorname{Im}\left(e^{(2+i)t}\vec{v}\right) \right\}$$

$$= \left\{ e^{2t}\begin{pmatrix} \cos t \\ -\operatorname{sen} t \end{pmatrix}, e^{2t}\begin{pmatrix} \operatorname{sen} t \\ \cos t \end{pmatrix} \right\}$$

$$Z(t) = \begin{pmatrix} e^{2t}\cos t & e^{2t}\operatorname{sen} t \\ -e^{2t}\operatorname{sen} t & e^{2t}\cos t \end{pmatrix}$$

$$W(t) = \det Z(t) = e^{4t}$$

$$Z(t)^{-1} = e^{-4t} \begin{pmatrix} e^{2t}\cos t & -e^{2t}\operatorname{sen} t \\ e^{2t}\operatorname{sen} t & e^{2t}\cos t \end{pmatrix}$$

$$Z(t)^{-1}\begin{pmatrix} e^{2t} \\ e^t \end{pmatrix} = e^{-4t}\begin{pmatrix} e^{4t}\cos t - e^{3t}\operatorname{sen} t \\ e^{4t}\operatorname{sen} t + e^{3t}\cos t \end{pmatrix} = \begin{pmatrix} \cos t - e^{-t}\operatorname{sen} t \\ \operatorname{sen} t + e^{-t}\cos t \end{pmatrix}$$

$$\int Z^{-1}(t)\begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix} dt = \begin{pmatrix} \int \cos t - \int e^{-t} \operatorname{sen} t \\ \int \operatorname{sen} t + \int e^{-t} \cos t \end{pmatrix} \ominus$$

$$\int e^{-t} \operatorname{sen} t \, dt = -e^{-t} \operatorname{sen} t + \int e^{-t} \cos t =$$

$$= -e^{-t} \operatorname{sen} t - e^{-t} \cos t - \int -e^{-t}(-\operatorname{sen} t)\, dt$$

$$= -e^{-t} \operatorname{sen} t - e^{-t} \cos t - \int e^{-t} \operatorname{sen} t \, dt$$

$$\Rightarrow \int e^{-t} \operatorname{sen} t \, dt = -\frac{1}{2} e^{-t} \operatorname{sen} t - \frac{1}{2} e^{-t} \cos t$$

$$\int e^{-t} \cos t \, dt = -e^{-t} \cos t - \int -e^{-t}(-\sin t)\, dt$$

$$= -e^{-t} \cos t - \int e^{-t} \sin t \, dt$$

$$= -e^{-t} \cos t + \frac{1}{2} e^{-t} \operatorname{sen} t + \frac{1}{2} e^{-t} \cos t$$

$$= \frac{1}{2} e^{-t} \operatorname{sen} t - \frac{1}{2} e^{-t} \cos t$$

$$\ominus \begin{pmatrix} \operatorname{sen} t + \frac{1}{2} e^{-t} \operatorname{sen} t + \frac{1}{2} e^{-t} \cos t \\ -\cos t + \frac{1}{2} e^{-t} \operatorname{sen} t - \frac{1}{2} e^{-t} \cos t \end{pmatrix}$$

$$\int_0^t Z^{-1}(s)\begin{pmatrix} e^{2s} \\ e^{s} \end{pmatrix} ds = \begin{pmatrix} \operatorname{sen} t + \frac{1}{2} e^{-t} \operatorname{sen} t + \frac{1}{2} e^{-t} \cos t - \frac{1}{2} \\ -\cos t + \frac{1}{2} e^{-t} \operatorname{sen} t - \frac{1}{2} e^{-t} \cos t + 1 + \frac{1}{2} \end{pmatrix}$$

$$u(t) = Z(t)\int_0^t Z^{-1}(s)\begin{pmatrix} e^{2s} \\ e^{s} \end{pmatrix} ds =$$

$$= \begin{pmatrix} e^{2t}\cos t & e^{2t}\,\text{sen}\,t \\ -e^{2t}\,\text{sen}\,t & e^{2t}\cos t \end{pmatrix}\begin{pmatrix} \sin t + \frac{1}{2}e^{-t}\,\text{sen}\,t + \frac{1}{2}e^{-t}\cos t - \frac{1}{2} \\ -\cos t + \frac{1}{2}e^{-t}\,\text{sen}\,t - \frac{1}{2}e^{-t}\cos t + \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{e^{2t}\cos t\sin t} + \cancel{\frac{1}{2}e^{t}\cos t\,\text{sen}\,t} + \frac{1}{2}e^{t}\cos^2 t - \frac{1}{2}e^{2t}\cos t + \\ \quad - \cancel{e^{2t}\,\text{sen}\,t\cos t} + \frac{1}{2}e^{t}\sin^2 t - \cancel{\frac{1}{2}e^{t}\,\text{sen}\,t\cos t} + \frac{3}{2}e^{2t}\,\text{sen}\,t \\ -e^{2t}\sin^2 t - \frac{1}{2}e^{t}\sin^2 t - \cancel{\frac{1}{2}e^{t}\,\text{sen}\,t\cos t} + \frac{1}{2}e^{2t}\,\text{sen}\,t \\ \quad -e^{2t}\cos^2 t + \cancel{\frac{1}{2}e^{t}\,\text{sen}\,t\cos t} - \frac{1}{2}e^{t}\cos^2 t + \frac{3}{2}e^{2t}\cos t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{t} - \frac{1}{2}e^{2t}\cos t + \frac{3}{2}e^{2t}\,\text{sen}\,t \\ -e^{2t} - \frac{1}{2}e^{t} + \frac{1}{2}e^{2t}\,\text{sen}\,t + \frac{3}{2}e^{2t}\cos t \end{pmatrix}$$

Int. completo:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{2t}\cos t + c_2 e^{2t}\,\text{sen}\,t - \frac{1}{2}e^{2t}\cos t + \frac{3}{2}e^{2t}\,\text{sen}\,t + \frac{1}{2}e^{t} \\ -c_1 e^{2t}\,\text{sen}\,t + c_2 e^{2t}\cos t + \frac{1}{2}e^{2t}\,\text{sen}\,t + \frac{3}{2}e^{2t}\cos t - \frac{1}{2}e^{t} - e^{2t} \end{pmatrix}$$

Con $c_1' = c_1 - \frac{1}{2}$, $c_2' = c_2 + \frac{3}{2}$ stessa sol. di prima!

Ex. 3

$$f(x,y) = \sqrt[6]{x^4 (y-3)^2} + 3$$

f è continua in $\mathbb{R}^2$

$f(0,3) = 3$

$$f_x(0,3) = \frac{d}{dx} f(x,3) \Big|_{x=0} = \frac{d}{dx} 3 \Big|_{x=0} = 0$$

$$f_y(0,3) = \frac{d}{dy} f(0,y) \Big|_{y=3} = \frac{d}{dy} 3 \Big|_{y=3} = 0$$

$$\frac{f(x,y) - f(0,3) - f_x(0,3)\,x - f_y(0,3)(y-3)}{\sqrt{x^2+(y-3)^2}} =$$

$$= \frac{\sqrt[6]{x^4(y-3)^2} + 3 - 3}{\sqrt{x^2+(y-3)^2}} =$$

coor. pol. $x = \rho\cos\theta$
$y = 3 + \rho\sin\theta$

$$= \frac{\sqrt[6]{\rho^4\cos^4\theta\ \rho^2\sin^2\theta}}{\sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}} = \frac{\sqrt[6]{\cos^4\theta\ \sin^2\theta}}{1} \quad \nexists \lim$$

f non è diff.^{le} in $(0,3)$.

$$\vec{v} = (v_1, v_2)$$

$$g(t) = f(tv_1, 3+tv_2) = \sqrt[6]{t^4 v_1^4 \; t^2 v_2^2} + 3$$

$$\frac{g(t)-g(0)}{t} = \frac{\sqrt[6]{t^6 \; v_1^4 v_2^2}}{t} = \frac{|t| \sqrt[6]{v_1^4 v_2^2}}{t}$$

Il limite per $t \to 0$ esiste (ed è nullo) se e solo se $\sqrt[6]{v_1^4 v_2^2} = 0$, ossia se e solo se $v_1 = 0$ oppure $v_2 = 0$.

Pertanto esistono in $(0,3)$ solo le derivate parziali.

Ex 4 : $\quad f(x,y) = x^2 y + y^3 - y + 1$

$$\nabla f = \left(2xy \; , \; x^2 + 3y^2 - 1 \right)$$

$$\nabla f = (0,0) \iff \begin{cases} 2xy = 0 \\ x^2 + 3y^2 - 1 = 0 \end{cases} \qquad \begin{cases} x = 0 \;\vee\; y = 0 \\ \text{———} \end{cases}$$

$$\begin{cases} x = 0 \\ y_{1/2} = \pm \frac{1}{\sqrt{3}} \end{cases} \vee \begin{cases} y = 0 \\ x_{1/2} = \pm 1 \end{cases}$$

Pti stazionari $\quad \overset{P_1}{\left(0, \frac{1}{\sqrt{3}}\right)} \; \overset{P_2}{\left(0, -\frac{1}{\sqrt{3}}\right)} \; \overset{P_3}{(1,0)} \; \overset{P_4}{(-1,0)}$

$$Hf = \begin{pmatrix} 2y & 2x \\ 2x & 6y \end{pmatrix}$$

$$\mathcal{H}f(P_1) = \begin{pmatrix} \frac{2}{\sqrt{3}} & 0 \\ 0 & \frac{6}{\sqrt{3}} \end{pmatrix}$$ def. pos $\Rightarrow$ P_1 pto di min rel.

$$\mathcal{H}f(P_2) = \begin{pmatrix} -\frac{2}{\sqrt{3}} & 0 \\ 0 & -\frac{6}{\sqrt{3}} \end{pmatrix}$$ def. neg $\Rightarrow$ P_2 pto di max stretto rel

$$\mathcal{H}f(P_3) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$ indef. $\Rightarrow$ P_3 sella

$$\mathcal{H}f(P_4) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$ indef $\Rightarrow$ P_4 sella

Ex 5

$$J(x) = \int_0^1 \left(\frac{x}{1+t^2} - \dot{x}^2 \right) dt$$

$$f_x - \frac{d}{dt} f_{\dot{x}} = \frac{1}{1+t^2} - \frac{d}{dt}\left(-2\dot{x}\right) = \frac{1}{1+t^2} + 2\ddot{x} = 0$$

$$\Rightarrow \quad 2\ddot{x} = -\frac{1}{1+t^2}$$

$$\dot{x} = -\frac{1}{2}\int \frac{1}{1+t^2} = -\frac{1}{2}\operatorname{arctg} t + c_1$$

$$x(t) = -\frac{1}{2}\int \operatorname{arctg} t + c_1 t$$

$$= -\frac{1}{2}\left[t \operatorname{arctg} t - \frac{1}{2}\int \frac{2t}{1+t^2}\, dt \right] + c_1 t$$

$$= -\frac{1}{2} t \operatorname{arctg} t + \frac{1}{4} \log(1+t^2) + c_1 t + c_2$$

$$\begin{cases} x(0) = c_2 = 0 \\ x(1) = -\frac{1}{2}\frac{\pi}{4} + \frac{1}{4}\log 2 + c_1 = 0 \end{cases} \qquad \begin{matrix} c_2 = 0 \\ c_1 = \frac{\pi}{8} - \frac{1}{4}\log 2 \end{matrix}$$

$$\hat{x}(t) = -\frac{1}{2} t \operatorname{arctg} t + \frac{1}{4}\log(1+t^2) + \frac{\pi}{8} t - \frac{1}{4}(\log 2)\, t$$

$$\begin{pmatrix} f_{xx} & f_{x\dot{x}} \\ f_{x\dot{x}} & f_{\dot{x}\dot{x}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{semid. neg}$$

$\Rightarrow \hat{x}$ è massimante

Ex 6

$$f(x,y,z) = y^2 e^{x^2} + z e^{yz} + x^2 + 2y - 3x = 0 \qquad f \in C^1(\mathbb{R}^3)$$

$$f(0,0,0) = 0$$

$$f_z = e^{yz} + yz e^{yz} \Big|_{\substack{x=0\\y=0\\z=0}} = 1 \neq 0 \qquad \begin{matrix} f_x = 2xy^2 e^{x^2} + 2x - 3 \\ f_y = 2y e^{x^2} + z^2 e^{yz} + 2 \end{matrix}$$

$\Rightarrow \exists!\ g = g(x,y)$ funz. implicita

$$g_x(0,0) = -\frac{f_x(0,0,0)}{1} = 3 \ ; \quad g_y(0,0) = \frac{-f_y(0,0,0)}{1} = -2$$

π : $z = 0 + 3(x-0) - 2(y-0)$

$z = 3x - 2y$