Analisi numerica — *Reissner-Mindlin plates with free boundary conditions .* Nota di
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ABSTRACT — It is well known that the solutions of Reissner-Mindlin equations can have, for small thickness t, severe boundary layers. In particular, near the part of the boundary where the so-called *free plate* boundary conditions are prescribed, the layer can be so strong that rotations are not uniformly bounded in H^2, for $t \to 0$. This is clearly a major drawback for numerical methods, as one cannot achieve error estimates of order h uniformly in t. Here we propose a new model for *free plate* boundary conditions that has less severe layers, and we propose a numerical method that provides a priori error estimates of order $O(h)$ uniformly in t.

KEY WORDS — Reissner-Mindlin, free plate, boundary laters.

RIASSUNTO — *È ben noto che le soluzioni delle equazioni di Reisser-Mindlin possono presentare forti strati limite vicino ai bordi della piastra. Tali strati limite sono particolarmente severi vicino a quelle parti della frontiera dove sono state assegnate condizioni ai limiti cosiddette di piastra libera. In tali casi si sa che le rotazioni non sono uniformemente limitate in H^2, e questo impedisce di avere stime dell'errore dell'ordine di h che siano uniformi nello spessore t. Qui proponiamo una diversa formulazione delle condizioni di piastra libera, che presentano strati limiti meno severi, e proponiamo anche un metodo numerico per il quale si possono dimostrare stime dell'ordine di $O(h)$ uniformemente in t.*

1. INTRODUCTION

In the last twenty years several good and reliable elements have been presented for the solution of Reissner-Mindlin plate equations. We just recall, for instance, [7], [9], [10], [3], [12], [4], [5]. See also [11], [13], [6] and the references therein. All the elements proposed in the above papers have been proved to be completely free from locking. In particular their convergence properties have been proved to be independent of the thickness t, and to be optimal *compared to interpolation estimates.* This implies optimal error estimates (in terms of powers of h), uniform in t, whenever the solution is regular enough, uniformly in t.

On the other hand, it has been proved by Arnold and Falk ([2]) that in general the solution of the Reissner Mindlin equations exhibits a strong boundary layer when t goes to zero. In particular for the simplest case of *clamped* boundary conditions one has that the rotations $\boldsymbol{\theta}(t)$ are uniformly bounded in $H^r(\Omega)$ only for $r < 5/2$. This implies that only the lowest order elements can have optimal estimates, of order $O(h)$ in the energy norm, uniformly in t.

Always following Arnold-Falk [2], the same bound on the regularity holds for the so-called *hard simply supported* boundary conditions, while for *soft simply supported* and for *free* boundary conditions the regularity bound goes down to $r < 3/2$. This puts an upper bound to error estimates in energy to $O(h^{r-1})$ so that no element can have even the "minimal" estimate $O(h)$.

It is not clear whether there are cases in which one *must* use soft simply supported boundary conditions instead of the hard ones. Hence using systematically the *hard* version one can think that, as far as regularity is concerned, simply supported boundary conditions are not worse than the (most studied) clamped ones.

The situation is different for *free* boundary conditions. There, traditionally, a single version is found in the literature (instead of a hard one and a soft one), and there seems to be no viable choice.

In the present paper we propose a different way of modeling the free boundary conditions for Reissner-Mindlin plates (that could be called, possibly, *hard free*). Roughly speaking it corresponds to minimizing the usual Reissner-Mindlin energy functional under the condition that the tangential component of rotations θ_s equals the tangential derivative $w_{/s}$ of the transversal displacement. We recall that in the limit for $t \to 0$ the solution will satisfy the Kirchhoff condition $\boldsymbol{\theta} = \boldsymbol{\nabla} w$, so that the condition $\theta_s = w_{/s}$ will always be true.

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From the general asymptotic analysis of [2] it can be easily obtained that with our *hard-free* boundary conditions the regularity bound goes back to $r < 5/2$, thus allowing, in principle, an $O(h)$ bound for the lowest order elements.

In particular we show here how it is possible to enforce the new boundary conditions for a particular choice of one of the classical low order elements, namely the Duran-Liebermann element [12]. We introduce a minor modification of this element, to be used near the boundary, that is very easy to implement. We show that such modification allows an easy treatment of hard-free boundary conditions, and we prove $O(h)$ a priori estimates in energy that are uniform in t.

An outline of the paper is as follows. In the next section we recall the Reissner-Mindlin equations, we introduce the hard-free boundary conditions and we analyze them from the regularity point of view. In Section 3 we introduce our example of discretization, and we prove the corresponding error bounds in Section 4. Some conclusions are drawn in Section 5.

Throughout the paper we shall use the following notation. $H^r(\mathcal{O})$ will denote the usual Sobolev space $W^{r,2}(\mathcal{O})$ of order r on the domain $\mathcal{O}$. For $r = 0$ we will often use the notation $L^2(\mathcal{O})$ as well. With an abuse of notation we shall use $\|\cdot\|_{r,\mathcal{O}}$, or simply $\|\cdot\|_r$ (when no confusion can arise) to denote the H^r norm of *both* scalar or vector-valued functions. The scalar product in $H^0(\mathcal{O}) \equiv L^2(\mathcal{O})$ will also be denoted by $(\cdot,\cdot)_{0,\mathcal{O}}$, or $(\cdot,\cdot)_0$, or even $(\cdot,\cdot)$, both for scalar and vector-valued functions.

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2. The Reissner-Mindlin equations

Let Ω be an open bounded domain in $\mathbb{R}^2$ and let g be given, say, in $L^2(\Omega)$; the Reissner–Mindlin equations require to find $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ such that

$$-\operatorname{div} \mathbf{C}\,\varepsilon(\boldsymbol{\theta}) - \boldsymbol{\gamma} = \mathbf{0} \quad \text{in } \Omega, \tag{2.1}$$

$$-\operatorname{div} \boldsymbol{\gamma} = g \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{\gamma} = \lambda t^{-2}(\nabla w - \boldsymbol{\theta}) \quad \text{in } \Omega. \tag{2.3}$$

In (2.1)-(2.3), $\mathbf{C}$ is the tensor of bending moduli, $\boldsymbol{\theta}$ represents the rotations, w the transversal displacement, and $\boldsymbol{\gamma}$ the scaled shear stresses. Moreover, ε is the usual symmetric gradient operator, $\lambda(= 5/6)$ is the shear correction factor, and t is the thickness.

Equations (2.1)- (2.3) have to be supplied by suitable boundary conditions. In order to introduce them, we set

$$\mathbf{M} := \mathbf{C}\,\varepsilon(\boldsymbol{\theta}), \quad \mathbf{M_n} := \mathbf{M}\cdot\mathbf{n} \tag{2.4}$$

and

$$M_{nn} := \mathbf{M_n}\cdot\mathbf{n}, \quad M_{ns} := \mathbf{M_n}\cdot\mathbf{s}, \tag{2.5}$$

where $\mathbf{n}$ and $\mathbf{s}$ are, respectively, the outward unit normal and counterclockwise unit tangent vector to $\partial\Omega$. We then assume that the boundary $\partial\Omega$ is split in three nonoverlapping parts $\partial\Omega = \Sigma_c \cup \Sigma_s \cup \Sigma_f$, corresponding to *clamped, (hard) simply supported* and *free* boundary conditions, that we make precise as follows. We require (formally)

$$\boldsymbol{\theta} = \mathbf{0}, \qquad w = 0 \qquad \text{on } \Sigma_c, \tag{2.6}$$

$$\theta_s = 0, \quad w = 0, \quad M_{nn} = 0 \qquad \text{on } \Sigma_s, \tag{2.7}$$

and

$$M_{nn} = 0, \quad M_{ns} = 0, \quad \gamma_n \equiv -(\operatorname{div}\mathbf{M})_n = 0 \qquad \text{on } \Sigma_f, \tag{2.8}$$

where, here and in all the sequel, we adopt the following notation: for every vector valued function $\boldsymbol{\eta}$ and for every scalar function v

$$\eta_n := \boldsymbol{\eta}\cdot\mathbf{n}, \quad \eta_s := \boldsymbol{\eta}\cdot\mathbf{s}, \quad v_{/n} := \frac{\partial v}{\partial \mathbf{n}}, \quad v_{/s} := \frac{\partial v}{\partial \mathbf{s}}. \tag{2.9}$$

We shall make only minor assumptions on the splitting of $\partial\Omega$: we assume that every part is the union of a finite number of connected components, and that every rigid movement $\mathbf{r}$ satisfying $\mathbf{r} = \mathbf{0}$ on Σ_c and $r_s = 0$ on Σ_s is necessarily $\mathbf{0}$. This, together with the usual ellipticity assumptions on $\mathbf{C}$ will give us the well known Korn inequality: there exists a constant $\alpha > 0$ such that for every $\boldsymbol{\eta} \in (H^1(\Omega))^2$ satisfying $\boldsymbol{\eta} = \mathbf{0}$ on Σ_c and $\eta_s = 0$ on Σ_s we have

$$\alpha \|\boldsymbol{\eta}\|^2_{(H^1(\Omega))^2} \leq \int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\eta}) : \varepsilon(\boldsymbol{\eta}) \mathrm{d}x. \tag{2.10}$$

It is known that, keeping g fixed, and letting $t \to 0$, the solution $(\boldsymbol{\theta}^t, w^t, \boldsymbol{\gamma}^t)$ of (2.1)-(2.3) with the boundary conditions (2.6)-(2.8) tends to a finite limit (θ^0, w^0) such that $\theta^0 = \nabla w^0$, and w^0 is the solution of the Kirchhoff model with the boundary conditions

$$w = 0 \quad \text{and} \quad w_{/n} = 0 \quad \text{on } \Sigma_c, \tag{2.11}$$

$$w = 0 \quad \text{and} \quad M_{nn} = 0 \quad \text{on } \Sigma_c, \tag{2.12}$$

$$M_{nn} = 0 \quad \text{and} \quad M_{ns/s} + (\operatorname{div} \mathbf{M})_n = 0 \quad \text{on } \Sigma_f, \tag{2.13}$$

where in the definition of $\mathbf{M}$ (2.4) we obviously have to replace $\boldsymbol{\theta}$ by $\boldsymbol{\nabla} w$. It is also known, however, that the convergence takes place only in Sobolev spaces of rather low order (see [2]). In particular, if Σ_f is not empty, we have that

$$\|\boldsymbol{\theta}(t)\|_r \leq C \tag{2.14}$$

holds, with C independent of t, only for $r < 3/2$. This is essentially due to the fact that the Reissner-Mindlin solution satisfies for every $t > 0$ the boundary condition $M_{ns} = 0$ which does not hold in the limit, hence causing a boundary layer in the first derivatives of $\boldsymbol{\theta}$ that forbids them to belong to $H^r(\Omega)$ for $r \geq 1/2$.

This is the reason why we propose to change (2.8) into

$$\theta_s = w_{/s}, \quad M_{nn} = 0 \quad \text{and} \quad M_{ns/s} - \gamma_n = 0 \quad \text{on } \Sigma_f. \tag{2.15}$$

Introducing the space

$$\begin{aligned} \mathcal{V} := \quad & \{(\boldsymbol{\theta}, w) \in (H^1(\Omega))^2 \times H^1(\Omega) \text{ such that } \boldsymbol{\theta} = \mathbf{0}, w = 0 \text{ on } \Sigma_c, \\ & \theta_s = w = 0 \text{ on } \Sigma_s, \quad \text{and } \theta_s = w_{/s} \text{ on } \Sigma_f\} \end{aligned} \tag{2.16}$$

we have the following result.

Proposition 2.1. *For every $t > 0$, any smooth solution of (2.1)-(2.3), with the boundary conditions (2.6), (2.7), (2.15) coincides with the unique minimizing argument on $\mathcal{V}$ of the functional*

$$J^t(\boldsymbol{\eta}, v) = \frac{1}{2} a(\boldsymbol{\eta}, \boldsymbol{\eta}) + \frac{\lambda t^{-2}}{2} ||\boldsymbol{\nabla} v - \boldsymbol{\eta}||^2_{0,\Omega} - (g, v), \tag{2.17}$$

where

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta}) \mathrm{d}x \equiv \int_\Omega \mathbf{M} : \varepsilon(\boldsymbol{\eta}) \mathrm{d}x. \tag{2.18}$$

Conversely, the unique minimizing argument of (2.17) satisfies (2.1)-(2.3) in the distributional sense, and if it is smooth enough it also satisfies the boundary conditions (2.6), (2.7), (2.15).

Proof. The proof is rather standard, and we do not detail it here. Essentially, we first remark that if $(\boldsymbol{\theta}, w)$ is the minimizing argument of (2.17), then setting $\boldsymbol{\gamma} = \lambda t^{-2}(\boldsymbol{\nabla} w - \boldsymbol{\theta})$ we have that $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ verifies the following variational problem:

$$\left\{ \begin{array}{ll} \quad \text{Find } ((\boldsymbol{\theta}, w), \boldsymbol{\gamma})) \in \mathcal{V} \times (L^2(\Omega))^2 \text{ such that } : & \\ a(\boldsymbol{\theta}, \boldsymbol{\eta}) + (\boldsymbol{\gamma}, \boldsymbol{\nabla} v - \boldsymbol{\eta}) = (g, v) & (\boldsymbol{\eta}, v) \in \mathcal{V}, \\ \lambda^{-1} t^2 (\boldsymbol{\gamma}, \boldsymbol{\delta}) - (\boldsymbol{\nabla} w, \boldsymbol{\delta}) + (\boldsymbol{\theta}, \boldsymbol{\delta}) = 0 & \boldsymbol{\delta} \in (L^2(\Omega))^2, \end{array} \right. \tag{2.19}$$

which implies clearly equations (2.1)-(2.3) in the distributional sense. We just have to check the boundary conditions on Γ_f, as the others are classical. For this we proceed more or less as usual. Let first $(\boldsymbol{\theta}, w)$ be a minimizer of (2.17): for every $(\boldsymbol{\eta}, v) \in \mathcal{V}$ we multiply (2.1) times $\boldsymbol{\eta}$ and we integrate over Ω, then we multiply (2.2) times v and we integrate over Ω, and finally we take the sum of the two. If the minimizer is smooth enough we can integrate by parts, compare with the first equation of (2.19), and obtain

$$\int_{\Sigma_s \cup \Sigma_f} M_{nn}\, \eta_n \,\mathrm{d}\Sigma + \int_{\Sigma_f} M_{ns}\, \eta_s + \gamma_n\, v \,\mathrm{d}\Sigma = 0 \qquad \forall (\boldsymbol{\eta}, v) \in \mathcal{V}. \tag{2.20}$$

Condition $M_{nn} = 0$ on $\Sigma_s \cup \Sigma_f$ follows immediately. Recalling that $\eta_s = v_{/s}$ on Σ_f and integrating by parts along Σ_f we easily get the last equation in (2.15). With similar arguments we prove the first part of the statement: if $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ is a smooth solution of (2.1)-(2.3) with the boundary conditions (2.6), (2.7), (2.15) we can easily see that it satisfies the variational problem (2.19) and hence it is a minimizer of (2.17). □

Remark 2.1. *Assuming, for simplicity, that the material is homogeneous (and hence the system has constant coefficients), that the load g is in $C^\infty(\bar{\Omega})$, and that Ω is a polygon (or that the boundary $\partial\Omega$ is piecewise C^∞), it is not difficult to prove, with the usual techniques (see for instance* [14]*), that for every $t > 0$ each component of the solution belongs to $C^\infty(\overline{D})$ where D is any open subset of Ω such that $\overline{D}$ that does not contain any vertex nor points where the boundary conditions change from one type to another. We do not address here the problem of the global regularity in Ω, as we are more interested in the question of the uniform regularity (in t) in domains that satisfy the same assumptions as the domain D above.*

Concerning the *uniform* regularity (in t) we have the following result.

Proposition 2.2. *In the same assumptions of Remark 2.1, and for every subdomain D (always as in Remark 2.1) there exists a constant C, independent of t, such that*

$$\|\boldsymbol{\theta}\|_{2,D} + \|w\|_{2,D} + t\|\boldsymbol{\gamma}\|_{1,D} + \|\boldsymbol{\gamma}\|_{0,D} \le C\|g\|_{0,\Omega} \tag{2.21}$$

Proof. The proof can be easily obtained by adapting the analysis of [2] to the present situation near Σ_f. We see that the most irregular term in the expansion of the solution (as computed by [2]) drops, leaving for $\boldsymbol{\theta}$ a limit regularity of order $r < 5/2$. □

In the sequel we are going to *assume* that the solution $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ satisfies

$$\|\boldsymbol{\theta}\|_{2,\Omega} + \|w\|_{2,\Omega} + t\|\boldsymbol{\gamma}\|_{1,\Omega} + \|\boldsymbol{\gamma}\|_{0,\Omega} \le C\|g\|_{0,\Omega} \tag{2.22}$$

uniformly in t. In order to show that the assumption is realistic, we consider a case in which $\Omega =]0, L_1[\times]0, L_2[$ and where $\Sigma_s = \{x = 0\} \cup \{x = L_1\}$ while $\Sigma_f = \partial\Omega \setminus \Sigma_s$. Assume that the material obeys the classical Hooke's law. Reflecting g "odd" around $\{x = 0\}$ (and observing that θ_1 and γ_1 reflect "even" and θ_2, w and γ_2 "odd"), we prove regularity in the neighborhood of $(0, 0)$ and $(0, L_2)$. In a similar way one proves regularity in the neighborhood of the other two vertices.

3. Discretization

In this section we are going to set the discretized problem. For this we consider first some finite element spaces. We assume therefore that we are given, as usual, a regular sequence of decompositions $\{\mathcal{T}_h\}_h$, satisfying the minimum angle condition. We denote by $\mathcal{L}_k^r$ the set of piecewise polynomials of degree $\le k$ that are globally in $H^r(\Omega)$. For every triangle $T \in \mathcal{T}_h$ we also define

$$TR(T) := \{\boldsymbol{\delta} | \quad \delta_1 = a + by, \quad \delta_2 = c - bx, \quad a, b, c \in \mathbb{R}\} \tag{3.1}$$

to be the usual rotated Raviart-Thomas element of lowest order. We also define a set of quadratic edge bubbles in the following way: for every edge e we denote by p_e the (unique) polynomial of degree 2 having value 1 at the midpoint of e and vanishing on the other two edges of T. Then we denote by $\boldsymbol{\eta}_e$ the vector valued function $\boldsymbol{\eta}_e = \mathbf{s}_e\, p_e$ where $\mathbf{s}_e$ is the tangent counterclockwise unit vector on e. Finally, we denote by B_T the space spanned by the three (vector valued) "bubbles" $\boldsymbol{\eta}_e$ obtained in that way in correspondence with the three edges of T. We can now set

$$B_h := \left(\bigoplus_{T \in \mathcal{T}_h} B_T \right) \cap (H^1(\Omega))^2, \tag{3.2}$$

$$\boldsymbol{\Theta}_h := (\mathcal{L}_1^1)^2 \oplus B_h, \tag{3.3}$$

$$W_h := \mathcal{L}_1^1, \tag{3.4}$$

$$\boldsymbol{\Gamma}_h := \sum_{T \in \mathcal{T}_h} TR(T). \tag{3.5}$$

Finally we define an interpolation operator Π_h from $\mathcal{V}$ to $\boldsymbol{\Gamma}_h$ as follows

$$\int_e \Pi_h \boldsymbol{\eta} \cdot \mathbf{s} \, \mathrm{d}s = \int_e \boldsymbol{\eta} \cdot \mathbf{s} \, \mathrm{d}s \qquad \forall \text{ edge } e. \tag{3.6}$$

Owing to the basic properties of Raviart-Thomas spaces (and their "rotated" counterpart) we have that (3.6) defines Π_h in a unique way. We also remark that, in particular, we have

$$\boldsymbol{\nabla} W_h \subset \boldsymbol{\Gamma}_h \qquad \text{so that} \qquad \Pi_h(\boldsymbol{\nabla} v_h) = \boldsymbol{\nabla} v_h \quad \forall v_h \in W_h. \tag{3.7}$$

Using the above definitions of $\boldsymbol{\Theta}_h$, W_h, and Π_h we then set

$$\begin{aligned} \mathcal{V}_h := \quad & \{(\boldsymbol{\eta}_h, v_h) \in \boldsymbol{\Theta}_h \times W_h \text{ such that } \boldsymbol{\eta}_h = \mathbf{0}, v_h = 0 \text{ on } \Sigma_c, \\ & (\boldsymbol{\eta}_h)_s = v_h = 0 \text{ on } \Sigma_s, \quad \text{and } (\Pi_h \boldsymbol{\eta}_h)_s = (v_h)_{/s} \text{ on } \Sigma_f\}. \end{aligned} \tag{3.8}$$

Note that the last boundary condition can be imposed by means of a simple condensation of the "tangential bubbles" which are different from zero on the external edges; therefore the space $\mathcal{V}_h$ can be used in practice without particular difficulties.

We can now define the discrete solution $(\boldsymbol{\theta}_h, w_h)$ as the unique minimizer of the functional

$$J_h^t(\boldsymbol{\eta}_h, v_h) = \frac{1}{2} a(\boldsymbol{\eta}_h, \boldsymbol{\eta}_h) + \frac{\lambda t^{-2}}{2} ||\boldsymbol{\nabla} v_h - \Pi_h \boldsymbol{\eta}_h||_{0,\Omega}^2 - (g, v_h) \tag{3.9}$$

over the discrete space $\mathcal{V}_h$. It is then elementary to see that, setting

$$\boldsymbol{\gamma}_h := \lambda t^{-2} (\boldsymbol{\nabla} w_h - \Pi_h \boldsymbol{\theta}_h), \tag{3.10}$$

the triple $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h)$ coincides with the unique solution of the variational problem

$$\left\{ \begin{array}{ll} \quad \text{Find } ((\boldsymbol{\theta}_h, w_h), \boldsymbol{\gamma}_h)) \in \mathcal{V}_h \times \boldsymbol{\Gamma}_h \text{ such that } : & \\ a(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) - (\boldsymbol{\gamma}_h, \Pi_h \boldsymbol{\eta}_h - \boldsymbol{\nabla} v_h) = (g, v_h) & (\boldsymbol{\eta}_h, v_h) \in \mathcal{V}_h, \\ \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \boldsymbol{\delta}_h) - (\boldsymbol{\nabla} w_h, \boldsymbol{\delta}_h) + (\Pi_h \boldsymbol{\theta}_h, \boldsymbol{\delta}_h) = 0 & \boldsymbol{\delta}_h \in \boldsymbol{\Gamma}_h. \end{array} \right. \tag{3.11}$$

4. A priori error estimates

In this section we shall prove a priori bounds for the error $(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)$. As a first step we define suitable interpolants of $\boldsymbol{\theta}$ and w:

$$\boldsymbol{\theta}_I(P) = \boldsymbol{\theta}(P) \text{ for all vertex } P \quad \text{ and } \int_e (\boldsymbol{\theta} - \boldsymbol{\theta}_I) \cdot \mathbf{s} \, \mathrm{d}s = 0 \text{ for all edge } e, \tag{4.1}$$

$$w_I(P) = w(P) \text{ for all vertex } P. \tag{4.2}$$

It is easy to check that both $\boldsymbol{\theta}_I$ and w_I are well defined. We recall that, on Σ_f, we have $\theta_s = w_{/s}$. We also note that for every element in $\boldsymbol{\Gamma}_h$ its tangential component on every edge is constant. Using this, (3.6), (4.1) and $\theta_s = w_{/s}$ we easily get, on every edge $e \in \Sigma_f$

$$(\Pi_h \boldsymbol{\theta}_I) \cdot \mathbf{s}_{|e} = \frac{1}{|e|} \int_e (\Pi_h \boldsymbol{\theta}_I) \cdot \mathbf{s} \, \mathrm{d}s = \frac{1}{|e|} \int_e (\boldsymbol{\theta}_I \cdot \mathbf{s}) \, \mathrm{d}s = \frac{1}{|e|} \int_e \boldsymbol{\theta}_s \, \mathrm{d}s = \frac{1}{|e|} \int_e w_{/s} \, \mathrm{d}s. \tag{4.3}$$

If P_1 and P_2 are the endpoints of e we can use (4.2), and the fact that w_I is piecewise linear, to continue (4.3) as follows

$$(4.4)\qquad (\Pi_h\boldsymbol{\theta}_I)\cdot\mathbf{s}_{|e} = \frac{1}{|e|}\int_e w_{/s}\,\mathrm{d}s = \frac{(w(P_2)-w(P_1))}{|P_2-P_1|} = \frac{(w_I(P_2)-w_I(P_1))}{|P_2-P_1|} = (w_I)_{/s} \text{ on } e,$$

so that actually $(\boldsymbol{\theta}_I, w_I) \in \mathcal{V}_h$, as the other requirements are obviously satisfied as well. We point out that, arguing as in (4.3) and using (3.6) we easily have

$$(4.5)\qquad (\Pi_h\boldsymbol{\theta}_I)\cdot\mathbf{s}_{|e} = \frac{1}{|e|}\int_e \boldsymbol{\theta}_s\,\mathrm{d}s = (\Pi_h\boldsymbol{\theta})\cdot\mathbf{s}_{|e},$$

that immediately implies

$$(4.6)\qquad \Pi_h\boldsymbol{\theta}_I = \Pi_h\boldsymbol{\theta}.$$

Similarly we can recall the steps in (4.4)

$$(4.7)\qquad \frac{1}{|e|}\int_e w_{/s}\,\mathrm{d}s = (w_I)_{/s} \text{ on } e,$$

and using again (3.6) and (3.7) we obtain

$$(4.8)\qquad \boldsymbol{\nabla} w_I = \Pi_h\boldsymbol{\nabla} w_I = \Pi_h\boldsymbol{\nabla} w.$$

We finally set

$$(4.9)\qquad \boldsymbol{\gamma}_I = \lambda t^{-2}(\boldsymbol{\nabla} w_I - \Pi_h\boldsymbol{\theta}_I)$$

Using (4.6) and(4.8) in (4.9) it is easy to check that

$$(4.10)\qquad \boldsymbol{\gamma}_I = \lambda t^{-2}(\Pi_h\boldsymbol{\nabla} w_I - \Pi_h\boldsymbol{\theta}_I) = \lambda t^{-2}(\Pi_h\boldsymbol{\nabla} w - \Pi_h\boldsymbol{\theta}) \equiv \Pi_h\boldsymbol{\gamma},$$

which will play a fundamental role for our proof.

Before deriving the error equations we first notice that the space $\mathcal{V}_h$, as defined in (3.8), *is not* a subspace of $\mathcal{V}$, defined in (2.16). As a consequence, for $(\boldsymbol{\eta}, v) \in \mathcal{V}_h$ we have, integrating by parts and using (2.1)-(2.2), using boundary conditions (2.6), (2.7), then using (2.15), and finally (3.8):

$$(4.11)\qquad \begin{aligned} a(\boldsymbol{\theta},\boldsymbol{\eta}) + {}& (\boldsymbol{\gamma}, \boldsymbol{\nabla} v - \boldsymbol{\eta}) - (g, v) = \\ & \textstyle\int_{\Sigma_f} M_{ns}\eta_s + \boldsymbol{\gamma}_n v\,\mathrm{d}s = \int_{\Sigma_f} M_{ns}(\eta_s - v_{/s})\,\mathrm{d}s = \int_{\Sigma_f} M_{ns}(\eta - \Pi_h\eta)_s\,\mathrm{d}s. \end{aligned}$$

We compare then (4.11) with the first equation of (3.11) to obtain, for $(\boldsymbol{\eta}, v) \in \mathcal{V}_h$,

$$(4.12)\qquad a(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\eta}) + (\boldsymbol{\gamma}, \boldsymbol{\nabla} v - \boldsymbol{\eta}) - (\boldsymbol{\gamma}_h, \boldsymbol{\nabla} v - \Pi_h\boldsymbol{\eta}) = \int_{\Sigma_f} M_{ns}(\eta - \Pi_h\eta)_s\,\mathrm{d}s,$$

that can also be rewritten as

$$(4.13)\qquad a(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\eta}) + (\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \boldsymbol{\nabla} v - \Pi_h\boldsymbol{\eta}) = (\boldsymbol{\gamma}, (I-\Pi_h)\boldsymbol{\eta}) + \int_{\Sigma_f} M_{ns}((I-\Pi_h)\eta)_s\,\mathrm{d}s.$$

Using Korn inequality and adding and subtracting $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ we have

$$(4.14)\qquad \begin{aligned} \alpha\|\boldsymbol{\theta}_I-\boldsymbol{\theta}_h\|_1^2 + \lambda^{-1}t^2\|\boldsymbol{\gamma}_I-\boldsymbol{\gamma}_h\|_0^2 = {}& a(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h,\boldsymbol{\theta}_I-\boldsymbol{\theta}_h) + \lambda^{-1}t^2(\boldsymbol{\gamma}_I-\boldsymbol{\gamma}_h,\boldsymbol{\gamma}_I-\boldsymbol{\gamma}_h) = \\ & a(\boldsymbol{\theta}_I-\boldsymbol{\theta},\boldsymbol{\theta}_I-\boldsymbol{\theta}_h) + a(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\theta}_I-\boldsymbol{\theta}_h) \\ & +\lambda^{-1}t^2(\boldsymbol{\gamma}_I-\boldsymbol{\gamma},\boldsymbol{\gamma}_I-\boldsymbol{\gamma}_h) + \lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\gamma}_I-\boldsymbol{\gamma}_h). \end{aligned}$$

On the other hand, as $\boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h = \lambda t^{-2}(\boldsymbol{\nabla}(w_I - w_h) - \Pi_h(\boldsymbol{\theta}_I - \boldsymbol{\theta}_h))$ we have from the error equation (4.13) (used with $\eta = \boldsymbol{\theta}_I - \boldsymbol{\theta}_h$ and $v = w_I - w_h$):

$$(4.15)\qquad \begin{aligned} & a(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\theta}_I-\boldsymbol{\theta}_h) + \lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\gamma}_I-\boldsymbol{\gamma}_h) = \\ & (\boldsymbol{\gamma}, (I-\Pi_h)(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)) + \textstyle\int_{\Sigma_f} M_{ns}(I-\Pi_h)(\theta_I-\theta_h)_s\,\mathrm{d}s. \end{aligned}$$

Combining (4.14) and (4.15) we have then

$$(4.16)\qquad \alpha\|\boldsymbol{\theta}_I-\boldsymbol{\theta}_h\|_1^2 + \lambda^{-1}t^2\|\boldsymbol{\gamma}_I-\boldsymbol{\gamma}_h\|_0^2 = I + II + III + IV$$

where

$$I = a(\boldsymbol{\theta}_I - \boldsymbol{\theta}, \boldsymbol{\theta}_I - \boldsymbol{\theta}_h), \tag{4.17}$$

$$II = \lambda^{-1} t^2 (\boldsymbol{\gamma}_I - \boldsymbol{\gamma}, \boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h), \tag{4.18}$$

$$III = (\boldsymbol{\gamma}, (I - \Pi_h)(\boldsymbol{\theta}_I - \boldsymbol{\theta}_h)), \tag{4.19}$$

$$IV = \int_{\Sigma_f} M_{ns} (I - \Pi_h)(\theta_I - \theta_h)_s \, \mathrm{d}s, \tag{4.20}$$

that we shall bound separately. We start by noticing that

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_I\|_1 \le C\, h\, \|\boldsymbol{\theta}\|_2, \tag{4.21}$$

and for every $\boldsymbol{\eta} \in (H^1(\Omega))^2$

$$\|\boldsymbol{\eta} - \Pi_h \boldsymbol{\eta}\|_0 \le C\, h\, \|\boldsymbol{\eta}\|_1. \tag{4.22}$$

Then we have first from (4.21)

$$I = a(\boldsymbol{\theta}_I - \boldsymbol{\theta}, \boldsymbol{\theta}_I - \boldsymbol{\theta}_h) \le C\, h\, \|\boldsymbol{\theta}\|_2\, \|\boldsymbol{\theta}_I - \boldsymbol{\theta}_h\|_1, \tag{4.23}$$

and from (4.10) and (4.22)

$$II = \lambda^{-1} t^2 (\boldsymbol{\gamma}_I - \boldsymbol{\gamma}, \boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h) \le C\, h\, t\, \|\boldsymbol{\gamma}\|_1\, t\, \|\boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h\|_0. \tag{4.24}$$

Similarly we can bound the third term by

$$III = (\boldsymbol{\gamma}, (I - \Pi_h)(\boldsymbol{\theta}_I - \boldsymbol{\theta}_h)) \le C\, h\, \|\boldsymbol{\gamma}\|_0\, \|\boldsymbol{\theta}_I - \boldsymbol{\theta}_h\|_1. \tag{4.25}$$

In order to bound the last term, we first recall the well known Agmon inequality [1]: if e is an edge of a triangle T (with the usual minimum angle condition), $\varphi \in H^1(T)$, and h_T is the diameter of T, then we have

$$\|\varphi\|_{0,e} \le C\, (h_T^{-1/2} \|\varphi\|_{0,T} + h_T^{1/2} \|\varphi\|_{1,T}). \tag{4.26}$$

For the fourth term we notice now that $(I - \Pi_h)(\boldsymbol{\theta}_I - \boldsymbol{\theta}_h)$ is orthogonal to constants on every edge. Hence for every boundary edge e belonging to a triangle T we can denote by $\overline{M_{ns}}$ the mean value of M_{ns} on e, and using (4.26) and usual approximation theory we have

$$\begin{aligned} \textstyle\int_e M_{ns} (I - \Pi_h)(\theta_I - \theta_h)_s \, \mathrm{d}s \le \;& \|M_{ns} - \overline{M_{ns}}\|_{0,e}\, \|(I - \Pi_h)(\theta_I - \theta_h)_s\|_{0,e} \\ & \le C (h_T^{1/2} \|M_{ns}\|_{1,T}\, h_T^{1/2} \|\theta_I - \theta_h\|_{1,T}. \end{aligned} \tag{4.27}$$

Using (4.27) on every edge, using (2.5) to see that $\|M_{ns}\|_{1,T} \le C \|\boldsymbol{\theta}\|_{2,T}$, and the usual Cauchy-Schwarz inequality we have then

$$IV = \int_{\Sigma_f} M_{ns} (I - \Pi_h)(\theta_I - \theta_h)_s \, \mathrm{d}s \le C\, h\, \|\boldsymbol{\theta}\|_2\, \|\theta_I - \theta_h\|_1 \tag{4.28}$$

Finally, inserting (4.23)-(4.25) and (4.28) in (4.16), and the usual arithmetic-geometric inequality we have

$$\alpha \|\boldsymbol{\theta}_I - \boldsymbol{\theta}_h\|_1^2 + \lambda^{-1} t^2 \|\boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h\|_0^2 \le C\, h^2\, \left(\|\boldsymbol{\eta}\|_2^2 + t^2 \|\boldsymbol{\gamma}\|_1^2 + \|\boldsymbol{\gamma}\|_0^2 \right). \tag{4.29}$$

From (4.29) we can then obtain easily an estimate for $\boldsymbol{\nabla}(w_I - w_h)$

$$\|\boldsymbol{\nabla}(w_I - w_h)\|_0 \le \lambda^{-1} t^2\, \|\boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h\|_0 + \|\boldsymbol{\theta}_I - \boldsymbol{\theta}_h\|_0. \tag{4.30}$$

Finally using (4.29), (4.30), the triangle inequality and absorbing λ in the constants, we can state the final result:

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|\boldsymbol{\nabla}(w - w_h)\|_0 \le C\, h\, \left(\|\boldsymbol{\eta}\|_2 + t \|\boldsymbol{\gamma}\|_1 + \|\boldsymbol{\gamma}\|_0 \right).$$

5. Conclusions

We presented an alternative model for the treatment of free boundary conditions for the Reissner-Mindlin plate bending equations. It consists of minimizing the usual energy functional under the kinematic constraint $\theta_s = w_{/s}$. This condition is already included in Kirchhoff equation $\boldsymbol{\theta} = \boldsymbol{\nabla} w$ that in the limit case $t = 0$ holds all over the domain Ω. For a thin plate, therefore, our modification is quite reasonable from the Mechanical point of view.

We have seen that with the new boundary conditions the boundary layer effect becomes less severe, reducing to the same order of the boundary layer that is present near the clamped part of the boundary, or near the hard simply supported part.

We also showed that the treatment of these boundary conditions from the numerical point of view requires some care, but can be done in a reasonably simple way. In particular, we considered the Duran-Liberman element, and we showed that the new boundary conditions for this element can be imposed in a simple and cheap way, namely by forcing a priori (that is, in the finite element space) that on each edge the *mean value* of θ_s is equal to $w_{/s}$. We also showed that in this way we can obtain a priori error estimates that are uniform in t, that are optimal with respect to the degree of the polynomial spaces, and that make use of the minimal regularity requirements.

A more extensive range of finite elements able to grant optimal and $t-$uniform estimates for this new free plate model will be treated in a future publication (see [8]).

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