Ex 1

$$x^3 y''' - 3xy' + 3y = x\log^2 x$$

Eq. omogenea:

$$\alpha(\alpha-1)(\alpha-2) - 3\alpha + 3 = 0$$
$$\alpha(\alpha-1)(\alpha-2) - 3(\alpha-1) = 0$$
$$(\alpha-1)(\alpha^2-2\alpha-3) = 0$$

$$\alpha_1 = 1, \quad \alpha_{2/3} = 1 \pm \sqrt{1+3} = 3 / -1$$

Int. gen. omogenea $\quad y(x) = c_1 x + c_2 x^3 + \frac{c_3}{x}$

Sol. particolare $\quad y_p(x) = x\log x\left(A\log^2 x + B\log x + C\right)$

$$= x\left(A\log^3 x + B\log^2 x + C\log x\right)$$

$$y_p' = A\log^3 x + B\log^2 x + C\log x + \cancel{x}\left(\frac{3A\log^2 x}{\cancel{x}} + \frac{2B\log x}{\cancel{x}} + \frac{C}{\cancel{x}}\right)$$

$$= A\log^3 x + (B+3A)\log^2 x + (C+2B)\log x + C$$

$$y_p'' = \frac{3A\log^2 x}{x} + \frac{2(B+3A)\log x}{x} + \frac{C+2B}{x}$$

$$y_p''' = \frac{6A\log x}{x^2} - \frac{3A\log^2 x}{x^2} + \frac{2(B+3A)}{x^2} - \frac{2(B+3A)\log x}{x^2} - \frac{C+2B}{x^2}$$

Sostituendo nell'eq. diff. completa:

$$6A\cancel{x\log x} - 3Ax\log^2 x + \cancel{2Bx} + 6Ax - 2Bx\log x - \cancel{6Ax\log x}$$
$$-Cx - \cancel{2Bx} - \cancel{3Ax\log^3 x} - \cancel{3Bx\log^2 x} - 9Ax\log^2 x$$
$$-\cancel{3Cx\log x} - 2Bx\log x - 3Cx + \cancel{3Ax\log^3 x}$$
$$+\cancel{3Bx\log^2 x} + \cancel{3Cx\log x} = x\log^2 x$$

$$\begin{cases} -12A = 1 \\ -4B = 0 \\ 6A - 4C = 0 \end{cases} \qquad \begin{cases} A = -\frac{1}{12} \\ B = 0 \\ C = \frac{3}{2}A = -\frac{1}{8} \end{cases}$$

Int. eq. completa:

$$y(x) = c_1 x + c_2 x^3 + \frac{c_3}{x} - \frac{1}{12} x\log^3 x - \frac{1}{8} x\log x$$
$$y'(x) = c_1 + 3c_2 x^2 - \frac{c_3}{x^2} - \frac{1}{12}\log^3 x - \frac{1}{4}\cancel{x}\frac{\log^2 x}{\cancel{x}} - \frac{1}{8}\log x - \frac{1}{8}$$
$$y''(x) = 6c_2 x + \frac{2c_3}{x^3} - \frac{1}{4}\frac{\log^2 x}{x} - \frac{1}{2}\frac{\log x}{x} - \frac{1}{8x}$$

$$\begin{cases} y(1) = c_1 + c_2 + c_3 = 0 \\ y'(1) = c_1 + 3c_2 - c_3 - \frac{1}{8} = 0 \\ y''(1) = 6c_2 + 2c_3 - \frac{1}{8} = 0 \end{cases} \qquad \begin{cases} c_1 + c_2 = -c_3 \\ 2c_1 + 4c_2 = \frac{1}{8} \\ 3c_2 + c_3 = \frac{1}{16} \end{cases}$$

$$\begin{cases} c_1 = -c_2 - \frac{1}{16} + 3c_2 \qquad c_1 = \frac{1}{16} - \frac{1}{16} = 0 \\ 4c_2 - \frac{1}{8} + 4c_2 = \frac{1}{8} \rightarrow c_2 = \frac{1}{32} \\ c_3 = \frac{1}{16} - 3c_2 = \frac{2-3}{32} = -\frac{1}{32} \end{cases}$$

Ex 2

$$\begin{cases} x' = -x + y - z \\ y' = 2x + y \\ z' = 3x + z \end{cases}$$

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

autovalori di A: $\begin{vmatrix} -1-\lambda & 1 & -1 \\ 2 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} =$

$$= (-1-\lambda)(1-\lambda)^2 - 2(1-\lambda) + 3(1-\lambda) =$$

$$= (1-\lambda)\left[(-1-\lambda)(1-\lambda) + 1\right] = (1-\lambda)(-1+\lambda^2+1) = 0$$

$\lambda = 1$ e $\lambda = 0$ con molt. 2

Autovettore rel. a $\lambda = 1$

$$\begin{pmatrix} -2 & 1 & -1 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -2x + y - z = 0 \\ x = 0 \\ x = 0 \end{cases}$$

$\vec{v}_1 = (0, 1, 1)$

Autov. relativo a $\lambda=0$

$$\begin{pmatrix} -1 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x+y-z=0 & \longrightarrow 0=0 \\ 2x+y=0 & \rightarrow y=-2x \\ 3x+z=0 & \rightarrow z=-3x \end{cases}$$

$$\vec{v}_2 = (1,-2,-3)$$

Autov. generalizzato:

$$\begin{pmatrix} -1 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

$$\begin{cases} -x+y-z=1 & 0=0 \\ 2x+y=-2 & y=-2-2x \\ 3x+z=-3 & z=-3-3x \end{cases}$$

$$\vec{v}_2^{\,1} = (0,-2,-3)$$

Soluzioni del sistema:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + c_3 \begin{pmatrix} t \\ -2t-2 \\ -3t-3 \end{pmatrix}$$

EX 3

$f = x^2 + y^2 - 3xy$

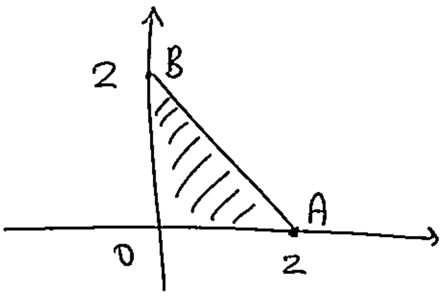
$\nabla f = \vec{0} \iff (x,y) = (0,0)$

$Hf(0,0)$ indefinita $\Rightarrow$ $(0,0)$ pto di sella

$\Rightarrow$ f non ha estremi liberi in $\mathbb{R}^2$

In T f ammette estremi assoluti, per il Teorema di Weierstrass.

Non ci sono pti critici interni a T.



su $\overline{OB}$ $\quad f(0,y) = y^2, \quad y \in [0,2]$

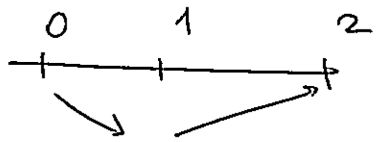
è sempre crescente

su $\overline{OA}$ $\quad f(x,0) = x^2, \quad x \in [0,2]$

è sempre crescente

su $\overline{AB}$ $\quad h(x) = f(x, 2-x) = 5x^2 - 10x + 4, \quad x \in [0,2]$

$h'(x) = 10x - 10 \geq 0 \iff x \geq 1$



Occorre testare la funzione nei vertici del triangolo e in $E(1,1)$.

$f(0,0) = 0$
$f(0,2) = f(2,0) = 4$
$f(1,1) = -1$

$\max_T f = 4 \qquad \min_T f = -1$

Ex 4

$$f(t,x,\dot{x}) = (4x - x^2 - \dot{x}^2 + 2\dot{x})e^{-t}$$

$$f_x = (4-2x)e^{-t} \qquad f_{\dot{x}} = (2-2\dot{x})e^{-t}$$

$$f_x - \frac{d}{dt} f_{\dot{x}} = 0 \iff \ddot{x} - \dot{x} - x = -3$$

$$\lambda^2 - \lambda - 1 = 0 \qquad \lambda_{1/2} = \frac{1 \pm \sqrt{5}}{2}$$

$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ int. gen. eq. omogenea

$x_p(t) = A \rightsquigarrow x_p(t) = 3$

$\hat{x}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + 3$ int. gen. eq. completa

Cond. di trasversalità:

$$f_{\dot{x}}\left(0,\ \hat{x}(0),\ \dot{\hat{x}}(0)\right)=0 \iff \dot{\hat{x}}(0)=1 \iff c_1\lambda_1+c_2\lambda_2=1$$

$$\lim_{t\to+\infty} \underbrace{f_{\dot{x}}\left(t,\ \hat{x}(t),\ \dot{\hat{x}}(t)\right)}_{\shortparallel}=0$$

$$2\left(1-c_1\lambda_1e^{\lambda_1 t}-c_2\lambda_2e^{\lambda_2 t}\right)e^{-t}=$$

$$=2\left(e^{-t}-c_1\lambda_1e^{(\lambda_1-1)t}-c_2\lambda_2e^{(\lambda_2-1)t}\right) \quad (*)$$

$$\lambda_1=\frac{1+\sqrt5}{2}>1 \quad \Rightarrow \quad \lambda_1-1>0$$

$$\lambda_2=\frac{1-\sqrt5}{2}<1 \quad \Rightarrow \quad \lambda_2-1<0$$

L'espressione $(*)$ è infinitesima per $t\to+\infty$ se e solo se $c_1=0$

Quindi $c_2\lambda_2=1 \Rightarrow c_2=\frac{1}{\lambda_2}=\frac{2(1+\sqrt5)}{(1-\sqrt5)(1+\sqrt5)}$

$$=-\frac{1+\sqrt5}{2}$$

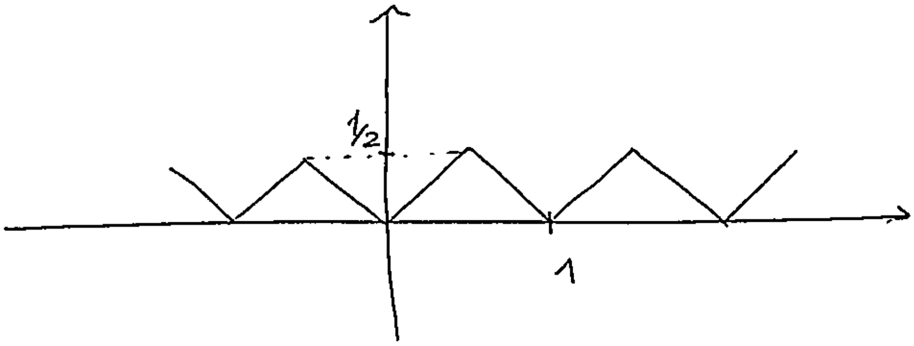
L'unica estremale è

$$\hat{x}(t)=-\frac{1+\sqrt5}{2}e^{\frac{1-\sqrt5}{2}t}+3$$

$$Hf_{x\dot{x}}=\begin{pmatrix}-2e^{-t} & 0\\ 0 & -2e^{-t}\end{pmatrix} \quad \text{def. negativa}$$

$\Rightarrow \hat{x}$ è massimante

5)



f è pari $\Rightarrow$ $b_k = 0$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2\pi}{T}kx\right)dx \qquad k = 0, 1, \dots$$

$T = 1$

$$= 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\cos(2\pi kx)\,dx$$

$$= 4\int_0^{\frac{1}{2}} x\cos(2\pi kx)\,dx$$

$$a_0 = 4\int_0^{\frac{1}{2}} x\,dx = \frac{1}{2}$$

$k \geq 1$

$$a_k = 4\left[x\,\frac{\mathrm{sen}(2\pi kx)}{2\pi k}\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\frac{\mathrm{sen}(2\pi kx)}{2\pi k}dx\right]$$

$$= \frac{\cancel{4}}{\cancel{2}\pi k}\left[\frac{\cos(2\pi kx)}{\cancel{2}\pi k}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{\pi^2 k^2}\left[(-1)^k - 1\right]$$

Serie di Fourier di f:

$$\frac{1}{4} + \sum_{k=1}^{+\infty}\frac{1}{\pi^2 k^2}\left[(-1)^k - 1\right]\cos(2\pi kx)$$

Siccome f è regolare a tratti e continua in $\mathbb{R}$ la serie di Fourier converge a $f(x)$, $\forall x \in \mathbb{R}$.

Per la conv. totale, basta osservare che

$$|a_k| \leq \frac{2}{\pi^2 k^2}$$

e quindi c'è convergenza totale in $\mathbb{R}$.

Infine, applicando l'uguaglianza di Parseval:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(x)\, dx = \frac{T}{2} \left\{ \frac{a_0^2}{2} + \sum_{k=1}^{+\infty} a_k^2 \right\}$$

$$a_k = \begin{cases} 0 & k = 2n \\ \dfrac{-2}{\pi^2 (2n+1)^2} & k = 2n+1 \end{cases}$$

Quindi

$$2 \int_0^{\frac{1}{2}} x^2 dx = \frac{1}{2} \left\{ \frac{1}{8} + \sum_{n=0}^{+\infty} \frac{4}{\pi^4 (2n+1)^4} \right\}$$

da cui $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$

6) $$f(\rho\cos\theta, \rho\sin\theta) = \rho^2 \cos\theta \operatorname{sen}\theta \, e^{\cos\theta \operatorname{sen}\theta}$$

$$|f(\rho\cos\theta, \rho\sin\theta)| \leq \rho^2 e \xrightarrow[\rho\to 0]{} 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$ f continua in $(0,0)$

$$f(x,0) = 0 = f(0,y) \Rightarrow \nabla f(0,0) = (0,0)$$

$$\frac{f(x,y) - f(0,0) - \nabla f(0,0)\cdot(x,y)}{\sqrt{x^2+y^2}} = \frac{f(x,y)}{\sqrt{x^2+y^2}}$$

$$= \rho\cos\theta \sin\theta \, e^{\cos\theta\sin\theta} =: (*)$$

in coord. polari

Prendendo il valore assoluto:

$$|(*)| \leq \rho e \xrightarrow[\rho\to 0]{} 0$$

dunque f è differenziabile in $(0,0)$.

$$\frac{\partial f}{\partial \vec{v}}(1,0) = \nabla f(1,0)\cdot\vec{v}$$

f è differenziabile in $(1,0)$ perché è di classe C^1 in un intorno di $(1,0)$

$$\nabla f(1,0) = (0,1) \Rightarrow \frac{\partial f}{\partial \vec{v}}(1,0) = v_2$$