Ist. Lombardo Accad. Sci. Lett. Rend. A **129** (1995), 97–105

A simple proof of the projectivity of Kontsevich's space of maps

Maurizio Cornalba[1]

SUNTO: *Si dà una semplice dimostrazione della proiettività della compattificazione dello spazio delle mappe da curve algebriche a spazi proiettivi recentemente introdotta da Kontsevich.*

The stacks of stable maps from curves to projective space have been introduced by Kontsevich [5][6]. It has been observed by several people that the underlying algebraic spaces are in fact projective. A proof can be found in [7]. Here we wish to present a simple proof based on the methods of [1]. We work over $\mathbb{C}$ throughout.

Consider a complete, connected, reduced curve C whose singularities are at worst nodes, n smooth numbered points $x_1, \ldots, x_n$ on C, and a morphism $\mu : C \to \mathbb{P}^r$. According to Kontsevich, one says that the datum of C, $x_1, \ldots, x_n$, and μ is a *stable map* if the following condition is satisfied. Let E be a smooth component of C such that $\mu(E)$ is a point; if the genus of E is zero (resp., one) then E contains at least three (resp., one) points which are among the x_i or are singular in C but not in E. An *isomorphism* between stable maps $(C, x_1, \ldots, x_n, \mu)$ and $(C', x'_1, \ldots, x'_n, \mu')$ is an isomorphism $\varphi : C \to C'$ such that $\varphi(x_i) = x'_i$ for $i = 1, \ldots, n$ and $\mu'\varphi = \mu$. A *family of stable maps* is a flat proper morphism $f : \mathcal{C} \to S$ together with n sections $\sigma_i : S \to \mathcal{C}$, $i = 1, \ldots, n$ and a morphism $\mu : \mathcal{C} \to \mathbb{P}^r$ such that, for every $s \in S$, $(f^{-1}(s), \sigma_1(s), \ldots, \sigma_n(s), \mu_{|f^{-1}(s)})$ is a stable map. One has obvious notions of pullback and of isomorphism between families of stable maps.

Let $F = (C, x_1, \ldots, x_n, \mu)$ be a stable map of degree d. If Q is a sufficiently general member of $|\mathcal{O}_{\mathbb{P}^r}(3)|$, then $\mu^*(Q) = \sum p_i$ is a divisor consisting of $3d$ smooth points of C, each occurring with multiplicity one. Furthermore, $\Gamma = (C, x_1, \ldots, x_n, p_1, \ldots, p_{3d})$ is a stable $(n + 3d)$-pointed curve. We then have an exact sequence of groups

$$1 \to G \to \mathrm{Aut}(F) \to G' ,$$

where $G = \mathrm{Aut}(\Gamma) \cap \mathrm{Aut}(F)$ and G' is the group of permutations of $p_1, \ldots, p_{3d}$. This shows that there is an upper bound for the order of $\mathrm{Aut}(F)$ which depends only on d, n, and the genus g of C. The fact that Γ is stable also implies that the number of singular points of C is bounded by $3g - 3 + n + 3d$.

Fix non-negative integers g, n, r, d. Then the functor

$$\mathcal{F}(S) = \left\{ \begin{matrix} \text{families of stable maps of degree } d \\ \text{from } n\text{-pointed genus } g \text{ curves to } \mathbb{P}^r \end{matrix} \right\} \Big/ \text{isomorphisms}$$

is coarsely represented by a complete separated algebraic space $\overline{M}_{g,n}(r, d)$ (cf. [5][7]). Clearly, $\overline{M}_{g,n}(r, d)$ is non-empty if and only if $2g - 2 + n + 3d > 0$, and $d = 0$ for

[1] Supported in part by the M.U.R.S.T. national project "Geometria Algebrica".

$r = 0$. We wish to show that $\overline{M}_{g,n}(r,d)$ is projective. This is clear if $d = 0$. In fact, $\overline{M}_{g,n}(r,0) = \overline{M}_{g,n} \times \mathbb{P}^r$, where $\overline{M}_{g,n}$ is the usual moduli space of stable n-pointed genus g curves, and we know that $\overline{M}_{g,n}$ is projective.

For $d > 0$ we argue as follows. For any family

$$\begin{array}{ccl} \mathcal{C} & \xrightarrow{\ \mu\ } & \mathbb{P}^r \\ f \big\downarrow \big\uparrow & \sigma_i, i = 1, \ldots, n & \\ S & & \end{array}$$

of stable maps of degree d from n-pointed curves of genus g to $\mathbb{P}^r$, which we denote by F, set

$$L_F = \omega_f(\textstyle\sum D_i) \otimes \mu^* \mathcal{O}(3)\,,$$

where $\omega_f = \omega_{\mathcal{C}/S}$ is the relative dualizing sheaf and $D_i = \sigma_i(S)$. We also set

$$\mathcal{L}_F = \langle L_F, L_F \rangle\,,$$

where $\langle\ ,\ \rangle$ is Deligne's bilinear symbol (cf. [2][3]); $\mathcal{L}_F$ is a line bundle on S which behaves nicely under base change. Therefore this construction defines a line bundle $\mathcal{L}$ on the moduli stack of stable maps of degree d from n-pointed curves of genus g to $\mathbb{P}^r$. Since, as we observed, the orders of the automorphisms groups of such maps are bounded, $\mathcal{L}$ can be viewed as a fractional line bundle on $\overline{M}_{g,n}(r,d)$. We shall prove the following result.

THEOREM 1. *For any choice of non-negative integers g, n, r, and d such that*

$$\begin{aligned} &2g - 2 + n + 3d > 0\,, \\ &d > 0 \text{ if } r > 0\,, \end{aligned}$$

$\mathcal{L}$ is ample on $\overline{M}_{g,n}(r,d)$.

Notice that, for $r = d = 0$, the theorem reduces to the well-known statement that Mumford's class κ_1 is ample on $\overline{M}_{g,n}$ (cf. [1], for instance). The first step in the proof is to observe that there is a family G of stable maps of degree d from n-pointed curves of genus g to $\mathbb{P}^r$ parametrized by a *scheme* Z such that the corresponding moduli map

$$\nu : Z \to \overline{M}_{g,n}(r,d)$$

is finite. A proof of this is sketched for instance in [7], based on a modification of a construction of Kollàr [4]. To show that $\mathcal{L}$ is ample it suffices to show that $\nu^*(\mathcal{L}) = \mathcal{L}_G$ is ample on Z. In order to prove this we shall use Seshadri's criterion. In other terms, we shall show that there is a positive constant α such that, for any integral complete curve Γ in Z, one has

$$(\mathcal{L}_G \cdot \Gamma) \geq \alpha\, m(\Gamma)\,,$$

where $m(\Gamma)$ stands for the maximum multiplicity of points of Γ. Since the intersection number $(\mathcal{L}_G \cdot \Gamma)$ is the degree of $\mathcal{L}_{G'}$, where G' is the pullback of G via the inclusion $\Gamma \subset Z$, we will be done if we can show that there is a positive constant α such that $\deg \mathcal{L}_F \geq \alpha\, m(S)$ for any non-isotrivial family F of stable maps of degree d from n-pointed genus g curves to $\mathbb{P}^r$ parametrized by an integral complete curve S. Here non-isotrivial means that the moduli map $S \to \overline{M}_{g,n}(r,d)$ does not send S to a point. Taking into account the definition of $\mathcal{L}_F$, what needs to be proved is

Lemma 2. *If $2g-2+n+3d>0$ and $d>0$ or $r=d=0$, there is a positive constant $\alpha=\alpha(g,n,r,d)$ such that, for any non-isotrivial family F of degree d stable maps from n-pointed genus g curves to $\mathbb{P}^r$ over an integral complete curve S,*

$$(L_F\cdot L_F)\geq \alpha\, m(S)\,.$$

The proof is essentially by reduction to the known case $r=d=0$. From now on we assume that $d>0$. Let the family F be given by maps $f:\mathcal{C}\to S$, $\mu:\mathcal{C}\to\mathbb{P}^r$ and sections $\sigma_i:S\to\mathcal{C}$, $i=1,\dots,n$. We begin by reducing to the case when the general fiber of f is smooth. Denote by $\Sigma(F)$ the union of all one-dimensional components of the locus of nodes in the fibers of f, and by $\pi_F:N(F)\to\mathcal{C}$ the normalization of $\mathcal{C}$ along $\Sigma(F)$. Let $\psi:S'\to S$ be a finite unramified base change, and let

$$\begin{array}{ccc}\mathcal{C}' & \xrightarrow{\ \mu'\ } & \mathbb{P}^r\\ f'\big\downarrow\big\uparrow \sigma'_i,\ i=1,\dots,n & & \\ S' & & \end{array}$$

be the pullback family, which we call F'. We can choose ψ in such a way that $\pi_{F'}^{-1}(\Sigma(F'))$ is a disjoint union of sections of $N(F')\to S'$. Moreover, since the number of singular points in the fibers of f is bounded independently of F, the degree of ψ can also be chosen to be bounded. Thus, in proving Lemma 2, we may assume that $\pi_F^{-1}(\Sigma(F))$ is a disjoint union of sections of $N(F)\to S$. Let $\mathcal{C}_1,\dots,\mathcal{C}_h$ be the connected components of $N(F)$, set $\pi_i=\pi_F\big|_{\mathcal{C}_i}$, $f_i=f\pi_i$, $\mu_i=\mu\pi_i$. Let $\sigma_{i,1},\dots,\sigma_{i,n_i}$ be the sections of f_i that come from components of $\pi_F^{-1}(\Sigma(F))$ lying on $\mathcal{C}_i$ or from sections σ_j such that $\sigma_j(S)$ lies on $\pi_F(\mathcal{C}_i)$. Then the datum of $f_i:\mathcal{C}_i\to S$, $\mu_i:\mathcal{C}_i\to\mathbb{P}^r$, and $\sigma_{i,1},\dots,\sigma_{i,n_i}$ is a family of stable maps of degree d_i with the property that the general fiber of f_i is smooth of genus g_i. It is clear from the definitions that

$$L_{F_i}=\pi_i^*(L_F)\,,$$

so that

$$(L_F\cdot L_F)=\sum(L_{F_i}\cdot L_{F_i})\,.$$

Moreover the invariants g_i, n_i, d_i satisfy the inequalities

$$g_i\leq g\,,\quad d_i\leq d\,,\quad n_i\leq n+2(3g-3+n+3d)\,.$$

This shows that it suffices to prove Lemma 2 for families whose general fiber is smooth; in fact, the possible objection that some of the families F_i might be such that $d_i=0$, so that Lemma 2 is false for them if $r\neq 0$, may be countered as follows. Suppose all the F_i with $d_i\neq 0$ are isotrivial, but F_j is not. Then $\mu_j(\mathcal{C}_j)$ is a single point, so $f_j:\mathcal{C}_j\to S$ is non-isotrivial as a family of stable curves, and one can apply to it Lemma 2 with $r=d=0$.

From now on we assume that the general fiber of $f:\mathcal{C}\to S$ is smooth. We set $D_i=\sigma_i(S)$ for $i=1,\dots,n$. A simple dimension count shows that, if $\mathcal{H}$ is a sufficiently general hyperplane, then

i) $\mu^{-1}(\mathcal{H})$ does not contain components of fibers of f;
ii) $\mu^{-1}(\mathcal{H})$ does not contain singular points of fibers of f;

iii) $\mu^{-1}(\mathcal{H})$ does not contain points of intersection between one of the D_i, $i = 1, \ldots, n$, and the fibers of f which are singular or lie above singular points of S.
iv) $\mu^{-1}(\mathcal{H})$ does not contain D_i for $i = 1, \ldots, n$;
v) $\mu^{-1}(\mathcal{H})$ cuts transversely all the fibers of f which are singular or lie over singular points of S.

Let $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ be distinct hyperplanes satisfying i), ii), iii), iv) and v). Possibly after a finite base change of bounded degree, $\mu^{-1}(\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3)$ becomes a sum of distinct sections. Moreover, since $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ satisfy v), we may choose a base change that does not affect $m(S)$. We may thus assume that

$$\begin{aligned}\mu^{-1}(\mathcal{H}_1) &= D_{n+1} + \cdots + D_{n+d}\,,\\ \mu^{-1}(\mathcal{H}_2) &= D_{n+d+1} + \cdots + D_{n+2d}\,,\\ \mu^{-1}(\mathcal{H}_3) &= D_{n+2d+1} + \cdots + D_{n+3d}\,,\end{aligned}$$

where $D_{n+1}, \ldots, D_{n+3d}$ are distinct sections, different from $D_1, \ldots, D_n$. The family of curves $f : \mathcal{C} \to S$, together with the sections $D_1, \ldots, D_{n+3d}$, has all the characters of a family of stable $(n+3d)$-pointed curves, except for the fact that some of the D_i may meet; however, by properties ii) and iii), this may occur only on smooth fibers of f not lying above singular points of S. To obtain a family $(f'' : \mathcal{C}'' \to S, D''_1, \ldots, D''_{n+3d})$ of semi-stable $(n+3d)$-pointed curves it is necessary to blow up, perhaps repeatedly, the points of intersection of two or more of the D_i and possibly, in genus zero, blow down some exceptional curves of the first kind. At each blow-up, the selfintersection of $\omega_f(\sum_{i=1}^{n+3d} D_i)$ decreases. If $g = 0$ and, at any stage of the process, the (proper transforms of the) D_i all meet at a point p of a smooth fiber Γ, the proper transform of Γ under the blow-up at p is an exceptional curve of the first kind not meeting sections, which needs to be blown down. The blow-down increases the selfintersection of $\omega_f\left(\sum D_i\right)$ exactly by one. Thus, in any case

$$\left(\omega_f\left(\textstyle\sum D_i\right)\cdot\omega_f\left(\textstyle\sum D_i\right)\right) \ge \left(\omega_{f''}\left(\textstyle\sum D''_i\right)\cdot\omega_{f''}\left(\textstyle\sum D''_i\right)\right)\,.$$

Now, if

$$F' = (f' : \mathcal{C}' \to S, D'_1, \ldots, D'_{n+3d})$$

is the stable model of $(f'' : \mathcal{C}'' \to S, D''_1, \ldots, D''_{n+3d})$, we have that

$$\left(\omega_{f'}\left(\textstyle\sum D'_i\right)\cdot\omega_{f'}\left(\textstyle\sum D'_i\right)\right) = \left(\omega_{f''}\left(\textstyle\sum D''_i\right)\cdot\omega_{f''}\left(\textstyle\sum D''_i\right)\right)\,,$$

so we conclude that

$$(L_F \cdot L_F) \ge (L_{F'} \cdot L_{F'})\,.$$

If F' is not isotrivial, we are done, since κ_1 is ample on $\overline{M}_{g,n+3d}$. From now on, we assume that F' is isotrivial. In particular, this implies that all the fibers of $f : \mathcal{C} \to S$ are smooth. When $g > 0$, $\mathcal{C}'$ dominates $\mathcal{C}$, so $\mathcal{C}' = \mathcal{C}$ and the D_i do not meet. Another consequence is that $\mu(\mathcal{C})$ is a surface. To see it, just combine the non-isotriviality of F with the following result.

Lemma 3. *Let X and Y be smooth curves, denote by p the genus of Y, let U be a disk, and let $y_1, \ldots, y_h$ be distinct points of Y. Suppose $2p - 2 + h > 0$. Let $\Psi : X \times U \to Y$ be a morphism such that the divisor $\Psi^{-1}(\sum y_i)$ is a sum $\sum \{x_j\} \times U$, where the x_j are k distinct points of X. Then, for any $x \in X$, $\Psi(x, u)$ is independent of u.*

To proof uses elementary deformation theory. Let $\psi : X \to Y$ be a morphism such that $\psi^{-1}(\sum y_i) = \sum x_j$. The first order deformations of ψ as a map from the k-pointed curve $(X, x_1, \ldots, x_k)$ to Y sending the x_j to the y_i are classified by $H^0(X, \mathcal{F})$, where $\mathcal{F}$ stands for $\psi^*(T_Y(-\sum y_i))/T_X(-\sum x_j)$, and those such that the moduli of $(X, x_1, \ldots, x_k)$ do not vary by the image of $H^0(X, \psi^*(T_Y(-\sum y_i)))$ in $H^0(X, \mathcal{F})$. The conclusion follows from the fact that the degree of $\psi^*(T_Y(-\sum y_i))$ is a multiple of $2 - 2p - h$, and hence negative.

For $g > 0$ we reach a contradiction establishing Lemma 2 by noticing that, since $\mu(\mathcal{C})$ is a surface, $\mathcal{H}_1 \cap \mathcal{H}_2 \cap \mu(\mathcal{C})$ is non-empty, so the D_i cannot be disjoint, contrary to what we established earlier. When $g = 0$, we argue somewhat differently. Since $\mu(\mathcal{C})$ is a surface, by choosing $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ to be sufficiently general, we may assume that $\mu(D_i)$ is not a point for $i > n$. Thus, if $i > n$ and p is any point of D_i, there is a hyperplane passing through $\mu(p)$ but not containing $\mu(D_i)$. It follows that

$$(\mu^*\mathcal{O}(1) \cdot D_i) \geq m(S) \qquad \text{for any } i > n\,.$$

Now set

$$\eta_h = \omega_f\Big(\sum_{i \leq h} D_i\Big)\,.$$

We wish to show that, for any $h \geq 2$ and any section D of f,

$$(\eta_h \cdot \eta_h) \geq 0\,, \quad (\eta_h \cdot D) \geq 0\,.$$

In fact $\eta_2 = \mathcal{O}(\sum a_i \Gamma_i)$, where the Γ_i are fibers of f, so $\sum a_i = (\eta_2 \cdot D_1) = (D_2 \cdot D_1) \geq 0$ and

$$(\eta_2 \cdot \eta_2) = 0\,, \quad (\eta_2 \cdot D) = \textstyle\sum a_i \geq 0\,.$$

In general

$$(\eta_h \cdot \eta_h) = (\eta_2 \cdot \eta_2) + \sum_{2 < i \leq h} (D_i \cdot \eta_2) + \sum_{2 < j \leq h} (D_j \cdot \omega_f(D_j)) + \sum_{\substack{i \leq h,\ 2 < j \leq h \\ i \neq j}} (D_i \cdot D_j) \geq 0\,,$$

while

$$(\eta_h \cdot D) = (\omega_f(D_j) \cdot D_j) + \sum_{\substack{0 < i \leq h \\ i \neq j}} (D_i \cdot D_j) \geq 0\,,$$

if $D = D_j$ for some $j \leq h$, and

$$(\eta_h \cdot D) = (\eta_2 \cdot D) + \sum_{2 < i \leq h} (D_i \cdot D) \geq 0$$

otherwise. Thus

$$(L_F \cdot L_F) = (\eta_{n+2d} \cdot \eta_{n+2d}) + 2 \sum_{i > n+2d} (D_i \cdot \eta_{n+2d}) + \sum_{i > n+2d} (D_i \cdot \mu^*\mathcal{O}(1)) \geq d\, m(S)\,.$$

This finishes the proof of Lemma 2, and hence of Theorem 1.

Bibliography

1. M.D.T. Cornalba, *On the projectivity of the moduli spaces of curves*, J. reine angew. Math. **443** (1993), 11–20.
2. P. Deligne, *La Formule de Dualité Globale*, in “Théorie des Topos et Cohomologie Étale des Schemas (S.G.A. 4)”, tome 3, Lecture Notes in Mathematics 305, 1973, pp. 481–587.
3. P. Deligne, *Le déterminant de la cohomologie*, in “Current Trends in Arithmetical Algebraic Geometry (K. Ribet, ed.)”, Contemporary Mathematics 67, 1987, pp. 93–177.
4. J. Kollàr, *Projectivity of complete moduli*, J. Differential Geometry **32** (1990), 235–268.
5. M. Kontsevich, *Enumeration of rational curves via torus actions*, preprint 1994.
6. M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, preprint 1994.
7. R. Pandharipande, *Notes on Kontsevich's compactification of the space of maps*, preprint 1995.

Maurizio Cornalba
Dipartimento di Matematica
Università di Pavia
via Abbiategrasso 215
27100 Pavia, Italia
e-mail: cornalba@ipv36.unipv.it