THE INF-SUP CONDITION, THE BUBBLE, AND THE SUBGRID

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PLAN OF THE TALK

- In the beginning there was STEFEN BANACH
- The CLOSED RANGE THEOREM
- Consequences of the closed range theorem
- Mixed formulations and their stability
- The Stokes war
- The message

STEFAN BANACH (March 30 1892 - August 31 1945), a great Polish mathematician, one of the moving spirits of the Lvov school of mathematics in pre-war Poland. He was largely self-taught in mathematics; his genius was accidentally discovered by Hugo Steinhaus. When World War II began, Banach was President of the Polish Mathematical Society and a full professor of Lvov University. Being on good terms with Soviet mathematicians, he was allowed to hold his chair during the Soviet occupation of Lvov. The German occupation of the city in 1941 resulted in the mass murder of Polish academics. Banach survived, but the only way he could work for a living was by feeding lice with his blood in a German institute where typhoid fever research was conducted. His health undercut during the occupation, Banach died before he could be repatriated from Lvov, which was incorporated into the Soviet Union, to Poland after the war.

DUAL NOTATION

Let W and Z be Banach spaces, and W' and Z' their dual spaces. We recall that the dual norms are defined as

$$\|f\|_{W'} := \sup_{w} \frac{{}_{W'}\langle f, w\rangle_W}{\|w\|_W} \qquad \|g\|_{Z'} := \sup_{z} \frac{{}_{Z'}\langle g, z\rangle_Z}{\|z\|_W}.$$

Given a linear continuous operator $M: \ W \to Z$ its dual operator $M^t: \ Z' \to W'$ is defined for all $g \in Z'$ as

$${}_{W'}\langle M^t g, w\rangle_W := \ {}_{Z'}\langle g, Mw\rangle_Z \quad \forall w \in W.$$

We also define $R(M)$ to be the image of M and $ker(M)$ to be its kernel (that is the set of $w \in W$ such that $Mw = 0$).

Finally we define the polar set of a $\Phi \subset W$ as

$$\Phi^0 := \{f \in W' \text{ such that } \ {}_{W'}\langle f, \phi\rangle_W = 0 \ \forall \phi \in \Phi\}.$$

THE BANACH CLOSED RANGE THEOREM

(Banach 1932, Yosida 1966)

Let W and Z be Banach spaces, and M a continuous linear operator $M: W \to Z$. Then the following conditions are all equivalent

$$R(M) \text{ is closed } \leftrightarrow R(M^t) \text{ is closed } \leftrightarrow$$
$$R(M) = (ker(M^t))^0 \leftrightarrow R(M^t) = (ker(M))^0.$$

Consequence: we note first that the property M is *bounding*, that is

$$\exists c > 0 \text{ such that } \|Mw\|_Z \geq c\|w\|_W \quad \forall w \in W,$$

is equivalent to say that $R(M)$ is closed and M is injective. Hence (CRT) M^t is surjective. In a similar way, if M^t is bounding, then M is surjective.

OPERATORS AND BILINEAR FORMS

Assume now that we have two reflexive Banach spaces W and S, and a bilinear continuous form $m : W \times S \to \mathbb{R}$. To m we associate in a natural way two operators $M : W \to S'$ and $M^t : S \to W'$ in the following way:

$$_{S'}\langle M\, w, s\rangle_S \equiv m(w,s) \equiv \; _W\langle w, M^t\, s\rangle_{W'}.$$

In terms of the bilinear form m we have then

$$\|M\, w\|_{S'} \equiv \sup_{s \in S} \frac{m(w,s)}{\|s\|_S} \quad \forall w \in W$$

and

$$\|M^t\, s\|_{W'} \equiv \sup_{w \in W} \frac{m(w,s)}{\|w\|_W} \quad \forall s \in S.$$

WRITING *BOUNDING* AS AN *INF-SUP*

Let the bilinear form m and the operators M and M^t be as before. We recall that $M : W \rightarrow S'$ is bounding if

$$\exists c > 0 \text{ such that } \|M\, w\|_{S'} \geq c\, \|w\|_W \qquad \forall w \in W,$$

and $M^t : S \rightarrow W'$ is bounding if

$$\exists c > 0 \text{ such that } \|M^t\, s\|_{W'} \geq c\, \|s\|_S \qquad \forall s \in S.$$

Hence, for instance, M^t is bounding means: there exists a $c > 0$ such that

$$c\, \|s\|_S \;\leq\; \|M^t\, s\|_{W'} \equiv \sup_w \frac{m(w,s)}{\|w\|_W} \;\; \forall s \in S,$$

that is (for $s \neq 0$)

$$c \;\leq\; \sup_w \frac{m(w,s)}{\|w\|_W \|s\|_S} \;\; \forall s \in S \text{ that is } c \;\leq\; \inf_s \sup_w \frac{m(w,s)}{\|w\|_W \|s\|_S}$$

THE LAX-MILGRAM LEMMA (1954)

Let W be a Hilbert space, and m a bilinear continuous form $W \times W \to \mathbb{R}$. Assume that there exists a $\mu > 0$ such that

$$m(w,w) \geq \mu \|w\|_W^2 \qquad \forall w \in W.$$

Then the associated operator M is an isomorphism $W \to W'$.

Proof: We have immediately, for every $w \in W$:

$$\|M^t w\|_{W'} \equiv \sup_v \frac{{}_W\langle v, M^t\, w\rangle_{W'}}{\|v\|_W} \equiv \sup_v \frac{m(v,w)}{\|v\|_W} \geq \frac{m(w,w)}{\|w\|_W} \geq \mu \|w\|_W .$$

Hence M^t is bounding, and (CRT) M is surjective. On the other hand ${}_{W'}\langle M\, w, w\rangle_W \geq \mu\, \|w\|_W^2$ implies that M is injective. QED

NEČAS (1956), BABUŠKA (1971)

Let W and S be Hilbert spaces, and m a bilinear continuous form $W \times S \to \mathbb{R}$. Then $M : W \to S'$ is an isomorphism if and only if there exist two positive constants μ_1 and μ_2 such that

$$\sup_{w} \frac{m(w,s)}{\|w\|_W} \geq \mu_1 \|s\|_S \;\; \forall s \in S \quad \text{and} \quad \sup_{s} \frac{m(w,s)}{\|s\|_S} \geq \mu_2 \|w\|_W \;\; \forall w \in W.$$

Proof: The two *sup*'s mean that $\|M^t s\|_{W'} \geq \mu_1 \|s\|_S \; \forall s \in S$ and that $\|M w\|_{S'} \geq \mu_2 \|w\|_W \; \forall w \in W$, respectively. The first tells us (CRT) that M is surjective. The second that M is injective. Hence M is an isomorphism. The converse is obvious. QED

BABUŠKA (1972)

Let W and S be Hilbert spaces, and m a bilinear continuous form $W \times S \to \mathbb{R}$. Then $M : W \to S'$ is an isomorphism if and only if

$$\sup_w \frac{m(w,s)}{\|w\|_W} > 0 \quad \forall s \neq 0 \text{ in } S \quad \text{and} \quad \inf_w \sup_s \frac{m(w,s)}{\|w\|_W \, \|s\|_S} \geq \mu > 0.$$

Proof: The second condition means $\|M\,w\|_{S'} \geq \mu \|w\|_W \ \forall w \in W$ and then that $R(M)$ is closed and M is injective. The first says that $\|M^t\, s\|_{W'} > 0 \ \forall s \neq 0$, so that M^t is injective. Then (CRT) M is surjective and hence it is an isomorphism. The converse is obvious. QED

STABILITY IN THE SUBSPACE ⇒ OPTIMAL ERROR BOUNDS

Assume that W is a Hilbert space and m is a continuous bilinear form $W \times W \to \mathbb{R}$. Assume that M is an isomorphism $W \to W'$.

For every $f \in W'$ and for every finite dimensional subspace W_h we can consider the approximated problem

$$\text{find } w_h \in W_h \text{ such that}$$
$$m(w_h, v_h) = {}_{W'}\langle f, v_h\rangle_W \quad \forall v_h \in W_h.$$

Note that, setting $w = M^{-1}f$, if the discrete problem has a solution w_h then we immediately have the so-called Galerkin Orthogonality :

$$\text{(GO)} \qquad m(w - w_h, v_h) = 0 \; \forall v_h \in V_h.$$

We are going to see that stability in W_h immediately implies optimal error bounds.

BABUŠKA (1972)

In the above hypotheses, assume also that we have stability in W_h, that is $\exists \mu > 0$ such that $\mu \, \|v_h\|_W \le \|M \, v_h\|_{W_h'} \quad \forall v_h \in W_h$ (that, if you prefer, you can write as an *inf-sup*). Then the discrete problem has a unique solution. Moreover for all $w_I \in W_h$ we have

$$\|w - w_h\|_W \le \left(1 + \frac{\|m\|}{\mu}\right) \|w - w_I\|_W .$$

Proof: We have for every $w_I \in W_h$

$$\mu \|w_I - w_h\|_W \le \|M(w_I - w_h)\|_{W_h'} \equiv \sup_{v_h} \frac{m(w_I - w_h, v_h)}{\|v_h\|_W}$$

$$(\text{ use GO}) \equiv \sup_{v_h} \frac{m(w_I - w, v_h)}{\|v_h\|_W} \le \|m\| \|w - w_I\|_W ,$$

and the result follows by triangle inequality. QED

MIXED FORMULATIONS

We are given two Hilbert spaces V and Q and two continuous bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$. For every right-hand side $f \in V', \ g \in Q'$ we consider the problem:

$$
\begin{aligned}
&\text{find } u \in V \text{ and } p \in Q \text{ such that:}\\
&a(u,v) + b(v,p) = {}_{V'}\langle f, v\rangle_V \qquad && \forall v \in V\\
&b(u,q) = {}_{Q'}\langle g, q\rangle_Q && \forall q \in Q.
\end{aligned}
$$

If we ask that our problem has a unique solution for every possible right-hand side $f \in V', \ g \in Q'$, we have to prove that the operator

$$
M := \begin{bmatrix} A & B^t \\ B & 0 \end{bmatrix}
$$

is an isomorphism from $V \times Q$ to $V' \times Q'$.

EXAMPLES OF MIXED FORMULATIONS

- Darcy: $-\Delta\, p = f \;\; in \;\Omega, \quad p = 0 \;\; on \;\; \partial\Omega. \quad \mathbf{u} = \nabla\, p, \quad -div\, \mathbf{u} = f$

$$V = H(div), \quad Q = L^2, \quad A = I, \quad B = div, \quad B^t = -\nabla$$

- Plates: $\Delta^2 p = f \;\; in\, \Omega, \;\; p = \frac{\partial p}{\partial n} = 0 \; on\, \partial\Omega. \;\; u = -\Delta p, \;\; -\Delta u = f$

$$V = H^1, \quad Q = H^1_0, \quad A = I, \quad B = \Delta, \quad B^t = \Delta$$

- Stokes: $-\Delta\mathbf{u} + \nabla p = \mathbf{f} \;\; in \;\; \Omega, \;\; div\,\mathbf{u} = 0 \;\; in \;\; \Omega, \quad \mathbf{u} = \mathbf{0} \;\; on \;\; \partial\Omega.$

$$V = (H^1_0)^2, \quad Q = L^2_{/\mathbb{R}}, \quad A = -\Delta, \quad B = div, \quad B^t = -\nabla$$

PRELIMINARY WORKS

At the beginning of the '70, several people had tackled problems in mixed form, and had succeeded to prove error estimates with ad hoc arguments. I will mention M. Fortin '72 ($P_2 - P_0$ element for Stokes) Babuška '72 and '73 (Dirichlet problem for $-\Delta + \lambda I$ with Lagrange mutlipliers), C. Johnson '73 (Hellan-Herrmann method for plates), Crouzeix-Raviart '73 ($P_1^{nc} - P_0$ and $(P_2 + B_3) - P_1$ for Stokes). The common instruments were

- The ellipticity of a ($\exists \alpha > 0$ s. t. $a(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V$).
- The bounding of B^t ($\exists \beta > 0$ s. t. $\|B^t q_h\|_{V_h'} \geq \beta \|q_h\|_Q^2 \quad \forall q_h \in Q_h$).

Different instruments were used (for plate problems) by Kikuchi-Ando '72 and Miyoshi '72/73.

B. (1974 CRAS) (another 2-lines proof)

Assume that you have the ellipticity of a and the bounding of B^t. Then the operator M is an isomorphism $V \times Q \to V' \times Q'$.

Proof: Consider for $\varepsilon > 0$ the penalized problem

$$\begin{aligned} a(u_\varepsilon, v) + b(v, p_\varepsilon) &= {}_{V'}\langle f, v\rangle_V && \forall v \in V \\ -b(u_\varepsilon, q) + \varepsilon((p_\varepsilon, q))_Q &= -{}_{Q'}\langle g, q\rangle_Q && \forall q \in Q. \end{aligned}$$

For every $\varepsilon > 0$ you have a unique solution (Lax-Milgram). From the first equation ($A\,u_\varepsilon + B^t p_\varepsilon = f$) and the bounding of B^t you have

$$\beta \|p_\varepsilon\|_Q \le \|f - A\,u_\varepsilon\|_{V'} \le C\,(\|f\|_{V'} + \|u_\varepsilon\|_V).$$

Taking $v = u_\varepsilon$ and $q = p_\varepsilon$, and using the ellipticity of a you have

$$\alpha \|u_\varepsilon\|_V^2 + \varepsilon \|p_\varepsilon\|_Q^2 = {}_{V'}\langle f, u_\varepsilon\rangle_V - {}_{Q'}\langle g, p_\varepsilon\rangle_Q \le C\,(\|f\|_{V'} + \|g\|_{Q'})\|u_\varepsilon\|_V,$$

so that u_ε, and hence p_ε as well, are bounded uniformly in ε. Then you let $\varepsilon \to 0$. QED

THE THEOREM THAT BABUŠKA COULD HAVE WRITTEN

Assume that you have the ellipticity of a and the bounding of B^t.
Then the operator M is an isomorphism $V \times Q \to V' \times Q'$.

Proof: You want to prove that M is bounding: $\forall (u, p) \in V \times Q$ there exists (v, q) in $V \times Q$ such that $\|(v, q)\|_{V \times Q} \leq C \, \|(u, p)\|_{V \times Q}$ and

$$ {}_{V' \times Q'}\langle M(u, p), (v, q) \rangle_{V \times Q} \geq \gamma \|(u, p)\|^2_{V \times Q}. $$

Use the bounding of B^t to find $v_p \in V$ such that

$$ b(v_p, p) \geq \beta \|p\|^2_Q \quad \text{and} \quad \|v_p\|_V \leq \|p\|_Q. $$

Then take $v = u + k \, v_p$ (with k to be chosen) and $q = -p$ so that

$$ {}_{V' \times Q'}\langle M(u, p), (v, q) \rangle_{V \times Q} = a(u, u + k \, v_p) + b(u + k \, v_p, p) - b(u, p) $$

$$ \geq \alpha \|u\|^2_V + k \, \beta \, \|p\|^2_Q - k \, \|a\| \, \|u\|_V \, \|p\|_Q \geq \gamma (\|u\|^2_V + \|p\|^2_Q) $$

for k small enough. QED

THE THEOREM THAT FORTIN COULD HAVE WRITTEN

Assume that you have the ellipticity of a and the bounding of B^t. Then the operator M is an isomorphism $V \times Q \to V' \times Q'$.

Proof: From the first equation ($A\,u + B^t p = f$) and the bounding of B^t you have

$$\beta\|p\|_Q \leq \|B^t\,p\|_{V'} \leq \|f - A\,u\|_{V'} \leq C\,(\|f\|_{V'} + \|u\|_V).$$

Now use the ellipticity of a and test the first equation against u,

$$\alpha\|u\|_V^2 \leq \ {}_{V'}\langle A\,u, u\rangle_V = \ {}_{V'}\langle f, u\rangle_V - \ {}_V\langle u, B^t p\rangle_{V'}.$$

From the second equation: ${}_V\langle u, B^t p\rangle_{V'} \equiv \ {}_{Q'}\langle Bu, p\rangle_Q = \ {}_{Q'}\langle g, p\rangle_Q$, so that

$$\alpha\|u\|_V^2 \leq \ {}_{V'}\langle f, u\rangle_V - \ {}_{Q'}\langle g, p\rangle_Q \leq C\,(\|f\|_{V'} + \|g\|_{Q'})\|u\|_V.$$

You end up with $\|u\|_V + \|p\|_Q \leq C\,(\|f\|_{V'} + \|g\|_{Q'})$. QED

B. (1974 RAIRO) THE NECESSARY AND SUFFICIENT CONDITION

The operator M is an isomorphism IF AND ONLY IF the following two conditions are satisfied:

- B^t is bounding
- A is an isomorphism $ker(B) \to (ker(B))'$.

The Proof was more than half a page. Then D.N. Arnold (1981) proposed to split $V = P \oplus K$ with $K = ker(B)$ and P its orthogonal complement in V. In the space $P \times K \times Q$ the operator M becomes

$$M := \begin{bmatrix} A_{PP} & A_{PK} & B_P^t \\ A_{KP} & A_{KK} & 0 \\ B_P & 0 & 0 \end{bmatrix}$$

From CRT you see that B^t is bounding iff $B_P : P \to Q'$ (and hence $B_P^t : Q \to P'$) is an isomorphism . QED (Aaarrrggghhh!!!)

MIXING UP THE MIXED FORMULATIONS

In its original abstract work (1971), Babuška used B to denote the whole bilinear form on $W \times S$ (that he called $U_1 \times U_2$). Then, in 1972, he applied the abstract result to a specific mixed formulation, taking $U_1 = U_2 = (V \times Q)$. His first *inf-sup* condition read then

$$\inf_{(u,p)} \sup_{(v,q)} \frac{B((u,p),(v,p))}{\|(u,p)\| \, \|(v,q)\|} \geq \beta > 0.$$

In order to prove this for his particular case, he used (as everybody) the V-ellipticity of a and the bounding of B^t in V_h' (that is a second *inf-sup*, this time for b).

As we have seen, I had an abstract result (iff) using that A was an isomorphism on the kernel of B, and the *inf-sup* condition on b.

From then on, whenever an inf and a sup appeared together, people started to talk of the BB condition (instead of one B (Banach)).

BACK TO OUR EXAMPLES

- Stokes: $-\Delta \mathbf{u} + \nabla p = \mathbf{f} \;\; in \;\; \Omega, \;\; div\, \mathbf{u} = 0 \;\; in \;\; \Omega, \quad \mathbf{u} = \mathbf{0} \;\; on \;\; \partial\Omega.$

$$V = (H^1_0)^2, \quad Q = L^2_{/\mathbb{R}}, \quad A = -\Delta, \quad B = div, \quad B^t = -\nabla$$

- Darcy: $-\Delta\, p = f \;\; in\; \Omega, \quad p = 0 \;\; on \;\; \partial\Omega. \quad \mathbf{u} = \nabla\, p, \quad -div\, \mathbf{u} = f$

$$V = H(div), \quad Q = L^2, \quad A = I, \quad B = div, \quad B^t = -\nabla$$

- Plates: $\Delta^2 p = f \;\; in\, \Omega, \;\; p = \frac{\partial p}{\partial n} = 0 \; on\, \partial\Omega. \;\; u = -\Delta p, \quad -\Delta u = f$

$$V = H^1, \quad Q = H^1_0, \quad A = I, \quad B = \Delta, \quad B^t = \Delta$$

THE MID-SEVENTIES(1974-76)

In the mid-seventies the times were ready to deal (mathematically) with problems more complicated than the standard form of the Laplace operator and its variants. Among zillions of other problems, we had mixed and hybrid methods for various applications:

- 2nd order elliptic problems (B. '74, Oden-Lee '75, Thomas '75, Raviart-Thomas '75 G. Fix '76, Falk '76)
- Kirchhoff plates and biharmonic problems (B.'75, B.-Marini '75, F. Kikuchi '75, B.-Raviart '76)
- Elasticity and elastoplasticity (Mercier '75, C. Johnson '76, Johnson-Mercier '76, Mercier-Falk '76, Falk '76)

For some problems, the ellipticity of a in the kernel would fail in an unrecoverable way, and interesting generalizations were developed (Mercier '74, Ciarlet-Raviart '74, Scholz '76, Miyoshi '76).

ADDING MORE CONFUSION TO THE *inf*'s AND THE *sup*'s

Although many people were working on mixed formulations for various applications, the problem that resisted more strenuously was Stokes problem, and a lot of researchers were fighting against it.

There, the ellipticity of a comes for free ($A = -\Delta, \quad V = (H^1_0)^2$). The *only* difficulty (there) was to prove the *inf-sup* condition for $B = div$ in the proper finite element spaces (or, more often, to find finite element spaces where the property is true).

As Olga Ladyzhenskaya had proved it for the continuous case (from $(H^1_0)^2$ to $L^2_{/\mathbb{R}}$), somebody thought that she also deserved to be mentioned: The BB condition became the LBB condition.

THE STOKES WAR

Stokes, as we saw, was strongly resisting. Some methods were proposed (M. Fortin '75 and '76, Bercovier '76, Girault-Raviart '76, Falk '76, Falk-King '76) but the task was difficult. The most commonly used elements ($Q_1 - P_0$, $Q_2 - Q_1$, Hood-Taylor) were beyond mathematical reach, and no new effective methods were proposed. Engineers were loughing... Four major steps had to arrive

- The Fortin's trick (Fortin '77)
- The analysis of Hood-Taylor's element (Bercovier-Pironneau '77 and Verfürth '84)
- The misterious birth of the $Q_2 - P_1$ element (Banff '80 (?))
- The analysis of the $P_1 - P_0$ element (Johnson-Pitkäranta '82) and the macroelement technique (Stenberg '84)

MATHEMATICIANS 1 - ENGINEERS 0

THE POWER OF THE BUBBLE

A powerful instrument in developing new elements for Stokes and proving their convergence was to increase the finite element velocity space with bubble functions. Here *bubble:=function having its support in a single element*.

It was proved that any possible choice (V_h, Q_h) of finite element spaces can be made stable by changing V_h into $V_h \oplus B_h$ (where B_h is a suitable space of bubbles) provided that

- V_h contains the piecewise linear continuous functions (if Q_h was made of continuous functions)
- V_h contains the piecewise quadratic functions (otherwise)

(B.-Pitkäranta, 1984)

ENGINEERS STRIKE BACK

Another powerful instrument appeared however in the middle of the eighties. The basic idea (inspired by a previous paper of Brooks-Hughes '82) became known as *stabilization à la Hughes-Franca* and can be summarized as adding to the bilinear form terms that equal zero when computed on the exact solution, tested on suitable modifications of the test functions. For instance, for Stokes problem, you can add a term of the form

$$\sum_K \tau_K (-\Delta \mathbf{u}_h + \nabla p_h - \mathbf{f}, \mathcal{S}(\mathbf{v}, q))_K$$

where $\mathcal{S}(\mathbf{v}, q)$ could be either ∇q or $\Delta \mathbf{v} + \nabla q$ or $-\Delta \mathbf{v} + \nabla q$ (and τ_K are suitably chosen positive coefficients $\simeq h_K^2$).

MATHEMATICIANS 1 - ENGINEERS 1

THE MESSAGE

The interesting solution to this battle has been to join forces in order to analyze the relationships among the two types of stabilizations, trying to get the best of each in each different problems.

In particular, the ability of the bubble (or, actually, of the Residual Free Bubble) to capture the subscale phenomena inside each element was underlined, together with its inability to capture the subscale phenomena crossing the interelement boundaries. But all this is *to recent* to be part of a *history* lecture...

From the above battle, we get however "the message": Instead of fighting your friends, fight the problem!

ENGINEERS & MATHEMATICIANS 3 - PROBLEM 0 !