SUBGRID SCALES, AUGMENTED PROBLEMS, AND STABILIZATIONS

FRANCO BREZZI

ABSTRACT. We present an overview of some recent approaches to deal with instabilities of numerical schemes and/or subgrid phenomena. The basic idea is that of *enlarging* (as much as one can) the finite element space, then to do an element-by element preprocessing, and finally solve a problem with the same number of unknowns as the one we started with, but having better numerical properties.

1. INTRODUCTION

In a number of applications, subgrid scales cannot be neglected. Sometimes, they are just a spurious by-product of a discretised scheme that lacks the necessary stability properties. In other cases, they are related to physical phenomena that actually take place on a very small scale but still have an important effect on the solution.

In recent times, it was discovered that some mathematical tricks to deal with these problems can help in both situations. One of these tricks is based on the so-called Residual Free Bubbles (RFB). In what follows, we are going to discuss its application, by considering two typical examples, one for each category: the case of *advection diffusion problems* and the case of *composite materials*. For dealing with these problems, in a typical mathematical fashion, we shall choose very simple *toy problems* that will however still retain some of the basic difficulties of their bigger industrial counterparts. In particular we consider:

Ex 1: **Advection-dominated scalar equations**: find u in $V := H^1_0(\Omega)$ such that

$$Lu := -\varepsilon\Delta u + \mathbf{c}\cdot\nabla u = f \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

Here Ω is, say, a convex polygon, $\mathbf{c}$ a given vector-valued smooth function (convective term), f a given smooth forcing term, and ε a positive scalar (diffusion coefficient). Clearly, $x = (x_1, x_2)$. The numerical approximation of the problem becomes nontrivial when the product of ε times a characteristic length of the problem (for instance, the diameter of Ω) is *much smaller* than $|\mathbf{c}|$ in a nonnegligible part of the domain. The variational formulation of (1.1) is

$$\begin{cases} \textit{find } u \ \in V \textit{ such that} \\ \mathcal{L}(u,v) := \int_\Omega \varepsilon\nabla u\cdot\nabla v\,dx + \int_\Omega \mathbf{c}\cdot\nabla u\,v\,dx = \int_\Omega f\,v\,dx \quad \forall v\in V. \end{cases} \tag{1.2}$$

1991 *Mathematics Subject Classification.* 65N30, 65N15, 65N50.
Key words and phrases. Residual free bubbles, stabilisations.

Ex 2: **Linear elliptic problems with composite materials**: find u in $V := H^1_0(\Omega)$ such that:

$$Lu := -\nabla \cdot (\alpha(x)\nabla u) = f \ \text{ in } \Omega; \quad u = 0 \ \text{ on } \partial\Omega. \tag{1.3}$$

As before, Ω is, say, a convex polygon, and f a given smooth forcing term. The (given) scalar function $\alpha(x)$ is assumed to be grater than a given positive constant α_0 in the whole domain Ω, and represents, somehow, the characteristics of a composite material. The numerical approximation of (1.3) becomes nontrivial when α has a *fine structure*, exhibiting sharp changes on a scale that is *much smaller* than the diameter of Ω. The variational formulation of (1.3) is

$$\begin{cases} \textit{find } u \ \in V \textit{ such that} \\ \mathcal{L}(u,v) := \int_\Omega \alpha(x)\nabla u \cdot \nabla v \, dx = \int_\Omega f\, v \, dx \quad \forall v \in V. \end{cases} \tag{1.4}$$

The first example corresponds to problems where an unsuited numerical scheme can generate spurious oscillations in the numerical solution, which are not present in the exact solution (that in general, will just exhibit a boundary layer near the part of the boundary where $\mathbf{c} \cdot \mathbf{n} \geq 0$, where $\mathbf{n}$ is the outward unit vector normal to $\partial\Omega$.

On the contrary, the second example corresponds to problems where a fine structure is already present, all over the domain, and needs to be captured by the numerical scheme, at an affordable cost.

In the sequel, we are going to give the basic idea of a general strategy that can prove useful, possibly in different ways, for both types of problems.

2. The residual free bubbles approach

We notice, to start with, that the two problems presented in the Introduction have variational formulations sharing the same structure:

$$\begin{cases} \textit{find } u \ \in V \textit{ such that} \\ \mathcal{L}(u,v) = (f,v) \quad \forall v \in V, \end{cases} \tag{2.1}$$

where, in both cases, $V := H^1_0(\Omega)$ and, from now on, $(\ ,\ )$ denotes the inner product in $L^2(\Omega)$. The difference is just in the type of bilinear form $\mathcal{L}(u,v)$ to be used for each problem.

Fixing our ideas on either one of the abstract formulations (2.1), we assume now that we are given a decomposition $\mathcal{T}_h$ of Ω into triangles, with the usual nondegeneracy requirements. For the sake of simplicity we assume that we start with finite element spaces V_h made of piecewise linear continuous functions vanishing on $\partial\Omega$. We also *play the game* that the dimension of V_h is the biggest one we are ready to afford, in the end, when we solve the final system of linear equations. However, we are ready to afford some extra work, as a pre-processor before building the stiffness matrix, provided that such work could be done *in parallel*, and in particular element-by-element.

Under these assumptions (that is given these *rules*) we can proceed as follows. We start by considering the space of *bubbles*

$$B_h := \Pi_K B_h(K), \qquad B_h(K) := H_0^1(K) \quad \forall K \in \mathcal{T}_h. \tag{2.2}$$

We consider now the *augmented space*

$$V_A := V_h \oplus B_h, \tag{2.3}$$

and the corresponding *augmented problem*

$$\begin{cases} find\ u\ \in V_A\ such\ that \\ \mathcal{L}(u_A, v_A) = (f, v_A) \quad \forall v_A \in V_A. \end{cases} \tag{2.4}$$

Notice that (2.4) is infinite dimensional, and therefore unsolvable. Still we can consider it, for the moment at the level of an abstract speculation. We then notice that, according to (2.3), we can split u_A as $u_A = u_h + u_B$. In its turn, u_B will be a sum of local bubble functions u_B^K, that is: $u_B = \sum_K u_B^K$. Therefore, in each $K \in \mathcal{T}_h$ we can take $v \in B_h(K)$ and obtain, from (2.4) that the restriction u_B^K of u_B to K is the unique solution of the following **local bubble equation**:

$$\begin{cases} find\ u_B^K\ \in B_h(K)\ such\ that \\ \mathcal{L}(u_B^K, v) = -\mathcal{L}(u_h, v) + (f, v) \quad \forall v \in B_h(K). \end{cases} \tag{2.5}$$

Equation (2.5), if solvable, would allow to express each u_B^K in terms of u_h. At the formal level, we can introduce the *solution operator* $\mathcal{S}_K$, that associates to every function g (for instance in $L^2(K)$) the solution $\mathcal{S}_K(g) \in H_0^1(K)$ of

$$\mathcal{L}(\mathcal{S}_K(g), v) = (g, v) \quad \forall v \in H_0^1(K) \tag{2.6}$$

and write the solution u_B^K of (2.5) as $u_B^K = \mathcal{S}_K(f - Lu_h)$. We are now ready to go back to (2.4), take $v = v_h$, and substitute in $u_A = u_h + u_B$ its expression as given by (2.5) and (2.6) to obtain

$$\mathcal{L}(u_h, v_h) - \sum_K \mathcal{L}(\mathcal{S}_K(Lu_h), v_h) = (f, v_h) - \sum_K \mathcal{L}(\mathcal{S}_K(f), v_h) \quad \forall v_h \in V_h. \tag{2.7}$$

This is the linear system that, in the end, we are going to solve. It can be seen (see e.g. [2], [7], [3], [4], [6]) that, for the firs t example, this corresponds to classical stabilized methods like SUPG (see e.g. [8], [9]). For the second exampl e, this would correspond to a two-level method of the type of the ones studied, for instance, in [13], [14]. Clearly, the major difficulty is in the actual solution of the local problems (2.5) that, in principle, present difficulties that look similar to solving the original problems. However, looking at (2.7), we notice that, in practice, we have to evaluate only terms of the type $\mathcal{L}(\mathcal{S}_K(g), v_h)$ that, in turn, can be written as $(\mathcal{S}_K(g), L^* v_h)$, where L^* is the adjoint operator of L. In our two examples we have $L^* v = -\varepsilon \Delta u - \mathbf{c} \cdot \nabla u$ for the first one, and $L* \equiv L$ for the second one (where L is self-adjoint). An important observation is now that, considering for instance the first example, $L^* v_h$ will be *constant* in each element. Hence, only the *mean value* of $\mathcal{S}_K(g)$ is needed. This implies that a rough approximate solution of (2.5) could still be acceptable. This will not be the case for our second example, where $\mathcal{S}_K(g)$ will be

integrated against a term depending on $\alpha(x)$. This term however will have a very definite structure, that we might think of to exploit. It is also possible to check that, in order to compute the terms depending on $\mathcal{S}_K$ appearing in (2.7), it is sufficient to compute the quantities

$$S^K_{i,j} := (\mathcal{S}_K(v^j_h), L^* v^i_h) \quad and \quad F_i := L^*(\mathcal{S}_K(f), v^i_h) \quad \forall i,j \quad \forall K \in \mathcal{T}_h, \tag{2.8}$$

where the v^i_h are the usual nodal basis for V_h. Clearly the terms appearing in (2.8) have to be computed in some *approximate* way, ; see for instance [7], [5], [10].

However, the implementation could also follow a path that is apparently quite different. Indeed, to every basis function $v^i_h \in V_h$ we can associate two other functions w_i and w^*_i that, in each K, are solutions of the problems

$$Lw_i = 0 \quad in\ K \qquad w_i = v^i_h \quad on\ \partial K, \tag{2.9}$$

and

$$L^* w^*_i = 0 \quad in\ K \qquad w^*_i = v^i_h \quad on\ \partial K. \tag{2.10}$$

Clearly $w_i = w^*_i$ whenever L is selfadjoint. It can be checked that the nodal values of the solution u_A of (2.4) *coincide* with the nodal values of the solution of the problem: find w_h, linear combination of the w_j's, such that

$$\mathcal{L}(w_h, w^*_i) = (f, w^*_i) \quad \forall i = 1, ..., dim(V_h). \tag{2.11}$$

On the other hand, the computation of the solution in the form (2.11) requires essentially the same amount of work as the computation in the form (2.7). It is also interesting to notice that, for the first example, this corresponds to the use of suitable basis functions (adapted to the operator) in the

Petrov-Galerkin formulation, as discussed, for instance, in [15]. For the second example, (2.11) is actually the original formulation of [13]. For applications of these concepts to different problems see for instance [11], [12], [10], [1].

References

[1] T. Arbogast, " Numerical subgrid upscaling of two-phase flow in porous media,"in "Multiphase flows and transport in porous media: State of the art", (Z. Chen, R.E. Ewing, and Z.-C. Shi eds.), Lecture Notes in Physics, Springer, Berlin, 2000.

[2] F. Brezzi, M.-O. Bristeau, L.P. Franca, M. Mallet, and G. Rogé, "A Relationship between Stabilized Finite Element Methods and the Galerkin Method with Bubble Functions," Comput. Methods Appl. Mech. Engrg. **96**, 117-129 (1992).

[3] F. Brezzi, L.P. Franca, T.J.R. Hughes, and A. Russo, " $b = \int g$," Comput. Methods Appl. Mech. Engrg. **145**, 329-339 (1997).

[4] F. Brezzi, T.J.R. Hughes, L.D. Marini, A. Russo, and E. Süli, "A priori error analysis of a finite element method with residual-free bubbles for advection-dominated equations." SIAM J. Num. Anal. **36**, 1933-1948 (1999)

[5] F. Brezzi, D. Marini, and A. Russo, "Applications of pseudo residual-free bubbles to the stabilization of convection-diffusion problems," Comput. Methods Appl. Mech. Engrg. **166**, 51-63 (1998).

[6] F. Brezzi, D. Marini, and E. Süli, "Residual-free bubbles for advection-diffusion problems: the general error analysis." To appear in Numer. Math.

[7] F. Brezzi, A. Russo, "Choosing bubbles for advection-diffusion problems," Math. Mod. and Meth. in Appl. Sci. **4**, 571-587 (1994).
[8] A.N. Brooks, T.J.R. Hughes, "Streamline Upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations," Comput. Methods Appl. Mech. Engrg. **32**, 199-259 (1982).
[9] L.P. Franca, S.L. Frey and T.J.R. Hughes, "Stabilized finite element methods: I. Applications to advective-diffusive model," Comput. Methods Appl. Mech. Engrg. **95**, 253-276 (1992).
[10] L.P. Franca, A.P. Macedo, "A Two-Level Finite Element Method and its Application to the Helmholtz Equation,"Int. J. Num. Meth. in Eng. **43**, 23-32 (1998).
[11] L.P. Franca, A. Russo, "Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles." Appl. Math. Lett. **9**, 83-88 (1996).
[12] L.P. Franca, A. Russo, "Approximation of the Stokes problem by residual-free macro bubbles," East-West J. Numer. Math. **4**, 265-278 (1996).
[13] T.Y. Hou, X.H. Wu, "A multiscale finite element method for elliptic problems in composite materials and porous media," J. Comput. Phys. **134**, 169-189 (1997).
[14] T.Y. Hou, X.H. Wu, and Z. Cai, "Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients." Math. of Comp. **68**, 913-943 (1999)
[15] A.R. Mitchell, D.F. Griffiths, "Generalised Galerkin methods for second order equations with significant first derivative terms." In: Proc. bienn. Conf., Dundee 1977, Lect. Notes Math **630**, 90-104 (1978).

Franco Brezzi, Dipartimento di Matematica and I.A.N.-C.N.R., Via Ferrata 1, 27100 Pavia, Italy

E-mail address: brezzi@dragon.ian.pv.cnr.it