Ex 1

Eq. caratt. dell' eq. omogenea : $t^2 - 2\lambda t + \lambda^2 = 0$

$$(t-\lambda)^2 = 0$$

$t = \lambda$ con molteplicità 2

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

$\lambda \neq -2$ $\quad -2$ non risolve l' eq. caratteristica

$$y_p(x) = A e^{-2x}$$

$$4A - 2\lambda(-2A) + \lambda^2 A = 1 \Rightarrow A = \frac{1}{\lambda^2 + 4\lambda + 4}$$

$\lambda = -2$ $\quad -2$ è sol. dell' eq. caratt.

$$y_p(x) = A x^2 e^{-2x}$$

$$y_p' = (2Ax - 2Ax^2) e^{-2x}$$

$$y_p'' = (2A + 4Ax^2 - 8Ax) e^{-2x}$$

$$2A + \cancel{4Ax^2} - \cancel{8Ax} + \cancel{8Ax} - \cancel{8Ax^2} + \cancel{4Ax^2} = 1$$

$$A = \frac{1}{2}$$

Integrale generale dell' eq. completa:

$$y(x) = \begin{cases} c_1 e^{\lambda x} + c_2 x e^{\lambda x} + \dfrac{e^{-2x}}{\lambda^2 + 4\lambda + 4} & \text{se } \lambda \neq -2 \\ c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2} x^2 e^{-2x} & \text{se } \lambda = -2 \end{cases}$$

Imponiamo le cond. ai limiti:

$\lambda \neq -2$
$$\begin{cases} y(0) = c_1 + \dfrac{1}{\lambda^2+4\lambda+4} = 0 \\[2ex] y'(1) = \lambda c_1 e^\lambda + c_2 e^\lambda + \lambda c_2 e^\lambda - \dfrac{2e^{-2}}{\lambda^2+4\lambda+4} = 0 \end{cases}$$

Matrice del sistema: $\begin{pmatrix} 1 & 0 \\ \lambda e^\lambda & e^\lambda + \lambda e^\lambda \end{pmatrix} = A_\lambda$

determinante $A_\lambda = e^\lambda(1+\lambda) = 0 \iff \lambda = -1$

Per $\lambda \neq -1$ esiste un'unica coppia $(\bar{c}_1, \bar{c}_2)$ soluzione del sistema:

$$\bar{c}_1 = -\frac{1}{\lambda^2+4\lambda+4}$$

$$\bar{c}_2 = \left\{ \frac{2e^{-2}}{\lambda^2+4\lambda+4} + \frac{\lambda e^\lambda}{\lambda^2+4\lambda+4} \right\} \cdot \frac{1}{\lambda e^\lambda + e^\lambda}$$

La soluzione del pb ai limiti è $y(x)$ con questi valori di c_1, c_2.

Se $\lambda = -1$ il sistema diventa $\begin{cases} c_1 + 1 = 0 \\ -c_1 e^{-1} - 2e^{-2} = 0 \end{cases}$ impossibile.

Se $\lambda = -2$
$$\begin{cases} y(0) = c_1 = 0 \\ y'(1) = -2c_1 e^{-2} + c_2 e^{-2} - 2c_2 e^{-2} + \cancel{e^{-2}} - \cancel{e^{-2}} = 0 \end{cases}$$
$$\begin{cases} c_1 = 0 \\ -c_2 e^{-2} = 0 \end{cases} \quad \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

Esiste un'unica soluzione $\frac{1}{2}x^2e^{-2x}$

$\boxed{\text{Ex 2}}$

$$y' + \frac{1}{x-3}y = \frac{1}{x+1}y^2$$

Il dato iniziale impone di scegliere $I = (-1, 3)$

$$z = \frac{1}{y} \qquad z' = -\frac{y'}{y^2}$$

$$-z' + \frac{1}{x-3}z = \frac{1}{x+1} \quad ; \quad z' - \frac{1}{x-3}z = -\frac{1}{x+1}$$

$$(3-x)\cdot\left(z\cdot\frac{1}{3-x}\right)' = -\frac{1}{x+1}$$

$$z' + \frac{1}{3-x}z = -\frac{1}{x+1}$$

$$\int\frac{1}{3-x}dx = -\log|3-x| = \log\frac{1}{3-x}$$

$$z\cdot\frac{1}{3-x} = -\int\frac{1}{(x+1)(3-x)} + c$$

$$\frac{1}{(x+1)(3-x)} = \frac{A}{x+1} + \frac{B}{3-x} = \frac{3A - Ax + Bx + B}{(x+1)(3-x)}$$

$$\begin{cases} 3A + B = 1 \\ -A + B = 0 \end{cases} \qquad \begin{cases} A = \frac{1}{4} \\ B = \frac{1}{4} \end{cases}$$

$$\frac{z(x)}{3-x} = -\frac{1}{4}\int\frac{dx}{x+1} - \frac{1}{4}\int\frac{dx}{3-x} + c$$

$$= -\frac{1}{4}\log(x+1) + \frac{1}{4}\log(3-x) + c$$

$$\frac{z(0)}{3} = \frac{\cancel{3}\log 3}{\cancel{3}\cdot 4} = +\frac{1}{4}\log 3 + c \implies c = 0$$

$$z(x) = \frac{1}{4}\left[\log\left(\frac{3-x}{x+1}\right)\right](3-x)$$

$$y(x) = \frac{4}{(3-x)\log\left(\frac{3-x}{x+1}\right)}$$

$x \in J = (-1, 1)$
(in quanto non si deve annullare il denominatore)

$\boxed{\text{Ex } 3}$

f è continua in $(0,0)$ (coord. polari)

$$f(x,0) = \frac{2x^4}{x^2} = 2x^2 \quad \Rightarrow \frac{d}{dx} f(x,0) = 4x \Big|_{x=0} = 0$$

$$f(0,y) = \frac{-y^3}{y^2} = -y \quad \Rightarrow \frac{d}{dy} f(0,y) = -1$$

$$\Rightarrow \nabla f(0,0) = (0,-1)$$

$$\frac{f(x,y)+y}{\sqrt{x^2+y^2}} = \frac{2x^4 \cancel{-y^3} \cancel{-x^2y} + \cancel{x^2y} + \cancel{y^3}}{(x^2+y^2)^{3/2}} = \frac{2x^4}{(x^2+y^2)^{3/2}} \xrightarrow[(x,y)\to(0,0)]{} 0$$

(coord. polari)

f è diff.le in $(0,0)$

$$\frac{\partial f}{\partial \vec{v}}(0,1) = \nabla f(0,1) \cdot \vec{v}$$

$$\frac{\partial f}{\partial x}(0,1) = \frac{d}{dx} f(x,1)\Big|_{x=0} = \frac{d}{dx}\left(\frac{2x^4-1-x^2}{x^2+1}\right)\Big|_{x=0} =$$

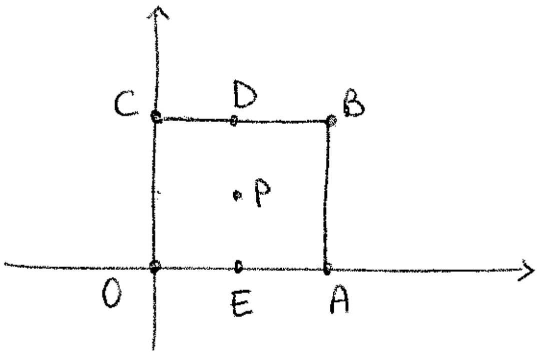
$$= \frac{(8x^3-2x)(x^2+1)-(2x^4-1-x^2)\cdot 2x}{(x^2+1)^2}\Big|_{x=0} = 0$$

$$\frac{\partial f}{\partial y}(0,1) = \frac{d}{dy} f(0,y)\Big|_{y=1} = -1$$

$$\frac{\partial f}{\partial \vec{v}}(0,1) = (0,-1)\cdot\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

EX 4

Unico punto stazionario interno $(1,1) = P$

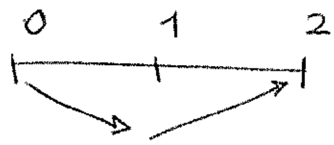


~~su $\overline{OA}$ $f(x,0)=0$~~

su $\overline{AB}$ $\quad f(2,y) = 0$

su $\overline{BC}$ $\quad f(x,2) = e\left(\frac{x^2}{2} - x\right) =: \varphi_1(x) \qquad x \in [0,2]$

$$\varphi_1' = e(x-1) \geq 0 \iff x \geq 1$$



$D(1,2)$

su $\overline{OC}$ $\quad f(0,y) = 0$

su $\overline{OA}$ $\quad f(x,0) = e\left(\frac{x^2}{2} - x\right) = \varphi_1(x)$

A questo punto:

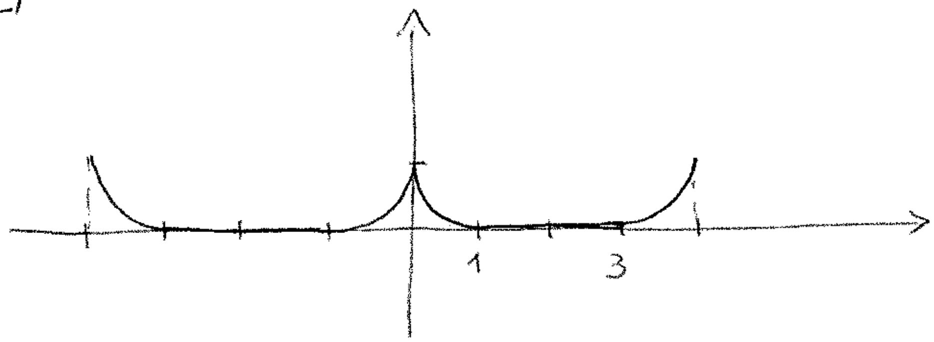
$f(1,1) = -\frac{1}{2}$ $\qquad f(D) = -\frac{e}{2}$

$f(0,0) = f(A) = f(B) = f(C) = 0$ $\qquad f(E) = -\frac{e}{2}$

E, D pti di minimo assoluto

tutti i punti dei segmenti $\overline{OC}$ e $\overline{AB}$ sono di max assoluto

$\boxed{\text{Ex 5}}$



f è pari $\Rightarrow$ $b_k = 0 \quad \forall k$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2\pi k x}{T}\right)dx$$

$$(T=4) \quad = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{\pi k x}{2}\right)dx \underset{f \text{ pari}}{=} \int_0^2 f(x)\cos\left(\frac{\pi k x}{2}\right)dx$$

$$= \int_0^1 (x-1)^2\cos\left(\frac{\pi k x}{2}\right)dx =$$

$$= \left[(x-1)^2 \operatorname{sen}\left(\frac{\pi k x}{2}\right)\cdot\frac{2}{k\pi}\right]_0^1 - \int_0^1 2(x-1)\operatorname{sen}\left(\frac{\pi k x}{2}\right)\frac{2}{k\pi}dx$$

$$= -\frac{4}{k\pi}\int_0^1 (x-1)\operatorname{sen}\left(\frac{\pi k x}{2}\right)dx$$

$$= \frac{4}{k\pi}\left[\left[(x-1)\cos\left(\frac{\pi k x}{2}\right)\cdot\frac{2}{k\pi}\right]_0^1 - \int_0^1 \cos\left(\frac{\pi k x}{2}\right)\cdot\frac{2}{k\pi}dx\right]$$

$$= \frac{8}{k^2\pi^2}\left(0+1\right) - \frac{8}{k^2\pi^2}\left[\operatorname{sen}\left(\frac{\pi k x}{2}\right)\cdot\frac{2}{k\pi}\right]_0^1 =$$

$$= \frac{8}{k^2\pi^2} - \frac{16}{k^3\pi^3} \operatorname{sen}\left(\frac{\pi k}{2}\right)$$

Se $k=0$ $\qquad a_0 = \int_0^1 (x-1)^2\,dx = \frac{(x-1)^3}{3}\Big|_0^1 = \frac{1}{3}$

La serie di Fourier di f è:

$$\frac{1}{6} + \sum_{k=1}^{+\infty} \left[\frac{8}{k^2\pi^2} - \frac{16}{k^3\pi^3} \operatorname{sen}\left(\frac{k\pi}{2}\right)\right] \cos\left(\frac{\pi k x}{2}\right)$$

Siccome f è regolare a tratti e continua in $\mathbb{R}$ la serie di Fourier converge puntualmente a f in ogni punto.

Inoltre $\qquad |b_k| \leq \frac{8}{k^2\pi^2} + \frac{16}{k^3\pi^3}$

e siccome $\sum \frac{1}{k^2}$ e $\sum \frac{1}{k^3}$ sono serie numeriche convergenti, si ottiene che la serie converge totalmente a f in $\mathbb{R}$.

Infine, per $x=0$

$$f(0) = 1 = \frac{1}{6} + \sum_{k=1}^{+\infty} \frac{8}{k^2\pi^2} - \frac{16}{\pi^3} \sum_{k=1}^{+\infty} \frac{\operatorname{sen}\left(\frac{k\pi}{2}\right)}{k^3}$$

$$\operatorname{sen}\left(\frac{k\pi}{2}\right) = \begin{cases} 0 & k=2n \\ (-1)^n & k=2n+1 \end{cases}$$

$$1 = \frac{1}{6} + \frac{\cancel{8}^{\,4}}{\cancel{\pi^2}} \cdot \frac{\cancel{\pi^2}}{\cancel{6}_{\,3}} - \frac{16}{\pi^3} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3}$$

$$\Rightarrow \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

Ex 6

Dall' eq. si ricava $y=0$

$$f(1,0)=0 \quad e \quad \partial_y f = 2xy+y^2+1+2y^2 \Big|_{\substack{x=1\\y=0}} = 1 \neq 0$$

Per il T. di Dini, esiste un' unica funz. $y(x)$ definita impl. dall' eq. $f(x,y)=0$

in un intorno di $x=1$ con $y(1)=0$ e

$$y'(x) = -\frac{\partial_x f(x,y(x))}{\partial_y f(x,y(x))} \qquad (*)$$

In particolare $y'(0) = -\frac{\cancel{0}\,1}{1} = \cancel{0}\,-1$

$$\lim_{x\to 1} \frac{y(x)+x-1}{(x-1)^2} = \left[\frac{0}{0}\right]$$

De L'Hopital: $\frac{y'(x)+1}{2(x-1)} \longrightarrow \frac{-1+1}{0} = \left[\frac{0}{0}\right]$

Bisogna applicare ancora de L'Hopital

~~xxx~~ Derivo $(*)$ rispetto a x:

$$y''(x) = -\frac{\left[\partial_{xx} f(x,y(x)) + \partial_{xy} f(x,y(x))\,y'(x)\right]\partial_y f(x,y(x))}{\left[\partial_y f(x,y(x))\right]^2} \cdots$$

$$\cdots \frac{-\partial_x f(x,y(x))\left[\partial_{xy} f(x,y(x)) + \partial_{yy} f(x,y(x))\,y'(x)\right]}{}$$

Per $x=\cancel{0}\,1$ ho $y(1)=0$ e $y'(1)=-1$ e :

$$y''(1) = -\frac{2\cdot 1 - 1\,(0 + 2(-1))}{1} = -\,2+2 = -4$$

$$\partial_x f = 2x + (y^2-1) \qquad \partial_y f(1,0) = 1$$

$$\partial_{xx} f = 2 \qquad \partial_{yy} f = 2x + 2y + 4y$$

$$\partial_{xy} f = 2y$$

$$\partial_{xy} f(1,0) = 0 \qquad \partial_x f(1,0) = 1$$

$$\partial_{xx} f(1,0) = 2 \qquad \partial_y f(1,0) = 1$$

$$\partial_{yy} f(1,0) = 2$$

Tornando al limite :

De L'Hopital $\qquad \dfrac{y''(x)}{2} \longrightarrow \dfrac{-4}{2} = -2$

è anche il valore del primo limite.