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Moduli of curves and theta-characteristics

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Introduction. This paper explains how to compactify the moduli space of pairs

(smooth genus g complex curve C, theta-characteristic on C)

and how to define natural classes in the Picard group of the compactified moduli space, as Mumford has done for the moduli space of curves [6]. Although I am convinced that most, if not all, of the facts I am going to present are known, in one form or another, still I believe it is worthwhile to give a unified account of them.

As will be apparent, once one hits upon the "right" generalization of the notion of theta-characteristic to the case of singular curves, the theory is quite simple and not really different, in any substantial way, from the theory of standard moduli spaces of curves.

During the preparation of this paper, I have been greatly stimulated by conversations with Roberto Catenacci, Maurizio Martellini, and Cesare Reina: to all of them go my sincere thanks.

1. Automorphisms of semi-stable curves. Let C be a compact connected noded curve. We shall say that a component E of C is *exceptional* if it is smooth and rational and meets the rest of C in at most two points: we will refer to these as the *endpoints* of E. We shall say that C is *decent* if it is semistable, of genus greater than one and, moreover, any two distinct exceptional components of C are disjoint. In the same way, one can also speak of decent 1-pointed curves of genus one: the only precaution to be taken is that a component containing the marked point should not be considered exceptional. In the sequel, we shall usually deal only with decent

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curves of genus greater that one, leaving it to the reader to work out the genus one case.

Consider a proper flat family of noded curves $\pi:\mathcal{C}\to T$. Suppose all fibers of π are decent. We shall say that $\pi:\mathcal{C}\to T$ is a family of decent curves over T, or, for brevity, a *decent curve over* T if, for any $t\in T$ and any exceptional component E of $\pi^{-1}(t)$, there is a neighbourhood of E which is the pullback of the blow-up at the origin of the surface $\{xy=z^2\}$ via $z=f$ for some function f defined on a neighbourhood of t in T.

Let $\pi:\mathcal{C}\to T$ be a decent curve over T. An automorphism of $\mathcal{C}$ over T is said to be *inessential* if it induces the identity on the stable model of $\pi:\mathcal{C}\to T$. Inessential automorphisms are easy to describe: more precisely, we shall describe the sheaf of inessential automorphisms, that is, the sheaf of groups on T whose sections over an open set U are the inessential automorphisms of $\pi^{-1}(U)$. Pick a point $t\in T$, set $C=\pi^{-1}(t)$, and let A be the local ring of t. Let E be an exceptional component of C, and let p,q be its endpoints. There is a function f on a neighbourhood of t such that $\mathcal{C}$ is, locally, of the following form:

$$xy = f \quad \text{near } p \quad , \qquad \xi\eta = f \quad \text{near } q \, ,$$

where x and ξ are local equations of E and $y\eta=1$. Let ϑ be an automorphism of $\mathcal{C}$. If ϑ is inessential,

$$\vartheta^*(x) = x \quad , \qquad \vartheta^*(\xi) = \xi \, .$$

On the other hand, any function in a neighbourhood of p can be written <u>uniquely</u> as a power series

$$\sum_{i\geq 0} a_i x^i + \sum_{i>0} b_i y^i \, ,$$

where a_i, b_i are functions on a neighbourhood of t. We may then write

$$\vartheta^*(y) = y + \sum_{i\geq 0} a_i x^i + \sum_{i>0} b_i y^i \, ,$$

where $1+b_1$ is a unit in A. Since $xy=\vartheta^*(x)\vartheta^*(y)=x\vartheta^*(y)$, we find that

$$0 = x\cdot\Big(\sum_{i\geq 0} a_i x^i + \sum_{i>0} b_i y^i \Big) = \sum_{i\geq 0} a_i x^{i+1} + \sum_{i>0} f b_i y^{i-1} \, ,$$

so

$$\vartheta^*(y) = y + \sum_{i>0} b_i y^i \quad , \quad f b_i = 0 .$$

Similarly,

$$\vartheta^*(\eta) = \eta + \sum_{i>0} c_i \eta^i \quad , \quad f c_i = 0 .$$

Since, however, $\vartheta^*(y)\cdot\vartheta^*(\eta)=1$, we must have that

$$(1+b_1)\cdot(1+c_1) = 1 \quad , \quad b_i = c_i = 0 \text{ for } i>1 .$$

In conclusion, a local espression for ϑ near E is

$$\vartheta^*(x)=x \;\; ; \;\; \vartheta^*(\xi)=\xi \;\; ; \;\; \vartheta^*(y)=u\cdot y \;\; ; \;\; \vartheta^*(\eta)=u^{-1}\cdot\eta ,$$

where u is a unit in A such that $f\cdot(u-1)=0$. Repeating this for all the exceptional components $E_1,\cdots,E_n$ of C, and writing $\mathcal{C}$, in a neighbourhood of E_i, as the pullback of the blow-up at the origin of $\{xy=z^2\}$ via some function f_i, we conclude that the stalk at t of the sheaf of inessential automophisms of $\mathcal{C}$ is isomorphic to $\prod A_i$, where

$$A_i = \{u \in A^\times \,|\, f_i\cdot(u-1)=0\} .$$

Notice that the isomorphism depends on the choice of an orientation on each E_i, that is, of an ordering between the two endpoints of E_i. Changing the ordering on the i-th exceptional component corresponds to passing to the reciprocal in the i-th factor of $\prod A_i$. For each element $\underline{k}=(k_1,\cdots,k_n)$ of $\prod A_i$ we shall denote by $\sigma_{\underline{k}}$ the corresponding germ of inessential automorphism.

On several occasions, we shall need the following simple remark.

(1.1) Lemma. *Let* $\pi:\mathcal{C}\to T$ *be a proper flat family of noded curves. Let* t *be a point of* T *and* E *an exceptional component of* $\pi^{-1}(t)$. *If* L *is a line bundle on* $\mathcal{C}$ *whose restriction to* E *is trivial, then* L *is trivial on a neighbourhood of* E.

Proof. Shrinking T, if necessary, we may find a union D of sections of π touching all components of $\pi^{-1}(t)$ except E. If m is a large enough integer, $L(mD)_{|\pi^{-1}(t)}$ is generated by global sections and its first cohomology group vanishes. Thus, after shrinking T, if necessary, we conclude that $R^1\pi_*L(mD)$ vanishes and $L(mD)$ is generated by global sections. In particular, $L(mD)$ is trivial on a neighbourhood of E, and so is L.

We now return to our family $\pi:\mathcal{C}\to T$ of decent curves. As before, we let t be a point of T, A its local ring, and set $C=\pi^{-1}(t)$. We denote by $E_1,\cdots,E_n$ the exceptional components of C and, for each i, we let p_i,q_i be the endpoints of E_i. Set $E_i^*=E_i-\{p_i,q_i\}$, and denote by $\tilde{C}$ the curve obtained from C by removing all the E_i^*. For each i, we let D_i (resp., D_i') be the connected component of $\tilde{C}$ that contains p_i (resp., q_i). We wish to attach to every n-tuple $\underline{k}=(k_1,\cdots,k_n)$ of elements of $A^\times$ a line bundle $M_{\underline{k}}$ on a neigbourhood of C; strictly speaking, $M_{\underline{k}}$ will be an element of the stalk at t of $R^1\pi_*\mathcal{O}_{\mathcal{C}}^\times$, that is, a germ of line bundle around C. Accordingly, we shall feel free to shrink T when necessary without mentioning it. For each connected component D of $\tilde{C}$, choose an open neighbourhood C_D. For each i, let V_i be a neighbourhood of p_i not containing q_i, and W_i a neighbourhood of q_i not containing p_i. We may set things up so that, for each i, V_i and W_i cover E_i, and moreover $V_i\cap C_{D_i}$ and $W_i\cap C_{D_i'}$ are disjoint. In addition, we may suppose that $V_i\cup W_i$ is disjoint from $V_j\cup W_j$ whenever $i\neq j$, and that the V_i, the W_i, and the C_D cover T. For each D, let U_D be the variety obtained by attaching to C_D a copy of V_i for each i such that $D=D_i$ and a copy of W_i for each i such that $D=D_i'$. The line bundle $M_{\underline{k}}$ is then obtained from the trivial bundle on $\coprod U_D$ by identifying the unit section on W_i to k_i times the unit section on V_i. Then $M_{\underline{k}}$ is trivial if and only if there are elements $h_D\in A^\times$, where D runs through the connected components of $\tilde{C}$, such that, for each i, $k_i=h_{D_i}/h_{D_i'}$. All this can be rephrased as follows. Let Γ be the graph whose vertices are the connected components of $\tilde{C}$ and whose edges are the exceptional components of C. After choosing an orientation on the edges of Γ (for example, we may stipulate that q_i precedes p_i for every i) an n-tuple $\underline{k}$ of elements of $A^\times$ may be viewed as an $A^\times$-valued 1-cochain on Γ: the corresponding line bundle is trivial if and only if $\underline{k}$ is a coboundary. In other words, the $M_{\underline{k}}$ are classified, up to isomorphism, by $H^1(\Gamma,A^\times)$. Notice, in addition, that the $M_{\underline{k}}$ are precisely the germs of those line bundles which are trivial on a neighbourhood of $\tilde{C}$ and on the exceptional components of C.

Let L be a germ of line bundle around C and $\sigma_{\underline{k}}$ a germ of inessential automorphism at t; denote by m_i be the degree of L on E_i. A simple computation using Lemma (1.1) shows that $\sigma_{\underline{k}}{}^*L$ is isomorphic to $L\otimes M_{\underline{k}^{\underline{m}}}$, where $\underline{k}^{\underline{m}}$ stands for the multi-index $(k_1{}^{m_1},\cdots,k_n{}^{m_n})$. A useful consequence of this is that, given two line bundles M and M' on a decent curve C with the same restrictions to $\tilde{C}$ and having the same nonzero degree on every exceptional component, there is an inessential automorphism σ of C such that $M' = \sigma^* M$. Another consequence is that L is isomorphic to $\sigma_{\underline{k}}{}^* L$ if and only if $\underline{k}^m$ is a coboundary.

Now let L be as above, and suppose that $\underline{k}$ is a coboundary, that is, that for each connected component D of $\tilde{C}$ there is $h_D \in A^\times$ such that, for each i, $k_i = h_{D_i}/h_{D_i'}$. Assume moreover that L has the same degree m on all exceptional components of C. We know that there is an isomorphism α between $\sigma_{\underline{k}}{}^* L$ and L. Up to multiplicative constants, in a neighbourhood of a general point of any component D of $\tilde{C}$, α is multiplication by $h_D{}^m$. By this we mean that, if s is a local generator of L, α maps $\sigma_{\underline{k}}{}^* s$ to $h_D{}^m \cdot s$.

2. Spin curves. Let C be a decent curve of genus $g \geq 2$. A *spin structure* on C is the datum of a line bundle ζ of degree $g-1$ on C and a homomorphism $\alpha : \zeta^2 \to \omega_C$, satisfying the following conditions:

i) ζ has degree 1 on every exceptional component of C,
ii) α is not zero at a general point of every non-exceptional component of C.

Condition i) forces α to vanish identically on exceptional components. Thus, if $\tilde{C}$ stands for the curve obtained from C by removing the exceptional components, α induces a homomorphism $\tilde{\alpha} : \zeta_{|\tilde{C}}{}^2 \to \omega_{\tilde{C}}$. Again by condition i), the degree of $\zeta_{|\tilde{C}}{}^2$ equals $2g-2-2N$, where N is the number of exceptional components of C. The degree of $\omega_{\tilde{C}}$ also equals $2g-2-2N$. By condition ii), $\tilde{\alpha}$ must be an isomorphism, so $\zeta_{|\tilde{C}}$ is a square root of $\omega_{\tilde{C}}$. In particular, when C is smooth, ζ is just a theta-characteristic on C.

A *spin curve* of genus $g \geq 2$ is a triple $X = (C, \zeta_X, \alpha_X)$ where C is a decent curve of genus g and (ζ_X, α_X) is a spin structure on C. We shall often write ω_X, $\mathcal{O}_X$, etc., to denote ω_C, $\mathcal{O}_C$, and so on; conversely, when no confusion is likely, we shall sometimes refer to C itself as a spin curve and write ζ_C, α_C instead of ζ_X, α_X. The *graph* Γ_X of the spin curve X is the graph we have associated to C in section 1, i. e., the graph whose vertices are the connected components of $\tilde{C}$ and whose edges are the exceptional components of C.

One may also speak of spin curves of genus one: these are just decent 1-pointed curves of genus one with a spin structure on. In the sequel, we shall usually restrict to spin curves of genus greater that one, leaving it to the reader to work out the genus one case.

Let X, X' be two spin curves of genus g and let C, C' be the underlying decent curves. An *isomorphism* between X and X' is an isomorphism

$\sigma: C \to C'$ such that there is an isomorphism between $\sigma^*\zeta_{X'}$ and ζ_X that is compatible with the canonical isomorphism between $\sigma^*\omega_{X'}$ and ω_X; notice that the isomorphism between $\sigma^*\zeta_{X'}$ and ζ_X is determined up to sign. We shall denote by $\mathrm{Aut}(X)$ the group of automorphisms of X, by $\mathrm{Aut}_0(X)$ the subgroup of inessential automorphisms, and by $\mathrm{Aut}_1(X)$ the quotient of $\mathrm{Aut}(X)$ by $\mathrm{Aut}_0(X)$, that is, the image of $\mathrm{Aut}(X)$ in the group of automorphisms of the stable model of C.

We will now determine $\mathrm{Aut}_0(X)$; in particular, we shall see that it is a finite group, so $\mathrm{Aut}(X)$ is finite, too. We use the same notations employed in section 1 to describe the inessential automorphisms of C.

(2.1) Lemma. *Let $X = (C, \zeta_X, \alpha_X)$ and $X' = (C, \zeta_{X'}, \alpha_{X'})$ be two spin curves with the same underlying decent curve. Assume that the restrictions of ζ_X and $\zeta_{X'}$ to $\tilde{C}$ are isomorphic. Then there is an inessential isomorphism $\sigma_{\underline{k}}$ between X and X'; the constants k_i are determined up to sign.*

To show existence we argue as follows. By the discussion in section 1 there is a $\sigma_{\underline{h}}$ such that $\sigma_{\underline{h}}{}^*\zeta_X$ and $\zeta_{X'}$ are isomorphic, $\underline{h}$ being determined up to coboundaries: we may then assume that $\zeta_{X'} = \zeta_X$. Thus the only difference between X and X' lies in the α: for each connected component D of $\tilde{C}$ there is a non-zero constant m_D such that $\alpha_{X'} = m_D \cdot \alpha_X$ on D. For each D pick a square root ℓ_D of m_D. Again by the discussion in section 1 we know that there is a unique inessential automorphism σ of C such that there is an isomorphism between $\sigma^*\zeta_X$ and ζ_X that restricts to multiplication by ℓ_D on each D: thus σ is an isomorphism between X and X'. The only ambiguity in the definition of σ lies in the signs of the ℓ_D. Since σ restricts on E_i to multiplication by $\ell_{D_i}/\ell_{D_i'}$, this proves the last assertion of the lemma.

Lemma (2.1) shows, among other things, that it is incorrect to view a spin curve X as a curve plus a square root of a well-specified line bundle, for the isomorphism class of $\zeta_X{}^2$ is not well determined: only the restriction of $\zeta_X{}^2$ to $\tilde{C}$ is. For us, however, the main consequence of the lemma is a description of $\mathrm{Aut}_0(X)$.

(2.2) Lemma. $\mathrm{Aut}_0(X)$ *is isomorphic to the group of coboundaries* $B^1(\Gamma_X, \mathbb{Z}_2)$*, or, which is the same, to* $C^0(\Gamma_X, \mathbb{Z}_2)/H^0(\Gamma_X, \mathbb{Z}_2)$.

We let C be the underlying decent curve of X, and $E_1, \cdots, E_n$ its exceptional components. By (2.1), the inessential automorphisms of X are those $\sigma_{\underline{k}}$ such that $\underline{k}$ is a coboundary and $k_i = \pm 1$ for every i. Put otherwise, if, for each i, we write η_i to denote the automorphism $\sigma_{\underline{k}}$ where $k_i=-1$ and $k_j=1$ for $j\neq i$, the elements of $\mathrm{Aut}_0(X)$ are the products

$$\prod \eta_i^{\varepsilon_i} \quad , \quad \varepsilon_i \in \mathbb{Z}_2,$$

such that $(\varepsilon_1,\cdots,\varepsilon_n)$ belongs to $B^1(\Gamma_X,\mathbb{Z}_2)$. This proves (2.2).

3. The number of theta-characteristics. It is well known that, on a smooth curve of genus g, there are 2^{2g} non-isomorphic theta-characteristics. We wish to show that, in a certain sense, the same is true also on a stable curve $\bar{C}$. What are we to take as a theta-characteristic on $\bar{C}$? The first answer that comes to mind is that one should consider, for all decent curves C having $\bar{C}$ as stable model, all line bundles ζ such that (C,ζ,α) is a spin curve for some $\alpha:\zeta^2\rightarrow\omega_C$. If we did this, however, we would get, in general, infinitely many non-isomorphic line bundles, since, as we observed in section 2, the isomorphism class of ζ^2 is not well-defined. The solution is to fix, for every C, one ζ^2 and consider all its square roots: this gives the correct count.

To see why this is so, let N be the set of nodes of $\bar{C}$ that are not blown up in passsing from $\bar{C}$ to C, and notice that, in order for a spin curve with C as underlying decent curve to exist, a necessary and sufficient condition is that, on the normalization of each irreducible component of $\bar{C}$, there should be an even number of points mapping to N. If we view the nodes as edges of the dual graph Γ of $\bar{C}$, this translates into the fact that the sum of the nodes in N should be a cycle modulo 2. In other words there are 2^h acceptable ways of blowing up $\bar{C}$, where $h=h_1(\Gamma)$. Fix one acceptable C, and let (C,ζ,α), (C,ζ',α') be two spin curves such that $\zeta^2=\zeta'^2$; then $\zeta'\otimes\zeta^{-1}$ is a point of order two in the Picard group of $\bar{C}$. Since the number of these points is 2^{2g-h}, we get, altogether, 2^{2g} "theta-characteristics".

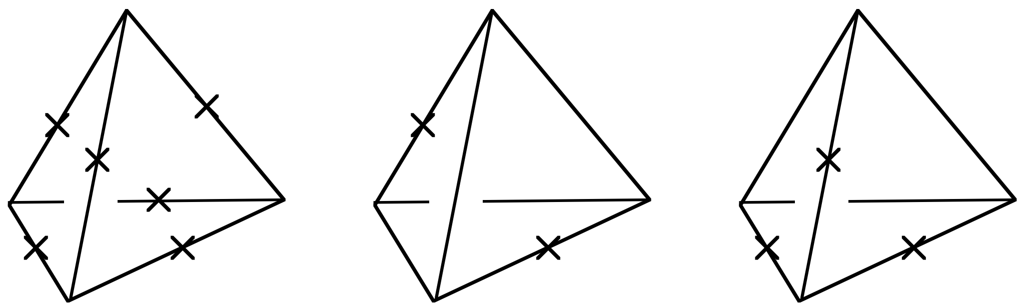
(3.1) Example. Let $\bar{C}$ be the union of two smooth components C_1 and C_2 of genera α and $g-\alpha$, meeting at one point p: in this case p has to be blown up and a "theta-characteristic" is a line bundle restricting to theta-characteristics on C_1 and C_2 and to $\mathcal{O}(1)$ on the exceptional component. In other words, to give a "theta-characteristic on $\bar{C}$" is equivalent to giving theta-characteristics on C_1 and C_2.

(3.2) Example. Let $\bar{C}$ be an irreducible curve with one node p: in this case there are two kinds of "theta-characteristics". The first kind are just the square roots of $\omega_{\bar{C}}$. These can be obtained as follows. Let $\hat{C}$ be the normalization of $\bar{C}$ and let q,r be the two points of $\hat{C}$ mapping to p. Then a square root of $\omega_{\bar{C}}$ is a square root of $\omega_{\hat{C}}(q+r)$, with the fibers over q and r suitably identified. As there are two possible identifications, this yields $2\cdot 2^{2g-2}=2^{2g-1}$ "theta-characteristics". The remaining 2^{2g-1} are obtained by blowing up $\bar{C}$ at p and suitably glueing a theta-characteristic on $\hat{C}$ to $\mathcal{O}(1)$ on the exceptional component.

(3.3) Example. Let $\bar{C}$ be the union of two components of genera $g-\alpha-1$ and α meeting at two points p and q. We have two choices. Either we take a square root of $\omega_{\bar{C}}$ – this accounts for 2^{2g-1} "theta-characteristics" – or we blow up both nodes and suitably glue theta-characteristics on the two components with $\mathcal{O}(1)$ on both exceptional components.

(3.4) Example. Let $\bar{C}$ be the union of two components of genera $g-\alpha-2$ and α meeting at three points p, q, and r. We may blow up one of these points, or all three. Each of these four alternatives produces $4\cdot 2^{2\alpha}\cdot 2^{2g-2\alpha-4}=2^{2g-2}$ "theta-characteristics".

(3.5) Example. Let $\bar{C}$ be the union of four components of genera α, β, γ, $g-\alpha-\beta-\gamma-3$, each meeting the other three at one point. The dual graph of $\bar{C}$ is a tetrahedron. Blowing up can be performed, up to automorphisms of the graph, according to one of the following schemes, the marked edges corresponding to the nodes to be blown up.



Altogether, there are 8 acceptable ways of blowing up $\bar{C}$, each yielding 2^{2g-3} "theta-characteristics".

4. Families of spin curves. Let $\pi:\mathcal{C}\to T$ be a family of noded curves. A *spin structure* on π is the datum of a line bundle ζ on $\mathcal{C}$ and a homomorphism $\alpha:\zeta^2\to\omega_\pi$ such that the triple $(\pi^{-1}(t),\zeta_{|\pi^{-1}(t)},\alpha_{|\pi^{-1}(t)})$ is a spin curve for each $t\in T$. A family of spin curves of genus g parametrized by a variety T, or, for brevity, a *spin curve of genus* g *over* T, is a triple

$\mathfrak{X}=(\pi:\mathcal{C}\to T,\zeta_{\mathfrak{X}},\alpha_{\mathfrak{X}})$, where $\pi:\mathcal{C}\to T$ is a family of noded curves of genus g, and $(\zeta_{\mathfrak{X}},\alpha_{\mathfrak{X}})$ is a spin structure on π. When no confusion is likely, we will often refer to $\pi:\mathcal{C}\to T$ itself as a family of spin curves and write ζ_π, α_π instead of $\zeta_{\mathfrak{X}}$, $\alpha_{\mathfrak{X}}$.

A *deformation* of a spin curve X is the datum of a family $\mathfrak{X}$ of spin curves parametrized by some variety T, a point $t_0\in T$, and an isomorphism from X to the fiber of $\mathfrak{X}$ over t_0.

Given a family of spin curves (or a deformation) as above and a morphism $f:S\to T$, there is an obvious notion of pullback of $\mathfrak{X}$ to a family (or deformation) over S. The notion of isomorphism of families of spin curves over T, or deformations over the pointed variety (T,t_0), is defined exactly as for ordinary spin curves.

(4.1) Lemma. *Let $\mathfrak{X}=(\pi:\mathcal{C}\to T,\zeta_{\mathfrak{X}},\alpha_{\mathfrak{X}})$ be a family of spin curves. Then $\pi:\mathcal{C}\to T$ is a family of decent curves.*

Proof. We fix our attention on a point $t\in T$ and an exceptional component E of $\pi^{-1}(t)$, with endpoints p and q. There are neighbourhoods V_p of p and V_q of q such that $\mathcal{C}$ is of the form:

$$xy=f \quad \text{in } V_p \quad , \quad \xi\eta=g \quad \text{in } V_q\,,$$

where f and g vanish at t and x and ξ vanish on E. We wish to show that one may assume that $y\eta=1$ on a neighbourhood of $E-\{p,q\}$. Let A' be a neighbourhood of E and B a neighbourhood of the complement of $E-\{p,q\}$ in $\pi^{-1}(t)$. Shrinking T, if necessary, we may assume that A' and B cover $\mathcal{C}$; we may also assume that $A'\cap B$ is contained in $V_p\cup V_q$, and that V_p and V_q are disjoint. Set

$$A=A'-(\{y=0\}\cup\{\eta=0\})\,,$$

and let M be the line bundle on $\mathcal{C}$ whose transition function, relative to the covering $\{A,B\}$, equals y on $A\cap B\cap V_p$ and η^{-1} on $A\cap B\cap V_q$: notice that the restriction of M to E is trivial. By Lemma (1.1), M is trivial on a neighbourhood of E. This means that there are nowhere vanishing holomorphic functions h_1, h_2, h_3, defined, respectively, on a neighbourhood of p, a neighbourhood of q, and A, such that

$$h_3=y\cdot h_1 \quad ; \quad h_3=\eta^{-1}\cdot h_2\,.$$

Replacing x, y, ξ, η with $h_1^{-1}\cdot x$, $h_1\cdot y$, $h_2\cdot\xi$, $h_2^{-1}\cdot\eta$ proves our assertion.

The homomorphism $\alpha_{\mathfrak{X}}$ yields a homomorphism $\beta: L \to \mathcal{O}_{\mathcal{C}}$, where L stands for $\zeta_{\mathfrak{X}}{}^2 \otimes \omega_\pi{}^{-1}$. The restriction of L to E has degree two. We denote by L' the line bundle wose transition function, relative to the cover $\{A,B\}$, equals y on $A \cap B \cap V_p$ and η on $A \cap B \cap V_q$. The restriction of L' to E also has degree two. It is then a consequence of Lemma (1.1), applied to $L' \otimes L^{-1}$, that there are nonzero sections s_1 and s_2 of L defined, respectively, on a neighbourhood of p and on a neighbourhood of q, such that $s_1 = \eta^2 \cdot s_2$. Write

$$\beta(s_1) = \sum_{i>0} a_i x^i + \sum_{i\geq 0} b_i y^i \quad , \quad \beta(s_2) = \sum_{i>0} c_i \xi^i + \sum_{i\geq 0} d_i \eta^i ,$$

where the a_i, b_i, etc., are functions on T. Clearly, a_1 and c_1 are units, so, replacing x, ξ, f, g with $a_1 x$, $c_1 \xi$, $a_1 f$, $c_1 g$, we may suppose that they are equal to 1. Using the fact that, away from $\{y=0\}$ and $\{\eta = 0\}$, one can make the substitutions

$$x = f \cdot \eta \quad , \quad \xi = g \cdot \eta^{-1} \quad , \quad y = \eta^{-1} ,$$

and equating terms of degree 1 in η in the identity

$$\beta(s_1) = \eta^2 \cdot \beta(s_2) , \tag{4.2}$$

we then conclude that $f = g$. As $y\eta = 1$, this finishes the proof of (4.1).

(4.3) Remark. The proof of (4.1) gives a bit more. In fact, by comparing terms of all degrees in (4.2), we find that there is a nowhere zero function γ in a neighbourhood of E such that $\beta(\gamma \cdot s_1) = x$, $\beta(\gamma \cdot s_2) = \xi$. Therefore there are local generators s_p and s_q of $\zeta_{\mathfrak{X}}{}^2$ at p and q such that

$$\alpha_{\mathfrak{X}}(s_p) = x \cdot r \quad , \quad \alpha_{\mathfrak{X}}(s_q) = \xi \cdot r ,$$

where r is a generator of ω_π in a neighbourhood of E.

(4.4) Lemma. *Let $\mathfrak{X} = (\pi: \mathcal{C} \to T, \zeta_{\mathfrak{X}}, \alpha_{\mathfrak{X}})$ be a family of spin curves, and let σ be an automorphism of $\mathfrak{X}$. If T is connected and, for some $t \in T$, the restriction of σ to $\pi^{-1}(t)$ is the identity, then σ is the identity.*

Proof. It suffices to deal with the case when T is the spectrum of an artinian local ring A. We shall adopt the same notation as in section 1. Clearly, σ is inessential, hence of the form $\sigma_{\underline{k}}$ for some elements $k_i = 1 + u_i$ of $A^\times$ such that $f_i \cdot u_i = 0$; moreover, our hypotheses imply that each u_i

belongs to the maximal ideal of A. Notice that $k_i^2 \neq 1$ unless $u_i = 0$. In fact, to say that $k_i^2 = 1$ means that

$$0 = 2 \cdot u_i + u_i^2 ;$$

since u_i is nilpotent, and we are not in characteristic two, we conclude by descending induction that $u_i = 0$. Since $\zeta_{\mathfrak{X}}$ and $\sigma^* \zeta_{\mathfrak{X}}$ are isomorphic, there are elements h_D of A such that $k_i = h_{D_i}/h_{D_i'}$ for every i: on D, the isomorphism between $\zeta_{\mathfrak{X}}$ and $\sigma^* \zeta_{\mathfrak{X}}$ is "multiplication by h_D". Since this isomorphism is compatible with the natural one between ω_π and $\sigma^* \omega_\pi$, and this, as σ is inessential, is the identity on each D, $h_D^2 = 1$ for every D. Thus, for every i, $k_i^2 = 1$. As we have observed, this implies that $k_i = 1$, so σ is the identity. This proves (4.4).

Let $X = (C, \zeta_X, \alpha_X)$ be a spin curve, and let $\bar{C}$ be the stable model of C. Let $E_1, \cdots, E_n$ be the exceptional components of C, and denote by $r_1, \cdots, r_n$ the corresponding nodes of $\bar{C}$. Consider the universal deformation $\bar{\varrho} : \bar{\mathcal{D}} \to \bar{B}$ of $\bar{C}$: here $\bar{B}$ is the unit policylinder in the space of $3g-3$ complex coordinates $\tau_1, \cdots, \tau_{3g-3}$, and we may suppose that the coordinates have been chosen so that, for i between 1 and n, the locus $\{\tau_i = 0\}$ is the locus where the node r_i persists. Let B be another copy of $\bar{B}$, with coordinates $t_1, \cdots, t_{3g-3}$, and let $\varrho' : \mathcal{D}' \to B$ be the pullback of $\bar{\varrho} : \bar{\mathcal{D}} \to \bar{B}$ via the base change

$$\tau_i = t_i^2 \ , \quad i = 1, \cdots, n \qquad ; \qquad \tau_i = t_i \ , \quad i = n+1, \cdots, 3g-3 \ .$$

For each i, $\varrho'|_{\{t_i = 0\}}$ has a section consisting entirely of nodes and passing through r_i. Blowing up these sections for $i = 1, \cdots, n$ yields a family of decent curves $\varrho : \mathcal{D} \to B$, with $\mathcal{D}$ smooth and C as central fiber. We write $\mathcal{E}_1, \cdots, \mathcal{E}_n$ to denote the exceptional divisors in $\mathcal{D}$, indexed in such a way that $\mathcal{E}_i \cap C = E_i$ for each i. We wish to put a structure of family of spin curves on $\varrho : \mathcal{D} \to B$. Set $L = \omega_\varrho(-\sum \mathcal{E}_i)$; by the discussion in section 1 we may alter the identification of C with the central fiber of ϱ by an inessential automorphism in such a way that the restriction of L to C is isomorphic to ζ_X^2. We may then extend ζ_X to a square root of L on a neighbourhood of the central fiber: call this $\zeta_{\mathfrak{U}}$, and let $\alpha_{\mathfrak{U}} : \zeta_{\mathfrak{U}}^2 \to \omega_\varrho$ be the composition of the isomorphism of $\zeta_{\mathfrak{U}}^2$ with L and the inclusion of L in ω_ϱ. By shrinking B if necessary, we may suppose that $\zeta_{\mathfrak{U}}$ is defined on all of $\mathcal{D}$. Clearly, the triple $\mathfrak{U} = (\varrho : \mathcal{D} \to B, \zeta_{\mathfrak{U}}, \alpha_{\mathfrak{U}})$ is a family of spin curves. Lemma (2.1) tells us that, altering the identification of C with the central fiber of ϱ by an inessential automorphism, we may suppose that the fiber of $\mathcal{U}$ over $0 \in B$ is isomorphic to X. We shall call the family $\mathcal{U}$, together with the identification of its central fiber with X, which we denote by ψ, a *universal deformation of* X. Clearly, this terminology needs to be justified. Before doing so, we need one observation concerning $\mathcal{U}$. The action of the group $\mathrm{Aut}(\bar{C})$ on $\bar{C}$ extends to equivariant actions on $\bar{\mathcal{D}}$ and $\bar{B}$. Let Γ' be the quotient of $\mathrm{Aut}(\bar{C})$ modulo the subgroup of those automorphisms that act trivially on $\bar{B}$, and let G' be the group of all the liftings of elements of Γ' to automorphisms of B. The group G' fits into an exact sequence

$$1 \to H \to G' \to \Gamma' \to 1 ,$$

where H is the group generated by all the automorphisms of B of the form

$$(\cdots, t_{i-1}, t_i, t_{i+1}, \cdots) \longmapsto (\cdots, t_{i-1}, -t_i, t_{i+1}, \cdots) .$$

Denote by G the fiber product of G' and $\mathrm{Aut}(\bar{C})$ over Γ'. The group G acts on the fiber product of B and $\bar{\mathcal{D}}$ over $\bar{B}$, i. e., on $\mathcal{D}'$, and this action lifts to an action on $\mathcal{D}$; moreover, there is an exact sequence

(4.5) $$1 \to H \to G \to \mathrm{Aut}(\bar{C}) \to 1 .$$

The group G acts on the central fiber of $\varrho : \mathcal{D} \to B$, that is, on C. The action of H is particularly easy to describe. The automorphism

$$(\cdots, t_{i-1}, t_i, t_{i+1}, \cdots) \longmapsto (\cdots, t_{i-1}, -t_i, t_{i+1}, \cdots)$$

of B lifts to an automorphism σ of $\mathcal{D}'$ restricting to the identity on $\bar{C}$: we want to see what this induces on $\mathcal{D}$. Near r_i, $\mathcal{D}'$ is of the form $\{(x, y, t) \mid xy = t_i{}^2\}$. Blowing up along $x = y = 0$ is performed by a substitution

$$x = z \quad ; \quad y = zv \quad ; \quad t_i = zw ,$$

and the proper transform of $\mathcal{D}'$ has equation $v = w^2$, while the equation of the exceptional divisor is $z = 0$. Thus a system of local coordinates on $\mathcal{D}$ is given by z, w, and the t_j with $j \neq i$. Also, σ lifts to $\mathcal{D}$, and its lifting is given, in local coordinates, by

$$(z, w, t_1, \cdots) \longmapsto (z, -w, t_1, \cdots) .$$

Since w restricts to a linear coordinate on E_i vanishing at one of the endpoints of E_i, the lifting of σ restricts, on C, to the inessential automorphism η_i (cf. section 2). We shall use the same symbol η_i to designate both the lifting of σ to $\mathfrak{D}$ and the automorphism of B it covers, namely $t_i \longmapsto -t_i$.

It follows from the above computation that G acts effectively on C, i. e., that it is a subgroup of $\mathrm{Aut}(C)$; it is also easy to show that $\mathrm{Aut}(X)$ is a subgroup of G and that

$$\mathrm{Aut}_0(X) = \mathrm{Aut}(X) \cap H .$$

However, since this is a formal consequence of Proposition (4.6) below, we omit the proof.

Let X be a spin curve with C as underlying decent curve. As usual, we denote by $E_1, \cdots, E_n$ the exceptional components of C. Consider the universal deformation of X constructed above. Recall that this consists of the family of spin curves $\mathfrak{U}$ and of the isomorphism ψ from X to the central fiber of $\mathfrak{U}$: the family of decent curves underlying $\mathfrak{U}$ is $\varrho : \mathfrak{D} \to B$. We wish to show that any small deformation of X is (uniquely) a pullback of $(\mathfrak{U}, \psi)$.

(4.6) Proposition. *Consider a deformation of* X*, consisting of a family of spin curves* $\mathfrak{X}$ *over* T *and of an isomorphism* φ *from* X *to the fiber of* $\mathfrak{X}$ *over* $t_0 \in T$. *Let* $\pi : \mathfrak{C} \to T$ *be the family of decent curves underlying* $\mathfrak{X}$. *Then, possibly after shrinking* T*, there is a unique cartesian diagram*

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\ \delta\ } & \mathfrak{D} \\ \downarrow \pi & & \downarrow \varrho \\ T & \xrightarrow{\ \gamma\ } & B \end{array}$$

such that:

i) $\gamma(t_0) = 0$,

ii) $\psi = \delta \circ \varphi$,

iii) The identity on $\mathfrak{C}$ *is an isomorphism between* $\mathfrak{X}$ *and the pullback of* $\mathfrak{U}$.

Proof. We first discuss uniqueness. Given γ, the uniqueness of δ follows from (4.4). The morphism γ is certainly unique up to composition with products of automorphisms of B of the form η_i. Uniqueness of γ follows from the fact that such a product acts non-trivially on X.

We next prove existence. Lemma (4.1) shows that, possibly after shrinking T, a diagram as in the statement of the proposition exists; the only trouble is that ii) and iii) are not necessarily satisfied. We can however assume that i) is satisfied and that $\psi^{-1} \circ \delta \circ \varphi$ induces the identity on the stable model of C. By pullback via γ and δ we get, on $\mathcal{C}$, another spin structure $\alpha:\zeta^2 \to \omega_\pi$. The homomorphisms α and α_X are isomorphisms along $\tilde{C}$, except at the endpoints of exceptional components. Let then E be an exceptional component, and p, q its endpoints. Using the same notation as in (4.1) and (4.3), Remark (4.3) says that there are local generators s_p, s_p' (resp., s_q, s_q') for ζ^2 and $\zeta_X{}^2$ at p (resp., at q) and a generator r for ω_π on a neighbourhood of E such that

(4.7) $$\alpha(s_p) = x\cdot r = \alpha_X(s_p') \quad , \quad \alpha(s_q) = \xi\cdot r = \alpha_X(s_q') \ .$$

The generator s_p is uniquely determined up to multiplication by functions of the form $1+y\cdot v$, where $f\cdot v=0$, and similar considerations apply to s_q, s_q', and s_p'. We may then extend the isomorphism $\alpha_X \circ \alpha^{-1}$ to neighbourhoods of p and q by sending s_p to s_p' and s_q to s_q'. Hence $\zeta_X{}^2 \otimes \zeta^{-2}$ is trivial on a neighbourhood of $\tilde{C}$, as well as on the exceptional components. There are units w and h such that, along $E-\{p,q\}$,

$$s_p = w\cdot s_q \quad , \quad s_p' = h\cdot w\cdot s_q' \ .$$

From this and (4.7) we conclude that $w\cdot\xi = x = h\cdot w\cdot\xi$, hence that

$$f\cdot h = \eta\xi\cdot h = \eta\xi = f \ .$$

Since $\zeta_X{}^2 \otimes \zeta^{-2}$ is trivial on E, we may find power series $1+\sum c_i y^i$ and $1+\sum d_i \eta^i$ with coefficients in the local ring of t_0 in T such that

$$h\cdot(1+\sum_{i>0} c_i y^i) = 1+\sum_{i\geq 0} d_i \eta^i \ .$$

Multiplying this identity by f, recalling that $y\eta=1$, and comparing terms of the same degree in η we find that

$$f\cdot c_i = f\cdot d_i = 0 \quad \text{for every } i \ .$$

Now set

$$\lambda = (1+d_0)^{-1}(1+\sum_{i\geq 0} d_i \eta^i) = 1+\sum_{i>0} d_i' \eta^i \ ,$$

$$\mu = 1+\sum_{i>0} c_i y^i \ .$$

Notice that $f\cdot d_i'=0$ for every i and, moreover,

$$(1+d_0)\cdot\lambda = h\cdot\mu \ .$$

Replacing s_q' and s_p' with $\lambda\cdot s_q'$ and $\mu\cdot s_p'$, we may therefore suppose that $h=1+d_0$. Repeating this for all the exceptional components $E_1,\cdots,E_n$ of C, and using the notation of section 1, we conclude that there is $\underline{h}\in\prod A_i$ such that $\zeta_{\mathfrak{X}}{}^2$ is isomorphic to $\zeta_{\mathfrak{X}}{}^2\otimes M_{\underline{h}}$. Each h_i can be written as the square of some $k_i\in A^\times$; since at least one among $1-k_i$ and $1+k_i$ is a unit, from $f\cdot h_i=f$ we get that either $f\cdot k_i=f$ or $f\cdot k_i=-f$, hence we may choose k_i so that $f\cdot k_i=f$. Replacing our original δ with $\delta\circ\sigma_{\underline{k}}$ we may therefore suppose that $\zeta_{\mathfrak{X}}{}^2$ and $\zeta^2=\delta^*\zeta_{\mathfrak{U}}{}^2$ are isomorphic. Keeping track of the way $\underline{k}$ was constructed, we may also suppose that the isomorphism between $\zeta_{\mathfrak{X}}{}^2$ and ζ^2 is compatible with $\alpha_{\mathfrak{X}}$ and α. Since $(\zeta_{\mathfrak{X}}\otimes\zeta^{-1})^2$ is trivial, $\zeta_{\mathfrak{X}}\otimes\zeta^{-1}$ is of the form $M_{\underline{m}}$, where $m_i=\pm1$ for every i. Composing γ and δ with an appropriate composition of automorphisms of B and $\mathfrak{D}$ of the form η_i, we may therefore suppose that the isomorphism between $\zeta_{\mathfrak{X}}{}^2$ and ζ^2 is induced by an isomorphism between $\zeta_{\mathfrak{X}}$ and ζ. To show that $\mathfrak{U}$ does indeed have the required universal property it remains to show that we can arrange things so that $\psi=\delta\circ\varphi$, where φ and ψ are the identifications of X with the central fibers of $\mathfrak{X}$ and $\mathfrak{U}$, respectively. What we can say at this stage is that $\psi^{-1}\circ\delta\circ\varphi$ is an inessential automorphism of the spin curve X, hence a product σ of automorphisms of C of the form η_i. Replacing γ and δ with $\sigma\circ\gamma$ and $\sigma\circ\delta$ concludes the proof of (4.6).

It follows from (4.6) and (4.4) that the action of $\mathrm{Aut}(X)$ on X extends to an action on all of $\mathfrak{U}$, that is, to compatible actions on $\mathfrak{D}$ and B that respect the spin structure.≠≠≠ In particular, this implies that $\mathrm{Aut}(X)$ is a subgroup of the group G, as announced before the statement of (4.6).

5. Moduli of spin curves. We denote by $\overline{S}_g$ the set of isomorphism classes of spin curves of genus g, and by S_g the subset consisting of classes of smooth curves. Given a spin curve X we denote by $[X]$ its class. We wish to define a natural structure of analytic variety on $\overline{S}_g$. Fix a spin curve X, and consider its universal deformation $\mathfrak{U}_X$; let B_X be the base of $\mathfrak{U}_X$. The natural way to introduce an analytic structure on a neighbourhood of $[X]$ is to transplant the structure of $B_X/\mathrm{Aut}(X)$ via the map

$$\beta_X: B_X/\mathrm{Aut}(X)\to\overline{S}_g$$

provided this can be shown to be injective. Assume, for the moment, that this has been proved, and let Y be another spin curve such that the images of β_X and β_Y intersect; let $[Z]$ be a point in the intersection. Since $\mathfrak{U}_X$ is a universal deformation of any one of its fibers, there is a commutative diagram

$$\begin{array}{ccccc} B_X & \xleftarrow{\ f\ } & B_Z & \xrightarrow{\ h\ } & B_Y \\ & \searrow & \downarrow & \swarrow & \\ & & \bar{S}_g & & \end{array}$$

after suitably shrinking B_Z, of course. As f and h are holomorphic open embeddings, this shows that the analytic structures induced by β_X and β_Y are compatible.

We now show that β_X is injective. We adopt the notations we used in section 4 when constructing universal deformations. Thus, the decent curve underlying X is C, its stable model is $\bar{C}$, the universal deformation of $\bar{C}$ is $\bar{\varrho}:\bar{\mathcal{D}}\to\bar{B}$, the family of curves underlying the universal deformation of X is $\varrho:\mathcal{D}\to B$, and so on. What we need is the following.

(5.1) Lemma. *Let* a *and* b *be points of* B *such that there is an isomorphism*

$$\gamma:\varrho^{-1}(a)\longrightarrow\varrho^{-1}(b)\,.$$

Then there is an automorphism σ *of* X *such that* $\sigma(a)=b$ *and that, viewing* σ *as an automorphism of* $\mathcal{D}$*, the restriction of* σ *to* $\varrho^{-1}(a)$ *is* γ.

Proof. Denote by $\bar{a}$ and $\bar{b}$ the images of a and b in $\bar{B}$, and let

$$\bar{\gamma}:\bar{\varrho}^{-1}(\bar{a})\longrightarrow\bar{\varrho}^{-1}(\bar{b})$$

be the isomorphism induced by γ. There is an element $\bar{\sigma}$ of $\mathrm{Aut}(\bar{C})$ such that $\bar{\sigma}(\bar{a})=\bar{b}$ and such that the restriction of $\bar{\sigma}$, viewed as an automorphism of $\bar{\mathcal{D}}$, restricts to $\bar{\sigma}$ on $\bar{\varrho}^{-1}(\bar{a})$. Then $\bar{\sigma}$ lifts to an element σ of the group G appearing in the exact sequence (4.5). Multiplying σ by a suitable element of H, we may suppose that $\sigma(a)=b$. Clearly $\sigma^{-1}\circ\gamma$ is an inessential automorphism of $\varrho^{-1}(a)$; moreover

$$\sigma^*\zeta_{\mathcal{U}}{}^2=\sigma^*(\omega_\varrho(-\textstyle\sum\mathcal{E}_i))=\omega_\varrho(-\textstyle\sum\mathcal{E}_i)=\zeta_{\mathcal{U}}{}^2\,,$$

so $\sigma^{-1}\circ\gamma$ is a product of automorphisms of the form η_{i_ℓ}, $\ell=1,\cdots,m$, where $t_{i_1},\cdots,t_{i_m}$ are those coordinates, among $t_1,\cdots,t_n$, such that $t_i(a)=0$. It follows that we may suppose that σ restricts to γ on $\varrho^{-1}(a)$: it remains to show that σ belongs to $\mathrm{Aut}(X)$. Since $(\sigma^*\zeta_{\mathcal{U}})^2=\zeta_{\mathcal{U}}{}^2$ and since $\sigma^*\zeta_{\mathcal{U}}=\zeta_{\mathcal{U}}$ on

$\varrho^{-1}(a)$, it follows that $\sigma^*\zeta_{\mathfrak{U}} = \zeta_{\mathfrak{U}}$ on all of $\mathfrak{D}$. This finishes the proof of (5.1).

It is apparent from the construction of $\overline{S}_g$ that the natural map

$$\chi : \overline{S}_g \to \overline{M}_g$$

from the moduli space of genus g spin curves to the moduli space of stable curves of genus g is holomorphic and, moreover, that

$$\partial\overline{S}_g = \overline{S}_g - S_g$$

is a closed proper analytic subvariety of $\overline{S}_g$, so S_g is an open subvariety of $\overline{S}_g$.

(5.2) Proposition. *The variety $\overline{S}_g$ is normal and projective. The natural morphism $\chi : \overline{S}_g \to \overline{M}_g$ is finite.*

Proof. Normality immediately follows from the construction of $\overline{S}_g$. As χ is certainly finite-to-one, it will suffice to show that it is proper and separated to prove it is finite, and hence that $\overline{S}_g$ is projective, since $\overline{M}_g$ is. Suppose χ is not separated. Since the restriction of χ to S_g is certainly separated, this means that there are two non-isomorphic spin curves X and X', with the same stable model, and neighbourhoods U and U' of X and X' in $\overline{S}_g$ such that $U \cap S_g = U' \cap S_g$. This, in turn, implies that there are two families of spin curves, $\mathfrak{X}$ and $\mathfrak{X}'$, over the unit disk Δ such that the central fiber of $\mathfrak{X}$ is X, the central fiber of $\mathfrak{X}'$ is X', all other fibers are smooth, and there is an isomorphism of spin curves $\varphi : \mathfrak{X}|_{\Delta - \{0\}} \to \mathfrak{X}'|_{\Delta - \{0\}}$; notice that φ extends to an isomorphism between the stable models of $\mathfrak{X}$ and $\mathfrak{X}'$. That χ is separated is then a consequence of the following auxiliary result.

(5.3) Lemma. *Let $\pi : \mathcal{C} \to \Delta$ and $\pi' : \mathcal{C}' \to \Delta$ be families of decent curves over the unit disk with the same stable model $\overline{\pi} : \overline{\mathcal{C}} \to \Delta$. Let (ζ, α) and (ζ', α') be spin structures on π and π'. Suppose that every fiber of $\overline{\pi}$, except possibly the central one, is smooth, and that (ζ, α) and (ζ', α') agree on $\overline{\pi}^{-1}(\Delta - \{0\})$. Then the identity on $\overline{\pi}^{-1}(\Delta - \{0\})$ extends to an isomorphism of spin curves over Δ between $\pi : \mathcal{C} \to \Delta$ and $\pi' : \mathcal{C}' \to \Delta$.*

Proof of the lemma. Let $\mathfrak{D}$ be the graph of the meromorphic mapping from $\mathcal{C}$ to $\mathcal{C}'$ obtained by composing $\mathcal{C} \to \overline{\mathcal{C}}$ with the inverse of $\mathcal{C}' \to \overline{\mathcal{C}}$, and let $\varrho : \mathfrak{D} \to \Delta$ be the natural projection. Clearly, ϱ is a family of decent curves.

Denote by L and L' the pullbacks of ζ and ζ' to $\mathcal{D}$. The isomorphism between ζ and ζ' away from the central fibers extends to an isomorphism between $L(D)$ and L', where D is a Cartier divisor supported on the union of the exceptional components of the central fiber of ϱ. Let E_i be one such component. Notice that the intersection number $(D\cdot E_i)$ is even; moreover the degree of the restriction of L^2 (resp., of L'^2) to E_i is 0 or 2 depending on whether E_i is contracted or not by the projection to $\mathcal{C}$ (resp., to $\mathcal{C}'$). Hence

$$(D\cdot E_i) = \frac{1}{2}((L'\otimes L^{-1})^2\cdot E_i)$$

can be even only if it is zero. Thus D vanishes, and the projections from $\mathcal{D}$ to $\mathcal{C}$ and $\mathcal{C}'$ are isomorphisms. The fact that the isomorphism between ζ and ζ' away from the central fiber extends to one between L and L' means that the isomorphism between $\mathcal{C}$ and $\mathcal{C}'$ is an isomorphism of families of spin curves. The proof of Lemma (5.3) is complete.

To prove (5.2) it remains to show that χ is proper. To do this, it will suffice, for every stable curve $\bar{C}$, to find a neighbourhood U of $[\bar{C}]$ in $\bar{M}_g$ and a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\ \delta\ } & \chi^{-1}(U) \\ \downarrow\beta & & \downarrow\chi \\ V & \xrightarrow{\ \gamma\ } & U \end{array}$$

such that δ is onto, and γ and β are proper. To construct these, let $\bar{\varrho}:\bar{\mathcal{D}}\to\bar{B}$ be a universal deformation of $\bar{C}=\bar{\varrho}^{-1}(0)$, let $r_1,\cdots,r_m$ be the nodes and Γ the dual graph of $\bar{C}$. We may suppose that there are coordinates $\tau_1,\cdots,\tau_{3g-3}$ on $\bar{B}$ such that, for i between 1 and m, the locus $\{\tau_i=0\}$ is the locus where the node r_i persists. We let U be the image of $\bar{B}$ in $\bar{M}_g$ and V the inverse image of $\bar{B}$ under the map $\varphi:\mathbb{C}^{3g-3}\to\mathbb{C}^{3g-3}$ defined by

$$\varphi(t_1,\cdots,t_{3g-3}) = (t_1{}^2,\cdots,t_m{}^2,t_{m+1},\cdots,t_{3g-3})\,.$$

We next construct 2^{2g} families of spin curves over as many copies of V, let W be the disjoint union of these copies, and let δ be the map that sends a point in a copy of V to the isomorphism class of the fiber. The construction is as follows. Let $\mathcal{D}'$ be the fiber product of $\bar{\mathcal{D}}$ and V, and $\varrho':\mathcal{D}'\to V$ the projection map. Pick nodes $r_{i_1},\cdots,r_{i_h}$ in $\bar{C}$ in such a way that the set of the remaining ones, viewed as an element of $C_1(\Gamma,\mathbb{Z}_2)$, is a cycle. For each ℓ

between 1 and h there is a section of ϱ' passing through r_{i_ℓ} and consisting entirely of nodes: blow up these, denote by $\mathcal{D}$ the resulting variety, by $\mathcal{E}_{i_1},\cdots,\mathcal{E}_{i_h}$ the exceptional divisors, and let $\varrho:\mathcal{D}\to V$ be the projection. Then choose a square root ζ of $\omega_\varrho(-\sum\mathcal{E}_{i_\ell})$ and let $\alpha:\zeta^2\to\omega_\varrho$ be the inclusion: this defines a spin structure on $\varrho:\mathcal{D}\to V$. By the count in section 3 there are, altogether, 2^{2g} possible choices of a mod. 2 cycle in Γ and of a square root, giving rise to 2^{2g} non-isomorphic families of spin curves over (copies of) V. Moreover, by Lemma (5.3) and by the discussion in section 3, the fibers of these families over any given $v\in V$ are, up to isomorphism, all the spin curves having the stable curve corresponding to v as stable model. This shows that δ is onto and finishes the proof of (5.2).

It follows from Proposition (4.6) that $\bar{S}_g$ is a coarse moduli variety for spin curves of genus g.

The explicit description we have given of the local structure of $\bar{S}_g$ makes it easy to determine how the covering $\chi:\bar{S}_g\to\bar{M}_g$ ramifies over the boundary of $\bar{M}_g$. More exactly, given a stable curve $\bar{C}$ and a universal deformation $\bar{\varrho}:\bar{\mathcal{D}}\to\bar{B}$ of it, what is easy is to describe is how the covering $B'\to\bar{B}$ ramifies, where B' is the (closure of) the set of couples

$$(\text{point } b\in\bar{B}, \text{theta-characteristic on } \bar{\varrho}^{-1}(b))\,.$$

In fact, if X is a spin curve having $\bar{C}$ as its stable model, and B is the base of its universal deformation, as described in section 4, in a neighbourhood of $[X]$ the covering in question is just

$$B/\mathrm{Aut}_0(X)\to\bar{B}\,.$$

We shall illustrate this by describing what happens in each of the examples of section 3; as usual, we let C be the decent curve underlying X.

(5.3) Example. Let $\bar{C}$ be the union of two smooth components C_1 and C_2 meeting at one point p (cf. Example (3.1)). In this case C is the blow-up of $\bar{C}$ at p and the generator of $\mathrm{Aut}_0(X)=\mathbb{Z}_2$ acts on B by

$$(t_1,\cdots,t_{3g-3})\longmapsto(-t_1,t_2,\cdots,t_{3g-3})$$

so B' is just the disjoint union of 2^{2g} copies of $\bar{B}$.

(5.4) Example. Let $\bar{C}$ be an irreducible curve with one node p (cf. Example (3.2)). There are two cases:

a) $C=\bar{C}$, hence $B=\bar{B}$.

b) C is $\bar{C}$ blown up at p, $\mathrm{Aut}_0(X)$ is trivial, and hence $B/\mathrm{Aut}_0(X)\to\bar{B}$ is of the form

$$(t_1,\cdots,t_{3g-3})\longmapsto(t_1{}^2,t_2,\cdots,t_{3g-3}).$$

Thus B' is the union of 2^{2g-1} copies of $\bar{B}$ and of 2^{2g-2} double coverings of $\bar{B}$ branched along $\{\tau_1=0\}$.

(5.5) Example. Let $\bar{C}$ be the union of two components of genera $g-\alpha-1$ and α meeting at two points p and q (cf. Example (3.3)). There are two cases:

a) $C=\bar{C}$, hence $B=\bar{B}$.

b) C is $\bar{C}$ blown up at p and q, the generator of $\mathrm{Aut}_0(X)=\mathbb{Z}_2$ acts on B by

$$(t_1,\cdots,t_{3g-3})\longmapsto(-t_1,-t_2,t_3,\cdots,t_{3g-3}),$$

hence $B/\mathrm{Aut}_0(X)$ is the quadric $\{xy=z^2\}$ in the $\mathbb{C}^{3g-2}$ with coordinates $x,y,z,t_3,\cdots$, and $B/\mathrm{Aut}_0(X)\to\bar{B}$ is the double covering

$$(x,y,z,t_3,\cdots,t_{3g-3})\longmapsto(x,y,t_3,\cdots,t_{3g-3}).$$

Thus B' is the union of 2^{2g-1} copies of $\bar{B}$ and of 2^{2g-2} copies of $\{xy=z^2\}$.

(5.6) Example. Let $\bar{C}$ be the union of two components of genera $g-\alpha-2$ and α meeting at three points p, q, and r (cf. Example (3.4)). There are two cases:

a) C is $\bar{C}$ blown up at one of the nodes, say p, $\mathrm{Aut}_0(X)$ is trivial, hence this case is like Example (5.4) b).

b) C is $\bar{C}$ blown up at p, q and r, the generator of $\mathrm{Aut}_0(X)=\mathbb{Z}_2$ acts on B by

$$(t_1,\cdots,t_{3g-3})\longmapsto(-t_1,-t_2,-t_3,t_4,\cdots,t_{3g-3}),$$

hence $B/\mathrm{Aut}_0(X)$ is the locus

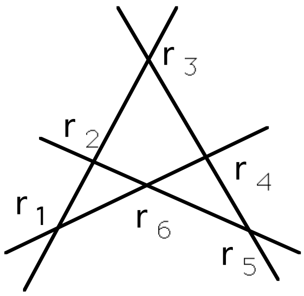
$$x_1x_2=x_4{}^2,\ x_1x_3=x_5{}^2,\ x_2x_3=x_6{}^2,\ x_1x_2x_3=x_4x_5x_6$$

in the $\mathbb{C}^{3g}$ with coordinates $x_1,x_2,\cdots,x_6,t_4,\cdots$, and $B/\mathrm{Aut}_0(X)\to\bar{B}$ is the four-sheeted covering

$$(x_1,x_2,\cdots,x_6,t_4,\cdots,t_{3g-3})\longmapsto(x_1,x_2,x_3,t_4,\cdots,t_{3g-3}).$$

Thus B' is the union of 2^{2g-4} copies of case b), of 2^{2g-3} double coverings of $\bar{B}$ branched along $\{\tau_1=0\}$, 2^{2g-3} double coverings branched along $\{\tau_2=0\}$, and 2^{2g-3} double coverings branched along $\{\tau_3=0\}$.

(5.7) Example. Let $\bar{C}$ be the union of four components of genera α, β, γ, $g-\alpha-\beta-\gamma-3$, each meeting the other three at one point (cf. Example (3.5)).



There are three cases:

a) C is $\bar{C}$ blown up at two of the nodes, not lying on the same component, say r_1 and r_5. Since $\mathrm{Aut}_0(X)$ is trivial, $B/\mathrm{Aut}_0(X)$ is smooth and $B/\mathrm{Aut}_0(X)\to\bar{B}$ is the four-sheeted covering

$$(t_1,\cdots,t_{3g-3})\longmapsto(t_1^2,t_2,\cdots,t_5^2,t_6,\cdots,t_{3g-3}).$$

b) C is $\bar{C}$ blown up at three of the nodes, lying on the same component, say r_1, r_2, and r_3. The generator of $\mathrm{Aut}_0(X)=\mathbb{Z}_2$ acts on B by

$$(t_1,\cdots,t_{3g-3})\longmapsto(-t_1,-t_2,-t_3,t_4,\cdots,t_{3g-3}),$$

hence this case is like Example (5.6) b).

c) C is $\bar{C}$ blown up at all the nodes, and $\mathrm{Aut}_0(X)\cong\mathbb{Z}_2^{\oplus 3}$ is generated by

$$(t_1,\cdots,t_{3g-3})\longmapsto(-t_1,-t_2,-t_3,t_4,\cdots,t_{3g-3}),$$
$$(t_1,\cdots,t_{3g-3})\longmapsto(t_1,t_2,-t_3,-t_4,-t_5,t_6,\cdots,t_{3g-3}),$$
$$(t_1,\cdots,t_{3g-3})\longmapsto(t_1,-t_2,t_3,t_4,-t_5,-t_6,t_7,\cdots,t_{3g-3}).$$

Thus $B/\mathrm{Aut}_0(X)$ is the locus

$$x_1x_3x_4=x_7^2,\ x_1x_2x_6=x_8^2,\ x_2x_3x_5=x_9^2,$$
$$x_4x_5x_6=x_{10}^2,\ x_7\cdots x_{10}=x_1\cdots x_6$$

in the $\mathbb{C}^{3g+1}$ with coordinates $x_1,x_2,\cdots,x_{10},t_7,\cdots$, and $B/\mathrm{Aut}_0(X)\to\bar{B}$ is the eight-sheeted covering

$$(x_1,x_2,\cdots,x_{10},t_7,\cdots,t_{3g-3})\longmapsto(x_1,\cdots,x_6,t_7,\cdots,t_{3g-3}).$$

Hence B' is the union of $15\cdot 2^{2g-6}$ components, $3\cdot 2^{2g-5}$ of them like in case a), $4\cdot 2^{2g-5}$ like in case b), and 2^{2g-6} like in case c).

6. Even and odd spin curves. It is classical that even and odd theta-characteristics do not mix; in other words, under deformation, even theta-characteristics stay even and odd ones stay odd. The same is true for spin curves. We say that a spin curve X is *even* (resp., *odd*) if $h^0(\zeta_X)$ is even (resp., odd). Mumford's proof [5] that the parity of a theta-characteristic is a deformation invariant extends, with very minor modifications, to our situation; accordingly, we will point out what the modifications are, instead of repeating the proof. Given a theta-characteristic ζ on a smooth C, Mumford's argument is based on the following observations. First of all, if $\Gamma = \sum p_i$ is an effective divisor of sufficiently high degree on C, with all the p_i distinct, then $H^0(\zeta(-\Gamma))$ and $H^1(\zeta(\Gamma))$ vanish, so $H^0(\zeta)$, $H^0(\zeta(\Gamma))$, and $H^0(\zeta/\zeta(-\Gamma))$ are subspaces of $H^0(\zeta(\Gamma)/\zeta(-\Gamma))$ such that

$$H^0(\zeta) = H^0(\zeta(\Gamma)) \cap H^0(\zeta/\zeta(-\Gamma)) .$$

Secondly, $H^0(\zeta(\Gamma))$ and $H^0(\zeta/\zeta(-\Gamma))$ are maximal isotropic subspaces with respect to the non-degenerate bilinear form on $H^0(\zeta(\Gamma)/\zeta(-\Gamma))$

$$(a,b) \longmapsto \sum \mathrm{res}_{p_i}(\bar{a}\bar{b}) ,$$

where $\bar{a}$ and $\bar{b}$ are liftings of a and b to sections of $\zeta(\Gamma)$ in a neighbourhood of Γ and the product is induced by $\zeta^2 \cong \omega_C$. The same argument applies to a possibly singular spin curve $X = (C, \zeta_X, \alpha_X)$ if we take $\zeta = \zeta_X$, choose Γ so that it is made up entirely of smooth points, has high degree on all non-exceptional components of C and does not touch any exceptional component, and, in the definition of the bilinear form on $H^0(\zeta(\Gamma)/\zeta(-\Gamma))$, interpret the product of $\bar{a}$ and $\bar{b}$ as coming from $\alpha_X : {\zeta_X}^2 \to \omega_C$.

(6.1) Example. Let $\bar{C}$ be the union of two smooth components C_1 and C_2 of genera α and $g-\alpha$ meeting at one point p, C its blow-up at p, and E the exceptional component of C. We know (Example (3.1)) that a spin structure on C comes from glueing theta-characteristics L_1 and L_2 on C_1 and C_2 to $\mathcal{O}(1)$ on E. Such a spin structure is even if L_1 and L_2 are both even or both odd, odd if one among L_1 and L_2 is even and the other odd. The spin structures that are even on both components number

$$2^{\alpha-1}(2^{\alpha}+1)\cdot 2^{g-\alpha-1}(2^{g-\alpha}+1) = 2^{g-2}(2^{g}+2^{g-\alpha}+2^{\alpha}+1)\ ,$$

those that are odd on both components

$$2^{\alpha-1}(2^{\alpha}-1)\cdot 2^{g-\alpha-1}(2^{g-\alpha}-1) = 2^{g-2}(2^{g}-2^{g-\alpha}-2^{\alpha}+1)\ ,$$

those that are even on C_1 and odd on C_2

$$2^{\alpha-1}(2^{\alpha}+1)\cdot 2^{g-\alpha-1}(2^{g-\alpha}-1) = 2^{g-2}(2^{g}+2^{g-\alpha}-2^{\alpha}-1)\ ,$$

those that are odd on C_1 and even on C_2

$$2^{\alpha-1}(2^{\alpha}-1)\cdot 2^{g-\alpha-1}(2^{g-\alpha}+1) = 2^{g-2}(2^{g}-2^{g-\alpha}+2^{\alpha}-1)\ .$$

Altogether, then, there are $2^{g-1}(2^{g}+1)$ even and $2^{g-1}(2^{g}-1)$ odd "theta-characteristics", as in the smooth case.

(6.2) Example. Let $\bar{C}$ be an irreducible curve with one node p. We know (Example (3.2)) that in this case there are two kinds of "theta-characteristics". The first kind come from theta-characteristics on the normalization $\hat{C}$ of $\bar{C}$, even (resp., odd) ones on $\hat{C}$ yielding even (resp., odd) ones on $\bar{C}$. Moreover each theta-characteristic on $\hat{C}$ corresponds to two on $\bar{C}$. This accounts for $2\cdot 2^{g-2}(2^{g-1}+1)$ even and $2\cdot 2^{g-2}(2^{g-1}-1)$ odd "theta-characteristics". The remaining ones come about by suitably identifying the fibers at q and r of square roots of $\omega_{\hat{C}}(q+r)$, where q and r are the points of $\hat{C}$ mapping to p. Let L be such a square root: by Riemann-Roch we have that

$$h^0(L) = h^0(L(-q-r))+1\ ,$$

so there is a section s of L that does not vanish at both q and r. In fact, s does not vanish at either one of these points since s^2, which is a section of $\omega_{\hat{C}}(q+r)$, must have opposite residues at q and r. This also shows that there are two possible identifications between the fibers of L at q and r, one sending $s(q)$ to $s(r)$, the other sending $s(q)$ to $-s(r)$. Clearly, the resulting "theta-characteristics" have opposite parity, so we get an equal number, 2^{2g-2}, of even and odd "theta-characteristics". Altogether, then, there are on $\bar{C}$

$$2^{2g-2}+2^{g-1}(2^{g-1}+1) = 2^{g-1}(2^{g}+1)$$

even "theta-characteristics" and

$$2^{2g-2}+2^{g-1}(2^{g-1}-1) = 2^{g-1}(2^{g}-1)$$

odd ones, as expected.

A consequence of the deformation invariance of the parity of a spin curve is that $\overline{S}_g$ is the disjoint union

$$\overline{S}_g = \overline{S}_g^+ \cup \overline{S}_g^-$$

of the two closed subvarieties consisting, respectively, of the even and odd spin curves of genus g. As is natural, we write S_g^+ to indicate $\overline{S}_g^+ \cap S_g$, and S_g^- to indicate $\overline{S}_g^- \cap S_g$.

(6.3) Lemma. *$\overline{S}_g^+$ and $\overline{S}_g^-$ are irreducible.*

This is classical. Here we sketch a proof by degeneration. We let $\overline{M}'_g$ be the complement of the branch locus of $\chi : \overline{S}_g \to \overline{M}_g$, and denote by χ_+ and χ_- the restrictions of χ to $\overline{S}_g^+$ and $\overline{S}_g^-$. It is enough to show that, given a point p in $\overline{M}'_g$, the monodromy action of the fundamental group of $\overline{M}'_g$ on $\chi_+^{-1}(p)$ and $\chi_-^{-1}(p)$ is transitive. The proof is by induction on g.

The cases $g=1,2$ are dealt with separately. View a smooth curve C of genus two as a double covering of the Riemann sphere ramified at six points $q_1, \cdots, q_6$. Then all odd theta-characteristics on C (there are six of them) are of the form $\mathcal{O}(q_i)$, while all even ones (there are ten of them) are of the form $\mathcal{O}(q_i + q_h - q_k)$, whith q_i, q_h, q_k distinct. Therefore the monodromy group acts transitively on even and odd theta-characteristics. Likewise, viewing a smooth elliptic curve C as a double covering of the Riemann sphere ramified at four points $q_1, \cdots, q_4$, and placing the origin at q_1, all even theta-characteristics on C are of the form $\mathcal{O}(q_i - q_1)$ whith $i > 1$, while there is a single odd one, and we conclude in the same way.

Now take $g > 2$. We begin by noticing that, if $i \geq 1$ and $[\overline{C}]$ is a general point of Δ_i, that is, if $\overline{C}$ is the union of two general smooth curves C_1 and C_2 of genera i and $g-i$ joined at a general point, then $[\overline{C}]$ belongs to $\overline{M}'_g$. In fact, for $i > 1$ such a $\overline{C}$ has no non-trivial automorphisms, while the only non-trivial automorphism of $\overline{C}$ for $i=1$ is the -1 involution on the elliptic tail, which acts trivially on spin structures. Taking $i=1$, Example (6.1) and the induction hypothesis show that $\chi_+^{-1}(p)$ is divided in two subsets A_1 and B_1, one consisting of $3 \cdot 2^{g-2}(2^{g-1}+1)$ points, the other of $2^{g-2}(2^{g-1}-1)$ points, on each of which a subgroup of the monodromy group acts transitively. If $g > 3$, taking $i=2$ shows that $\chi_+^{-1}(p)$ is also divided in two subsets A_2 and B_2, one consisting of $5 \cdot 2^{g-2}(2^{g-2}+1)$ points, the other of $3 \cdot 2^{g-2}(2^{g-2}-1)$ points, on each of which a subgroup of the monodromy group acts transitively. As A_1 must overlap both A_2 and B_2, this proves that the monodromy group acts transitively on $\chi_+^{-1}(p)$. If $g=3$, this argument has to be modified slightly. Taking $g=3$, $\alpha=\beta=\gamma=0$ in example (5.7), we see that $\chi^{-1}(p)$ is divided in fifteen subsets, fourteen of them consisting of four points and one of eight, on each of which a subgroup of the monodromy group acts transitively; since in this case A_1 consists of 30 points and B_1 of 6, one of these must overlap both A_1 and B_1. This proves that the monodromy group acts transitively on $\chi_+^{-1}(p)$ for $g\geq 3$; the same argument also works for $\chi_-^{-1}(p)$. The proof of (6.3) is thus complete.

7. Natural divisor classes on $\overline{S}_g$. Mumford (cf. [6], [4]) has defined a Picard group $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ for the moduli <u>stack</u> of genus g stable curves. Its elements are isomorphism classes of line bundles on the moduli stack. Such a line bundle is the datum, for every algebraic family $h:X\to S$ of stable curves of genus g, of an algebraic line bundle L_h (often written L_S) on S, and, for every cartesian diagram of algebraic families of stable curves

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow h & & \downarrow k \\ S & \xrightarrow[\ell]{} & T \end{array}$$

of an isomorphism between $\ell^*(L_T)$ and L_S. This isomorphism is required to be "natural", in a precise technical sense; the reader is referred to [6], [4], or [2] for details. Line bundles on $\overline{M}_g$ give rise, by pullback, to line bundles on the moduli stack; this yields a homomorphism from the ordinary Picard group of $\overline{M}_g$ to $\mathrm{Pic}(\overline{\mathcal{M}}_g)$. The homomorphism is known to have finite cokernel; moreover, for $g>2$, it is injective and $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ is a free abelian group. The operation in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ is normally written additively.

If we replace, in the definition of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, the words "family of stable curves" with "family of spin curves" we get the definition of a Picard group of the "moduli stack of spin curves", which we shall denote by $\mathrm{Pic}(\overline{\mathcal{S}}_g)$; if we limit ourselves to even or odd spin curves, we get groups $\mathrm{Pic}(\overline{\mathcal{S}}_g^+)$ and $\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$. Clearly, $\mathrm{Pic}(\overline{\mathcal{S}}_g)$ is the direct sum of $\mathrm{Pic}(\overline{\mathcal{S}}_g^+)$ and $\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$: given a class α in $\mathrm{Pic}(\overline{\mathcal{S}}_g)$, we will denote by α^+ and α^- its restrictions to $\overline{\mathcal{S}}_g^+$ and to $\overline{\mathcal{S}}_g^-$, so that $\alpha=\alpha^++\alpha^-$. Forgetting the spin structure and passing to the stable model yields a homomorphism

$$\chi^*:\mathrm{Pic}(\overline{\mathcal{M}}_g)\longrightarrow \mathrm{Pic}(\overline{\mathcal{S}}_g)\,.$$

We shall see later that χ^* is injective for $g>2$; for brevity, if α is a class in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, we shall normally use the letter α also to denote $\chi^*(\alpha)$.

The prototypical classes in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ are the Hodge class λ and the boundary classes δ_i, $i=0,\cdots,[g/2]$ (cf. [2]): one knows that they form an integral basis of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ for $g>2$, and that they generate it in every case [1]. We recall that, given a family of stable curves $h:X\to S$, the line bundle giving rise to λ is $\det(h_*\omega_h)$; since the first direct image sheaf $R^1h_*\omega_h$ is canonically trivial, this can also be written $d(\omega_h)$, which stands for the "determinant line bundle" of $h_!\omega_h$, as defined by Knudsen and Mumford [3]. If h is a family of spin curves, we can use the same procedure to define another natural line bundle M_h by setting

$$M_h = d(\zeta_h)\ .$$

There is one subtle point concerning the naturality of M_h. Given a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \downarrow h & & \downarrow k \\ S & \xrightarrow[\ell]{} & T \end{array}$$

of families of spin curves, the isomorphism between ζ_h and $m^*(\zeta_k)$ is defined only up to sign, as we observed in section 2, so that there seems to be a sign ambiguity in the isomorphism between M_h and $\ell^*(M_k)$. However, since the fiber of M_h over $s\in S$ is

$$\det(H^0(\zeta_h|_{h^{-1}(s)}))\otimes\det(H^1(\zeta_h|_{h^{-1}(s)}))^{-1}\ ,$$

and $H^0(\zeta_h|_{h^{-1}(s)})$ and $H^1(\zeta_h|_{h^{-1}(s)})$ have the same dimension, all ambiguities cancel. The class in $\mathrm{Pic}(\overline{\mathcal{S}}_g)$ determined by $\{M_h\}$ will be denoted by the letter μ.

In addition to μ, there are other natural classes in $\mathrm{Pic}(\overline{\mathcal{S}}_g)$, one for each component of the boundary $\partial\overline{S}_g=\overline{S}_g-S_g$. These components have already been implicitly determined in examples (6.1) and (6.2). When g is odd there are, altogether, $4\cdot([g/2]+1)$ components, half of them, A_i^+, B_i^+, $i=0,\cdots,[g/2]$, contained in $\overline{S}_g^+$, and half of them, A_i^-, B_i^-, $i=0,\cdots,[g/2]$, contained in $\overline{S}_g^-$. When g is even, the situation is the same except that B_i^- is defined only for $i<g/2$, so the components of the boundary number

$4\cdot(g/2+1)-1$. We shall describe the components of $\partial\overline{S}_g$ by describing their general members.

- A_i^+, $i>0$: two smooth components C_1 and C_2 of genera i and $g-i$ joined at points $p\in C_1$ and $q\in C_2$ by a $\mathbb{P}^1$, with even theta-characteristics on C_1 and C_2.
- B_i^+, $i>0$: as above, but with odd theta-characteristics on C_1 and C_2.
- A_i^-, $i>0$: as above, but with an even theta-characteristic on C_1 and an odd one on C_2.
- B_i^-, $i>0$: as above, but with an odd theta-characteristic on C_1 and an even one on C_2.
- A_0^+: an irreducible curve of genus g with only one node, with an even spin structure.
- A_0^-: as above, but with an odd spin structure.
- B_0^+: an irreducible curve of genus g with only one node, blown up at the node, with an even spin structure.
- B_0^-: as above, but with an odd spin structure.

That these subvarieties of $\overline{S}_g$ are irreducible follows from Lemma (6.3). The corresponding divisor classes in $\mathrm{Pic}(\overline{\mathcal{S}}_g)$ will be denoted $\alpha_i^+, \alpha_i^-, \beta_i^+, \beta_i^-$; the class $\beta_{g/2}^-$ is defined to be zero. These classes are related to the boundary classes in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ by the relations:

$$\delta_i = 2\cdot(\alpha_i+\beta_i)\ ,\quad i>0\ ,$$
$$\delta_0 = \alpha_0+2\cdot\beta_0\ .$$

This seems to be a good point to show that the homomorphism

$$\chi^*: \mathrm{Pic}(\overline{\mathcal{M}}_g) \longrightarrow \mathrm{Pic}(\overline{\mathcal{S}}_g)$$

is injective for $g>2$, as already announced. One way of showing that the classes $\lambda, \delta_0, \cdots, \delta_{[g/2]}$, which generate $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, are independent, is to construct families of stable curves $h: X\to S$, with S a smooth complete curve, such that the vectors $(\deg_h(\lambda), \deg_h(\delta_0), \cdots)$ are independent, where $\deg_h(L)$ stands for the degree of the line bundle L_h (cf. [1], for instance). Now, after a finite base change and blowing up suitable nodes, we can put on $h: X\to S$ a spin structure; since the effect of a finite base change on the vector $(\deg_h(\lambda), \deg_h(\delta_0), \cdots)$ is to multiply all entries by the degree of the base change itself, the same argument shows that $\lambda, \delta_0, \cdots, \delta_{[g/2]}$ are independent in $\mathrm{Pic}(\overline{\mathcal{S}}_g)$ as well. Since we may limit ourselves to even, or odd, spin structures throughout, this shows that, in fact, $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ also injects in $\mathrm{Pic}(\overline{\mathcal{S}}_g^+)$ and $\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$.

We can actually do better, and show that λ^+, the α_i^+, and the β_i^+ (resp., λ^-, the α_i^-, and the β_i^-) are independent. Here is a sketch of the argument for the even case, the one for the odd case being no different. Suppose there is a relation

$$(7.1) \qquad 0 = \ell \cdot \lambda^+ + \sum a_i \cdot \alpha_i^+ + \sum b_i \cdot \beta_i^+ .$$

Given an integer i such that $0 < i \leq [g/2]$, pick two smooth curves T and S of genera i and $g-i$, and consider the family of stable curves $h: X \to S$ whose fiber over $p \in S$ is the union of S and T with the point p identified to a fixed point q of T. For this family the degrees of the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ are as follows [2]:

$$\deg_h(\delta_i) = 2 - 2(g-i) \neq 0 \quad , \quad \deg_h(\lambda) = \deg_h(\delta_j) = 0 \quad \text{for } j \neq i .$$

If $\hat{X}$ stands for the blow-up of X along the locus of nodes in the fibers, and $\hat{h}$ for its projection to S, then we can make $\hat{h}: \hat{X} \to S$ into a family of spin curves by putting an even theta-characteristic, or an odd one, on both S and T. On these families the degrees of λ^+, of the α_i^+, and of the β_i^+ all vanish, with the exception of $\deg_h(\alpha_i^+)$ for the first family, and of $\deg_h(\beta_i^+)$ for the second one. It follows that all the coefficients in (7.1) vanish, except possibly for ℓ, a_0, and b_0. To handle these, one can proceed in essentially the same way, with a different family of stable curves, for example with the one obtained from a smooth genus $g-1$ curve C by identifying a variable point p of C with a fixed one [2]. After a suitable base change and blow-up, one obtains a family of spin curves all lying in A_0, or, depending on choices, in B_0. Calculating degrees shows that there is a rational constant k such that

$$\ell = 2k \cdot a_0 = k \cdot b_0 .$$

In conclusion, (7.1) is a relation between λ and δ_0. As we have observed, its coefficients must vanish. We have proved most of the following result.

(7.2) Proposition. *If* $g>2$ *is odd, the classes* $\mu^+, \mu^-, \alpha_i^+, \alpha_i^-, \beta_i^+, \beta_i^-$, $i=0,\cdots,(g-1)/2$, *are independent. If* $g>2$ *is even, the same is true of the classes* $\mu^+, \mu^-, \alpha_i^+, \alpha_i^-, \beta_i^+$, $i=0,\cdots,g/2$, *and* β_i^-, $i=0,\cdots,g/2-1$. *The following relations hold in* $\mathrm{Pic}(\overline{\mathcal{S}}_g)$:

i) $\delta_0 = \alpha_0 + 2\cdot\beta_0$,

ii) $\delta_i = 2\cdot(\alpha_i+\beta_i)$, $i>0$.

The following holds in $\mathrm{Pic}(\overline{\mathcal{S}}_g)\otimes\mathbb{Q}$:

iii) $\alpha_0 = 4\cdot\lambda + 8\cdot\mu$.

Obviously, it suffices to prove iii). We begin by showing that, for any family $h:X\to S$ of spin curves, one has

$$c_1(\alpha_{0_h}) = 4\cdot c_1(\lambda_h) + 8\cdot c_1(\mu_h) \qquad (7.3)$$

in $A^1(S)\otimes\mathbb{Q}$; for simplicity, we shall drop the suffix h throughout. We set $\mathcal{E}=\omega\otimes\zeta^{-2}$, and notice that

$$h_*(c_1(\mathcal{E})\cdot c_1(\zeta)) = \sum_{i>0}(c_1(\alpha_i)+c_1(\beta_i))+c_1(\beta_0) ,$$

$$h_*(c_1^2(\mathcal{E})) = -2\cdot(\sum_{i>0}(c_1(\alpha_i)+c_1(\beta_i))+c_1(\beta_0)) = c_1(\alpha_0)-c_1(\delta) ,$$

$$h_*(c_1(\mathcal{E})\cdot c_1(\omega)) = 0 .$$

Since h is a l. c. i. morphism, the Riemann-Roch theorem holds for it in Grothendieck's form. If we apply it to ζ, compare terms of degree one, and use the fact that $h_*(c_1^2(\omega))=12\cdot c_1(\lambda)-c_1(\delta)$ (cf. [6]), we find that

$$\begin{aligned}
8\cdot c_1(\mu) &= 4\cdot h_*(c_1^2(\zeta)) - 4\cdot h_*(c_1(\zeta)\cdot c_1(\omega)) + 8\cdot c_1(\lambda)\\
&= -2\cdot h_*(c_1(\mathcal{E})\cdot c_1(\zeta)) - 2\cdot h_*(c_1(\omega)\cdot c_1(\zeta)) + 8\cdot c_1(\lambda)\\
&= -2\cdot h_*(c_1(\mathcal{E})\cdot c_1(\zeta)) - h_*(c_1^2(\omega)) + 8\cdot c_1(\lambda)\\
&= -c_1(\delta)+c_1(\alpha_0) - 12\cdot c_1(\lambda) + c_1(\delta) + 8\cdot c_1(\lambda)\\
&= c_1(\alpha_0) - 4\cdot c_1(\lambda) ,
\end{aligned}$$

as desired. Now that (7.3) has been proved, we are essentially done. There are several different ways to conclude: we choose a lowbrow approach. Take as S the moduli space of smooth spin curves together with a level n structure, for sufficiently high n, and as $h:X\to S$ the universal family of spin curves on S. Then S is smooth, so (7.3) says that

$$\alpha_{0_h} = 4\cdot\lambda_h + 8\cdot\mu_h .$$

Since S is a finite covering of S_g, this in turn says that

$$(7.4) \qquad 4\cdot\lambda + 8\cdot\mu - \alpha_0 = \sum(r_i^+\alpha_i^+ + s_i^+\beta_i^+ + r_i^-\alpha_i^- + s_i^-\beta_i^-)\,,$$

where the $r_i^\pm$ and the $s_i^\pm$ are rational numbers. If we "evaluate" (7.4) on the families of spin curves constructed during the proof of the independence of the boundary classes in $\mathrm{Pic}(\overline{\mathcal{S}}_g)$, and use (7.3), we find, for every i, equalities

$$0 = r_i^\pm \deg(\alpha_i^\pm) \quad , \quad 0 = s_i^\pm \deg(\beta_i^\pm)\,,$$

so all the $r_i^\pm$ and all the $s_i^\pm$ vanish, since the degrees appearing in the above equalities are non-zero. The proof of Proposition (7.2) is now complete.

(7.5) Remark. With more work, one can actually show that part iii) of (7.2) holds in $\mathrm{Pic}(\overline{\mathcal{S}}_g)$, and not just modulo torsion.

(7.6) Remark. One interesting consequence of part iii) of (7.2) is that $\mathrm{Pic}(\mathcal{S}_g)$, the Picard group of the moduli stack of <u>smooth</u> spin curves of genus g, contains an element of order 4, in contrast with what happens for $\mathrm{Pic}(\mathcal{M}_g)$, which is infinite cyclic. In fact, iii) says that $h\cdot(4\cdot\lambda + 8\cdot\mu - \alpha_0)$ is trivial, for some positive integer h, so $h\cdot(4\cdot\lambda + 8\cdot\mu)$ is trivial on $\mathcal{S}_g$. Let k be the least positive integer such that $k\cdot(\lambda + 2\cdot\mu)$ is trivial on $\mathcal{S}_g$. Then $k\cdot(\lambda + 2\cdot\mu)$ is a linear combination, with integral coefficients, of boundary classes. Since these classes have been shown to be independent, and iii) holds, k must be divisible by 4. The same argument shows, more precisely, that both $\mathrm{Pic}(\mathcal{S}_g^+)$ and $\mathrm{Pic}(\mathcal{S}_g^-)$ contain elements of order 4. If we take into account Remark (7.5), we see that the restriction to $\mathcal{S}_g$ of $\lambda + 2\cdot\mu$ itself has order 4.

It would be of interest to completely determine the structure of $\mathrm{Pic}(\overline{\mathcal{S}}_g)$. For $g > 2$, the simplest answer consistent with what we already know would be that it is the free abelian group generated by the classes

$$\mu^+, \mu^-, \lambda^+, \lambda^-\,;$$
$$\alpha_i^+, \alpha_i^-\,,\ i = 1, \cdots, (g-1)/2\,;$$
$$\beta_i^+, \beta_i^-\,,\ i = 0, \cdots, (g-1)/2$$

if g is odd, and by the classes

$$\mu^+, \mu^-, \lambda^+, \lambda^- ;$$
$$\alpha_i^+, \alpha_i^-, \ i=1,\cdots, g/2 ;$$
$$\beta_i^+, \ i=0,\cdots, g/2 ; \quad \beta_i^-, \ i=0,\cdots, g/2-1$$

if g is even. If this were the case, then $\mathrm{Pic}(\mathfrak{S}_g^+)$ (resp., $\mathrm{Pic}(\mathfrak{S}_g^-)$) would be the direct sum of an infinite cyclic group generated by μ^+ (resp., by μ^-) and of a cyclic group of order four generated by $\lambda^+ + 2\cdot\mu^+$ (resp., by $\lambda^- + 2\cdot\mu^-$).

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