④ Determinare punti critici e classificarli

$$f = \frac{1}{3}x^3 + 4xy + 2y^2 + 4x$$

$$\begin{cases} \partial_x f = x^2 + 4y + 4 = 0 \\ \partial_y f = 4x + 4y = 0 \end{cases} \qquad \begin{cases} x^2 - 4x + 4 = 0 \\ y = -x \end{cases} \qquad \begin{matrix} x = 2 \\ y = -2 \end{matrix} \quad P$$

$$Hf = \begin{pmatrix} 2x & 4 \\ 4 & 4 \end{pmatrix} \qquad Hf(P) = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

determ. hessiano $= 0$

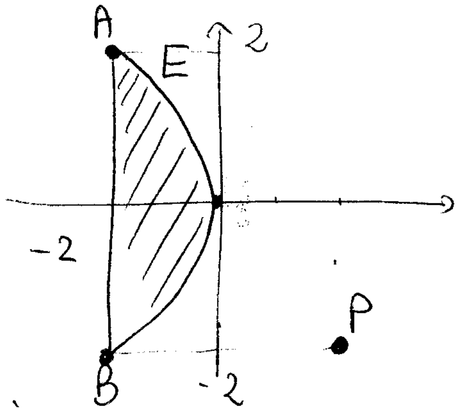
$$f(x,-x) = h(x) = \frac{1}{3}x^3 - 4x^2 + 2x^2 + 4x$$

$$h'(x) = x^2 - 4x + 4 = (x-2)^2 \geq 0 \quad \forall x$$

$\Rightarrow x = 2$ non è né di min né di max

$h''(x) = 2(x-2) \geq 0$ sse $x \geq 2$ $\quad$ $\underline{x = 2 \text{ pto di flesso}}$

Estremi in $E = \{(x,y) \mid -2 \leq x \leq -\frac{1}{2}y^2\}$



$\nexists$ pti critici interni

$$f(-2,y) = -\frac{8}{3} - 8y + 2y^2 - 8 \xrightarrow[\frac{d}{dy}]{} -8 + 4y = 0 \iff y = 2$$

$$\geq 0 \iff y \geq 2$$

sulla parabola:

$[-2, 2]$

$$-\frac{1}{3} \cdot \frac{1}{8}y^6 - 2y^3 + \cancel{2y^2} - \cancel{2y^2} = -\frac{1}{24}y^6 - 2y^3$$

$$\frac{d}{dy}: \quad -\frac{1}{4}y^5 - 6y^2 = -\frac{1}{4}y^2(y^3 + 24) \geq 0 \iff y \leq -\sqrt[3]{24} < -2$$

Restano individuati solo gli estremi A e B

$$f(-2,2) = -\frac{8}{3} - 16 \cancel{+8} \cancel{-8} \qquad \text{Min}$$

$$f(-2,-2) = -\frac{8}{3} + 16 \cancel{+8} \cancel{-8} \qquad \text{Max}$$

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3) Det. le estremali ammissibili del funzionale:

$$J(x) = \int_1^2 t^3 \dot{x}^2 dt$$

soggette al vincolo $\int_1^2 x\, dt = 2$ e alle cond. ai limiti

$x(1) = 4, \quad x(2) = 1$

$$\mathcal{L}(x) = \int_1^2 (t^3 \dot{x}^2 - \lambda x)\, dt$$

Il funzionale $\int_1^2 x\, dt$ non ha estremali e quindi possiamo procedere alla ricerca di estremali di $\mathcal{L}$.

$$f = t^3 \dot{x}^2 - \lambda x$$

$$f_x = -\lambda \; ; \quad f_{\dot{x}} = 2t^3 \dot{x} \; ; \quad \frac{d}{dt} f_{\dot{x}} = 6t^2 \dot{x} + 2t^3 \ddot{x}$$

Eq. di Eul - Lagr. per $\mathcal{L}$:

$$6t^2 \dot{x} + 2t^3 \ddot{x} + \lambda = 0 \longrightarrow 2t^2 \ddot{x} + 6t \dot{x} = -\frac{\lambda}{t}$$

eq. di Eulero non omog.

x la parte omogenea:

$$2\alpha(\alpha - 1) + 6\alpha = 0 \implies 2\alpha^2 + 4\alpha = 0 \quad \begin{cases} \alpha = 0 \\ \alpha = -2 \end{cases}$$

$c_1 + c_2 t^{-2}$ int. gen. omog. associata

Cerchiamo la soluz. particolare nella forma

$$\bar{x}(t) = \frac{k}{t} \qquad \dot{\bar{x}} = -\frac{k}{t^2}$$

$$\ddot{\bar{x}} = 2\frac{k}{t^3}$$

e sostituiamo

$$\frac{4k}{t} - \frac{6k}{t} = -\frac{\lambda}{t} \implies k = \frac{\lambda}{2}$$

Int. gen. dell' eq. di Eulero : $c_1 + c_2 t^{-2} + \frac{\lambda}{2t} = \hat{x}(t)$

Imponiamo

$$\begin{cases} \hat{x}(1) = 4 \\ \hat{x}(2) = 1 \end{cases} \quad \begin{cases} c_1 + c_2 + \frac{\lambda}{2} = 4 \\ c_1 + \frac{c_2}{4} + \frac{\lambda}{4} = 1 \end{cases} \quad \begin{cases} c_1 = -\frac{\lambda}{6} \\ c_2 = 4 - \frac{\lambda}{3} \end{cases}$$

$$\hat{x}(t) = -\frac{\lambda}{6} + \left(4 - \frac{\lambda}{3}\right) t^{-2} + \frac{\lambda}{2t}$$

Imponiamo l'ultimo vincolo

$$\int_1^2 \hat{x}(t)\,dt = 2 \iff -\frac{\lambda}{6} - \left(4 - \frac{\lambda}{3}\right) t^{-1} \Big|_1^2 + \frac{\lambda}{2} \log t \Big|_1^2 = 2$$

$$\iff -\frac{\lambda}{6} + \frac{1}{2}\left(4 - \frac{\lambda}{3}\right) + \frac{\lambda}{2}\log 2 = 2$$

$$-\frac{\lambda}{3} + \cancel{2} + \frac{\lambda}{2}\log 2 = \cancel{2} \qquad \boxed{\lambda = 0}$$

$$\hat{x}(t) = \frac{4}{t^2} \qquad t \in [1,2]$$

1 $f(x,y) = \begin{cases} \dfrac{x^3+x^2y+2y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

- f è cont. in $(0,0)$: si vede passando in coord. polari

- $f(x,0) = x$
  $f(0,y) = 2y$ $\Rightarrow$ $\nabla f(0,0) = (1,2)$

- $$\frac{\frac{x^3+x^2y+2y^3}{x^2+y^2} - x - 2y}{\sqrt{x^2+y^2}} = \frac{\cancel{x^3} + x^2y + \cancel{2y^3} - \cancel{x^3} - xy^2 - 2x^2y - \cancel{2y^3}}{(x^2+y^2)^{3/2}}$$

$$= \frac{-x^2y - xy^2}{(x^2+y^2)^{3/2}} = \frac{-\cancel{\rho^3}(\cos^2\theta \, \mathrm{sen}\theta + \cos\theta \, \mathrm{sen}^2\theta)}{\cancel{\rho^3}}$$

$\not\rightarrow 0$

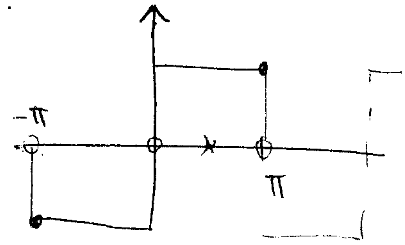
f non è diff.^{le} in $(0,0)$

$$f(t,t) = \frac{4t^3}{2t^2} = 2t \Rightarrow \frac{d}{dt} f(t,t)\Big|_{t=0} = \boxed{2 = \frac{\partial f}{\partial \vec{v}}(0,0)}$$

$\vec{v} = (1,1)$

5) $f : \mathbb{R} \to \mathbb{R}$ dispari, 2π - periodica t.c.

$$f(x) = \frac{\pi}{2} \qquad x \in (0, \pi]$$



$a_k = 0$

$$b_k = \frac{2}{\pi} \int_0^{\pi} \frac{\pi}{2} \operatorname{sen} kx \, dx = -\frac{\cos kx}{k} \Big|_0^{\pi} = \frac{1}{k} - \frac{(-1)^k}{k}$$

$$S(x) = \sum_{k=1}^{+\infty} \left(\frac{1}{k} - \frac{(-1)^k}{k} \right) \operatorname{sen} kx$$

$$\underset{k=2n+1}{=} \sum_{n=0}^{+\infty} \frac{2}{2n+1} \operatorname{sen} (2n+1)x$$

(*)

$$x = \frac{\pi}{2} \qquad \operatorname{sen}\left[(2n+1)\frac{\pi}{2}\right] = \operatorname{sen}\left(n\pi + \frac{\pi}{2}\right) = (-1)^n$$

$$S\left(\frac{\pi}{2}\right) = \sum_{n=0}^{+\infty} \frac{2(-1)^n}{2n+1} = \frac{\pi}{2}$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

(*)

$$S(x) = \begin{cases} \frac{\pi}{2} = f(x) & x \neq k\pi \\ 0 & x = k\pi \end{cases}$$

(2)

$f(x,y,z) = 2x^3 + y^3 + z^4 - xy - 2x = 0$

$f(1,0,0) = 0$ $\qquad\qquad f \in C^\infty(\mathbb{R}^3)$

$\partial_y f = 3y^2 - x$

$\partial_z f = 4z^3$

$\partial_x f = 6x^2 - y - 2 \quad ; \quad \partial_x f(1,0,0) = 4 \neq 0$

$\Rightarrow \quad \exists! \quad g = g(y,z) \quad$ t.c. $\quad f(g,y,z) = 0$

$g(0,0) = 1 \qquad f($

$$\partial_y g = \frac{-\partial_y f(g,y,z)}{\partial_x f(g,y,z)} \qquad\qquad H_g(0,0) = \begin{pmatrix} -\frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\partial_y g(0,0) = -\frac{\partial_y f(1,0,0)}{4} = +\frac{1}{4}$$

$$\partial_z g(0,0) = -\frac{\partial_z f(1,0,0)}{4} = 0$$

$$2g^3(y,z) + y^3 + z^4 - g(y,z)\,y - 2g(y,z) = 0$$

$$6g^2\partial_y g + 3y^2 - \partial_y g\, y - g - 2\partial_y g = 0$$

- $12g(\partial_y g)^2 + 6y - \partial_{yy} g\, y - \partial_y g - \partial_y g - 2\partial_{yy} g = 0$
  $+ 6g^2\partial_{yy} g$

  $12 - 2 + 4\partial_{yy} g(0,0) = 0 \Rightarrow \partial_{yy} g(0,0) = -\frac{5}{2}$

- $12g\,\partial_y g\,\partial_z g + 6g^2\partial_{yz} g - \partial_{yz} g\, y - \partial_z g - 2\partial_{yz} g = 0$

$$12\cdot 0 + 6\,\partial_{yz} g(0,0) - 2\,\partial_{yz} g(0,0) = 0 \Rightarrow \partial_{yz} g(0,0) = 0$$

$$6g^2\,\partial_z g + 4z^3 - \partial_z g\, y - 2\,\partial_z g = 0$$

$$12g(\partial_z g)^2 + 6g^2\,\partial_{zz} g + 12z^2 - \partial_{zz} g\, y - 2\,\partial_{zz} g = 0$$

$$6\partial_{zz} g(0,0) - 2\,\partial_{zz} g(0,0) = 0 \qquad \partial_{zz} g(0,0) = 0$$

6

$$Z(t) = \begin{pmatrix} 1 & -t \\ t & \frac{1}{t} \end{pmatrix}$$ risolve il sistema

$$Z'(t) = A(t)Z(t)$$

da cui $\quad A(t) = Z'(t)Z(t)^{-1}$

$$Z'(t) = \begin{pmatrix} 0 & -1 \\ 1 & -\frac{1}{t^2} \end{pmatrix} \quad ; \quad Z^{-1}(t) = \frac{1}{\frac{1}{t} + t^2} \begin{pmatrix} \frac{1}{t} & t \\ -t & 1 \end{pmatrix}$$

$$A(t) = \frac{t}{t^3+1} \begin{pmatrix} 0 & -1 \\ 1 & -\frac{1}{t^2} \end{pmatrix} \begin{pmatrix} \frac{1}{t} & t \\ -t & 1 \end{pmatrix} =$$

$$= \frac{t}{t^3+1} \begin{pmatrix} t & -1 \\ \frac{1}{t} + \frac{1}{t} & t - \frac{1}{t^2} \end{pmatrix} = \frac{t}{t^3+1} \begin{pmatrix} t & -1 \\ \frac{2}{t} & \frac{t^3-1}{t^2} \end{pmatrix}$$

$$= \frac{1}{t^3+1} \begin{pmatrix} t^2 & -t \\ 2 & \frac{t^3-1}{t} \end{pmatrix}$$

L' int. gen. del s. omog. è $\quad c_1\begin{pmatrix}1\\t\end{pmatrix}+c_2\begin{pmatrix}-t\\\frac{1}{t}\end{pmatrix}$

Una sol. part. si determina con la formula di variazione delle costanti

$$\bar{y}(t)=Z(t)\int_{t_0}^{t}Z^{-1}(s)f(s)\,ds \;=$$

$$Z^{-1}(s)f(s)=\frac{s}{s^3+1}\begin{pmatrix}\frac{1}{s} & s\\ -s & 1\end{pmatrix}\begin{pmatrix}0\\ s(s^3+1)\end{pmatrix}=$$

$$=s\begin{pmatrix}\frac{1}{s} & s\\ -s & 1\end{pmatrix}\begin{pmatrix}0\\ s\end{pmatrix}$$

$$=\begin{pmatrix}1 & s^2\\ -s^2 & s\end{pmatrix}\begin{pmatrix}0\\ s\end{pmatrix}=\begin{pmatrix}s^3\\ s^2\end{pmatrix}$$

$$=\begin{pmatrix}1 & -t\\ t & \frac{1}{t}\end{pmatrix}\int_1^t\begin{pmatrix}s^3\\ s^2\end{pmatrix}ds=\begin{pmatrix}1 & -t\\ t & \frac{1}{t}\end{pmatrix}\begin{pmatrix}\frac{t^4-1}{4}\\ \frac{t^3-1}{3}\end{pmatrix}=$$

$$=\begin{pmatrix}\frac{t^4-1}{4}-\frac{t(t^3-1)}{3}\\ t\frac{(t^4-1)}{4}+\frac{1}{t}\frac{t^3-1}{3}\end{pmatrix}=\begin{pmatrix}\frac{3t^4-3-4t^4+4t}{12}\\ \frac{3t^2(t^4-1)+4(t^3-1)}{12t}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{-t^4+4t-3}{12}\\ \frac{3t^6+4t^3-3t^2-4}{12t}\end{pmatrix}=\begin{pmatrix}-\frac{t^4}{12}+\frac{1}{3}t-\frac{1}{4}\\ \frac{1}{4}t^5+\frac{1}{3}t^2-\frac{1}{4}t-\frac{1}{3t}\end{pmatrix}\ \checkmark$$