Convergence of Mimetic Finite Difference Method for Diffusion Problems on Polyhedral Meshes with Curved Faces

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Abstract

New mimetic finite difference discretizations of diffusion problems on unstructured polyhedral meshes with strongly curved (non-planar) faces are developed. The material properties are described by a full tensor. The optimal convergence estimates, the second order for a scalar variable (pressure) and the first order for a vector variable (velocity), are proved.

1 Introduction

The mimetic finite difference (MFD) method preserves the essential properties of continuum differential operators such as conservation laws, solution symmetries, and the fundamental identities and theorems of vector and tensor calculus. The MFD method has been successfully employed to solve electromagnetic [13], gas dynamic [10], and diffusion [14, 16, 19, 12, 15, 17] problems. For the linear diffusion problem, the MFD method mimics the Gauss divergence theorem to enforce the local conservation law, the symmetry between the continuous gradient and divergence operators to have symmetry and positivity of the resulting discrete problem, and the null spaces of the involved operators to guarantee stability of the discretization.

The convergence of the MFD method has been proved for simplicial and quadrilateral meshes [3, 4], and for unstructured polyhedral meshes [8, 9] consisting of elements with *planar* faces. However, the meshes appearing in many applications (meshing of complex geometries, moving mesh methods, mesh reconnection methods, etc) have usually elements with *curved* (non-planar) faces. It was shown in [17] that the MFD method does not converge on meshes consisting of such elements. One possible remedy is to approximate a strongly curved face by triangles to get a polyhedral mesh where all elements have planar faces so that we may apply the MFD method from [8, 9]. The number of additional degrees of freedom will be proportional to the number of

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the triangles. In this paper, we propose a new MFD method which does not require additional topological operations with strongly curved faces and uses only three degrees of freedom for every such face.

There are other discretization schemes (see, e.g. [1, 18]) on polyhedral meshes; however, to the best of our knowledge, the convergent schemes result in non-symmetric discrete problems which significantly reduces the number of available efficient solution methods. The MFD method, by its nature, gives always the symmetric problem. We wonder whether the use of additional degrees of freedom is the only way to preserve symmetry in the discrete problem.

The outline of the paper is as follows. In section 2, we formulate a few theoretical assumptions on the problem and the mesh. In Section 3, we describe briefly the mimetic finite difference method. The stability of the method and the convergence estimates are proved in Sections 4 and 5, respectively.

2 Assumptions on the problem and the mesh

Let us consider a model elliptic boundary value problem:

$$\operatorname{div} \vec{F} = b, \qquad \vec{F} = -\mathbf{K} \operatorname{grad} p \tag{2.1}$$

where p denotes a scalar function that we refer to as the pressure, $\vec{F}$ denotes a vector function that we refer to as the velocity, $\mathbf{K}$ denotes a *full symmetric* tensor, and b denotes a source function. The problem is posed in a bounded domain $\Omega \subset \mathbb{R}^3$ and is subject to appropriate boundary conditions on $\partial\Omega$. For simplicity, we assume that the homogeneous Dirichlet boundary conditions are imposed on $\partial\Omega$. The other types of boundary conditions are easily incorporated into the general scheme of the MFD method [12]. We assume also that $\mathbf{K}$ satisfies the following regularity and ellipticity property.

(P1) *(Regularity and ellipticity of* $\mathbf{K}$*).* Every component of $\mathbf{K}$ is in $W^1_\infty(\Omega)$ and $\mathbf{K}$ is strongly elliptic, meaning that there exist two positive constants κ_* and κ^* such that

$$\kappa_* \|\mathbf{v}\|^2 \leq \mathbf{v}^T \mathbf{K}(\mathbf{x})\, \mathbf{v} \leq \kappa^* \|\mathbf{v}\|^2 \qquad \forall\, \mathbf{v} \in \mathbb{R}^3 \quad \forall\, \mathbf{x} \in \Omega. \tag{2.2}$$

Before we make precise the assumptions on Ω and on the subdomains, we introduce the definition of *pseudo-pyramid*.

Definition 2.1 Let $k \geq 3$ be an integer, and γ and τ be positive real numbers, with $\gamma < 1$. A *pseudo-pyramid with* k *lateral faces and shape constants* γ *and* τ is a subset $\mathcal{P}$ of $\mathbb{R}^3$ that can be constructed with the following three steps:

1. Take a pyramid $\mathcal{P}_0$ whose base b_0 is a convex polygon with k edges. Let V_0 be the vertex of this pyramid, d_0 be its diameter, and h_0 be its height. Up to a rigid displacement, we can assume that V_0 is in the origin and b_0 is a subset of the plane $z = h_0$. We also assume that $\mathcal{P}_0$ contains a sphere of radius $r \geq \gamma d_0$ (see Fig. 1).

2. Define a *radial* one-to-one C^1 mapping Φ of the pyramid $\mathcal{P}_0$ into itself. We recall that in a radial map a point $\mathbf{P}$ and its image $\mathbf{P}' = \Phi(\mathbf{P})$ lie on the same ray starting at the origin. We assume that, at every point $\mathbf{P}$ of $\mathcal{P}_0$ and at every point $\mathbf{P}'$ of $\Phi(\mathcal{P}_0)$, we have

$$\|D(\Phi)\| \leq \tau \quad \text{and} \quad \|D(\Phi^{-1})\| \leq \tau, \tag{2.3}$$

respectively. The norms in (2.3) are the usual Euclidean norms of 3×3 matrices.

3. Define the pseudo-pyramid $\mathcal{P}$ as: $\mathcal{P} \equiv \Phi(\mathcal{P}_0)$. The image of the base b_0 is a C^1 surface e, $e \equiv \Phi(b_0)$, that we refer to as *the base* of the pseudo-pyramid. Accordingly, the images of the k lateral faces of $\mathcal{P}_0$ will be referred to as the lateral faces of $\mathcal{P}$.

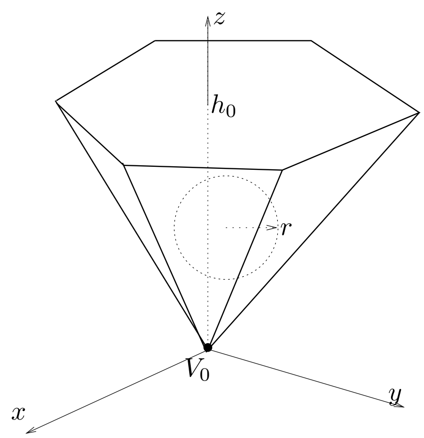

Figure 1: Pyramid $\mathcal{P}_0$ containing a sphere of radius r.

Note that the convexity assumption could be replaced with the star-shaped assumption (see [8] for more details). However, for simplicity of the presentation, we shall not do it here.

According to the above definition, at each point of the base e, we can define a normal unit vector $\mathbf{n}$ pointing outward of $\mathcal{P}$ and varying continuously with the point. Thus, we can define the *average normal vector* $\tilde{\mathbf{n}}$ as

$$\tilde{\mathbf{n}} = \frac{1}{|e|} \int_e \mathbf{n} \, \mathrm{d}S \tag{2.4}$$

where $|e|$ denotes the area of e. It is not difficult to see that $|\tilde{\mathbf{n}}| \leq 1$. A lower bound for $|\tilde{\mathbf{n}}|$ (depending on γ and τ) is contained in the following technical lemma, whose proof is reported in Appendix A.

Lemma 2.1 *Let $\mathcal{P}$ be a pseudo-pyramid with shape constants γ and τ. Let e be its base and let $\tilde{\mathbf{n}}$ be the average normal to e, as defined in (2.4). Then*

$$|\tilde{\mathbf{n}}| \geq \frac{2\gamma}{\tau^4}. \tag{2.5}$$

We are now ready to list our assumptions on the domain Ω and its partition Ω_h. We allow unstructured partitions which are quite general, in order to satisfy the needs of complex engineering applications.

(M1) *(Domain).* We assume that Ω is a bounded connected subset of $\mathbb{R}^3$ with a Lipschitz continuous boundary.

(M2) *(Mesh elements)*. Let Ω_h be a non-overlapping conformal partition of Ω into elements E. We assume that there exist two positive constants γ_* and τ_*, and two integer numbers N_e and N_k, independent of the partition, such that every element E is the union of at most N_e pseudo-pyramids with at most N_k lateral faces and shape constants $\gamma \geq \gamma_*$ and $\tau \leq \tau_*$. We assume that the pseudo-pyramids have the same vertex, and the boundary ∂E is the union of the bases of the pseudo-pyramids. These bases will be referred to as *the faces of* E. The element E will be still referred to as the polyhedron.

(M3) *(Moderately and strongly curved faces)*. We fix a constant σ_* which will be independent of the partition. Then, for each face e, we say that e is *moderately curved* if at every point of e we have

$$|\mathbf{n} - \tilde{\mathbf{n}}| \leq \sigma_* \, |e|^{1/2} \tag{2.6}$$

where $\mathbf{n}$ is the normal to e and $\tilde{\mathbf{n}}$ is its average normal as defined in (3.2). Otherwise, we say that the face e is *strongly curved*.

For every element E, we denote by $|E|$ its volume and by h_E its diameter. We finally set as usual

$$h = \sup_E h_E.$$

The meshes generated by smooth mappings or by uniform refinement of a coarse mesh contain typically elements with moderately curved faces. On the other hand, the meshes generated by moving mesh methods contain frequently elements with strongly curved faces. Assumption (**M3)** draws a theoretical boundary between two types of faces; therefore, is rather a *definition* than an assumption.

Assumption **(M2)** implies that every element E is *star-shaped* with respect to the common vertex V of the pseudo-pyramids that form it. From shape regularity of the pseudo-pyramids, we have that E is star-shaped with respect to every point of a little sphere with center in V and radius $\rho_* \, h_E$ where ρ_* depends solely on the constants γ_*, τ_*, and N_e. Moreover, we may prove that there exist two positive constants, v_* and a_*, which depend only on the constants γ_*, τ_*, and N_e and such that

$$v_* \, h_E^3 \leq \, |E|, \qquad a_* \, h_E^2 \leq \, |e| \tag{2.7}$$

for all faces e of every element E.

Depending on context, we shall use ∂E either for the boundary of the element E or for the set of its faces.

3 Mimetic finite difference method

In this section, we recall briefly the main steps in the MFD method (see [8] for more details). Let us introduce an operator $\mathcal{G}$, $\mathcal{G}\, p = -\mathbf{K} \mathrm{grad}\, p$, which we refer to as the velocity operator. Then, the Green formula reads

$$\int_\Omega \mathbf{F} \cdot (\mathbf{K}^{-1} \mathcal{G}\, p) \, \mathrm{d}V = \int_\Omega p \, \mathrm{div} \mathbf{F} \, \mathrm{d}V. \tag{3.1}$$

This states clearly that the velocity and divergence operators are adjoint to each other, i.e.

$$\mathcal{G} \, = \mathrm{div}^*.$$

The mimetic finite difference (MFD) method produces discretizations of these operators which are adjoint to each other with respect to inner products in the discrete velocity and pressure spaces.

The *first* step of the MFD method is to specify the degrees of freedom for physical variables p and $\vec{F}$ and their location.

We consider the space Q^d of discrete pressures that are constant on each polyhedron E. For $\mathbf{q} \in Q^d$, we denote by q_E (or by $(\mathbf{q})_E$) its value on E. The dimension, N_Q, of Q^d is obviously equal to the number of polyhedrons in Ω_h. In what follows, we shall denote by Q^d either the vector space $\mathbb{R}^{N_Q}$ or the space of piecewise constant functions, depending on context. The identification will be obvious and no confusion should arise.

The definition of the space of discrete velocities requires some additional considerations. To every element E in Ω_h and to every face e of E, we associate a *vector* $\mathbf{F}_E^e$ with three components (since we are considering a three-dimensional problem). Moreover, for every element E in Ω_h and every face e of E, we define the vector $\mathbf{n}_E^e$ as the unit normal (at each point of e) pointing outside of E and the average normal vector $\tilde{\mathbf{n}}_E^e$ as

$$\tilde{\mathbf{n}}_E^e = \frac{1}{|e|} \int_e \mathbf{n}_E^e \,\mathrm{d}S. \tag{3.2}$$

Assumption **(M2)** and Lemma 2.1 give the following lower bound:

$$|\tilde{\mathbf{n}}_E^e| \geq \frac{2\,\gamma_*}{\tau_*^4}. \tag{3.3}$$

In addition to the above notation we assign, to each face e of Ω_h, a pair of arbitrary unit vectors $\mathbf{a}^{e,1}$ and $\mathbf{a}^{e,2}$ orthogonal to $\tilde{\mathbf{n}}_E^e$ and orthogonal to each other (see Fig. 2).

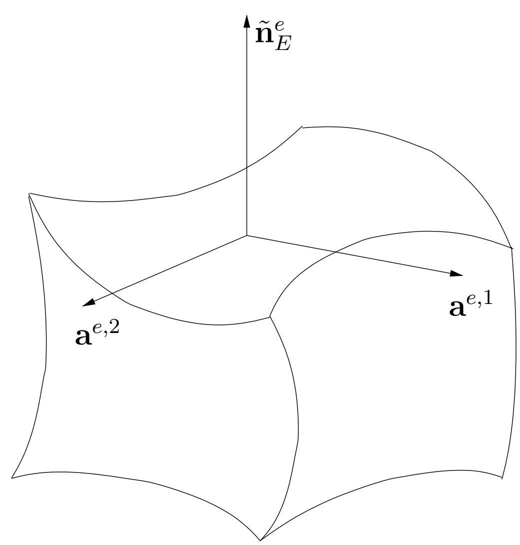

Figure 2: The local coordinate system for a strongly curved (top) face e of E.

For a discrete velocity field $\mathbf{G}$ we will denote by $\mathbf{G}_E$ its restriction to the boundary of E, and by $\mathbf{G}_E^e$ (or by $(\mathbf{G}_E)^e$) the restriction of $\mathbf{G}_E$ to a face e of ∂E. We impose the following *continuity* of the face-based velocity unknowns.

(C1) *(Continuity of velocities)*. We assume that for each face e, shared by two polyhedrons E_1 and E_2, we have

$$\mathbf{F}^e_{E_1} \cdot \tilde{\mathbf{n}}^e_{E_1} = -\mathbf{F}^e_{E_2} \cdot \tilde{\mathbf{n}}^e_{E_2}. \tag{3.4}$$

Moreover, we assume that on *strongly curved faces* we have the full continuity of the discrete velocity vector. This means that together with (3.4) we also have

$$\mathbf{F}^e_{E_1} \cdot \mathbf{a}^{e,i} = \mathbf{F}^e_{E_2} \cdot \mathbf{a}^{e,i}, \qquad i = 1, 2. \tag{3.5}$$

We denote the vector space of face-based velocity unknowns by X^d. The number, N_X, of our discrete velocity unknowns is equal to three times the number of boundary faces plus *six times* the number of internal faces. In our theoretical discussion, we shall consider X^d as the subspace of $\mathbb{R}^{N_X}$ which verifies (3.4) on all faces and (3.5) on strongly curved faces.

In practice, for each face e of every element E, we use the local coordinate system given by vectors $\mathbf{a}^{e,1}$, $\mathbf{a}^{e,2}$, and $\tilde{\mathbf{n}}^e_E$ (see Fig. 2). Then, on moderately curved faces, only the third component of $\mathbf{F}^e_E$ will be subject to the continuity requirements, and the other two components will be treated as *internal degrees of freedom*.

If k^m_E is the number of moderately curved faces of E, we will have $2\,k^m_E$ internal degrees of freedom, that could be eliminated during the assembly process by *static condensation*. Hence, in the final matrix, after static condensation, the total number of velocity unknowns will be equal to the total number of moderately curved faces, plus three times the number of strongly curved faces.

Necessity to use three velocity components on strongly curved faces is possibly the intrinsic difficulty (see, e.g. Lemma 5.1) and the reason why nobody succeeded in doing a reasonable job on strongly curved faces.

It is clear that the parameter σ_* defined in (2.6) is at our choice. If we choose a huge number for σ_*, then most faces will be classified as *moderately curved* and the asymptotically optimal convergence rate will be observed only on very fine meshes. Indeed, as we shall see later on, the value of σ_* enters our *a priori* estimates. Hence, in practice, we are likely to face the usual trade-off between cost of the method and quality of the results. Still, more knowledge has to be gained from experience in order to decide how to choose σ_*.

To summarize, one pressure unknown is defined on each polyhedron and one velocity vector-unknown is defined on each face; its component in the direction of $\tilde{\mathbf{n}}^e$ is continuous on all faces, while the other two components are continuous only if the face is strongly curved.

Once we got the degrees of freedom in Q^d and in X^d, we can define interpolation operators from the spaces of smooth enough scalar and vector-valued functions to the discrete spaces Q^d and X^d, respectively. To every function q in $L^1(\Omega)$, we associate the element $\mathbf{q}^I$ in Q^d by

$$(\mathbf{q}^I)_E = \frac{1}{|E|}\int_E q\,\mathrm{d}V \qquad \forall E \in \Omega_h. \tag{3.6}$$

It is immediate to check that

$$\int_E (\mathbf{q}^I)^2_E\,\mathrm{d}V \le \int_E q^2\,\mathrm{d}V \qquad \forall E \in \Omega_h, \quad \forall q \in L^2(E). \tag{3.7}$$

For every vector-valued function $\vec{G} \in (H^1(\Omega))^3$, we define $\mathbf{G}^I \in X^d$ as follows. For every element E and for every face e of E, we consider once more the average normal vector $\tilde{\mathbf{n}}^e_E$ and set, for convenience of notation,

$$\mathbf{a}_E^{e,3} = \frac{\tilde{\mathbf{n}}_E^e}{|\tilde{\mathbf{n}}_E^e|}.$$

To define the components of $(\mathbf{G}_E^I)^e$ in the three orthogonal directions, we set

$$(\mathbf{G}_E^I)^e \cdot \mathbf{a}_E^{e,3} := \frac{1}{|e|\,|\tilde{\mathbf{n}}_E^e|}\int_e \vec{G}\cdot\mathbf{n}_E^e\,\mathrm{d}S \quad \text{and} \quad (\mathbf{G}_E^I)^e \cdot \mathbf{a}^{e,i} := \frac{1}{|e|}\int_e \vec{G}\cdot\mathbf{a}^{e,i}\,\mathrm{d}S \quad (i=1,2). \tag{3.8}$$

In the next section, we shall prove that this interpolation operator is well defined and uniformly bounded.

If $\vec{G}$ is continuous across the interior mesh faces, it is easy to see that the resulting vector $\mathbf{G}^I$ will satisfy the continuity property **(C1)**. Hence $\mathbf{G}^I \in X^d$. Our interpolation operator have the following three important properties.

1. Whenever $\vec{G}$ is constant on e, we obtain easily from (3.8) that $(\mathbf{G}_E^I)^e \cdot \mathbf{a}_E^{e,3} = \vec{G}\cdot\mathbf{a}_E^{e,3}$ and $(\mathbf{G}_E)e \cdot \mathbf{a}^{e,i} = \vec{G}\cdot\mathbf{a}^{e,i}$ for $i = 1, 2$. Thus
$$(\mathbf{G}_E^I)^e = \vec{G} \qquad \text{when } \vec{G} \text{ is constant on } e. \tag{3.9}$$

2. Definition (3.8) implies the following crucial equality
$$\int_e (\mathbf{G}_E^I)^e \cdot \mathbf{n}_E^e\,\mathrm{d}S = \int_e (\mathbf{G}_E^I)^e \cdot \tilde{\mathbf{n}}_E^e\,\mathrm{d}S = |\tilde{\mathbf{n}}_E^e|\,|e|\,(\mathbf{G}_E^I)^e\cdot\mathbf{a}_E^{e,3} = \int_e \vec{G}\cdot\mathbf{n}_E^e\,\mathrm{d}S. \tag{3.10}$$

3. Using (3.3), we have easily
$$|\mathbf{G}_E^e| \le \frac{\nu_*}{|e|^{1/2}}\left(\int_e |\vec{G}|^2\,\mathrm{d}S\right)^{1/2} \tag{3.11}$$
where
$$\nu_* = \frac{\tau_*^4}{2\gamma_*}.$$

   COMMENT: WHY DID YOU PUT A $\sqrt{2}$? If you have two vectors $\mathbf{v}$ and $\mathbf{w}$ with, say $|v_1| \le |w_1|$, $|v_2| \le |w_2|$ and $|v_3| \le 17\,|w_1|$, then $|\mathbf{v}| \le 17\,|\mathbf{w}|$...

The *second* step of the MFD method is to equip the spaces of discrete pressures and velocities with inner products. The inner product on the vector space Q^d is given by

$$[\mathbf{p},\,\mathbf{q}]_{Q^d} = \sum_{E\in\Omega_h} p_E\,q_E|E| \qquad \forall\,\mathbf{p},\,\mathbf{q}\in Q^d. \tag{3.12}$$

The inner product on X^d is a sum of elemental inner products $[\mathbf{F},\,\mathbf{G}]_E$ defined for every element E in Ω_h. Let k_E be the total number of faces in E, so that the total number of scalar components of $\mathbf{F}_E$ and $\mathbf{G}_E$ is $3k_E$. Let us denote them by $\{\mathbf{F}_E\}_1, ..., \{\mathbf{F}_E\}_{3k_E}$ and $\{\mathbf{G}_E\}_1, ..., \{\mathbf{G}_E\}_{3k_E}$, respectively. We assume that we are given (for each E) a symmetric positive definite $3\,k_E \times 3\,k_E$ matrix $M_E \equiv \{M_E\}_{i,j}$. Then, we set

$$[\mathbf{F},\,\mathbf{G}]_E = \sum_{i,j=1}^{3\,k_E} \{M_E\}_{i,j}\,\{\mathbf{F}_E\}_i\,\{\mathbf{G}_E\}_j. \tag{3.13}$$

From (3.13) we can easily construct the inner product in X^d by setting

$$[\mathbf{F},\,\mathbf{G}]_{X^d} = \sum_{E\in\Omega_h}[\mathbf{F},\,\mathbf{G}]_E \qquad \forall\,\mathbf{F},\,\mathbf{G}\in X^d. \tag{3.14}$$

Some minimal approximation properties for the inner product (3.13) are required. The construction of the matrix M_E is a non-trivial task for a polyhedral element (see [9] where elements with *planar* faces were analyzed). We shall return to this problem in Section 5. For the time being, we assume that the inner product (3.13) has the following property.

(S1) *(Stability)*. We assume that there are two positive constants s_* and S^* independent of the partition Ω_h such that, for every $\mathbf{G}\in X^d$ and every E in Ω_h, one has

$$s_*|E|\sum_{e\in\partial E}|\mathbf{G}_E^e|^2 \le\ [\mathbf{G},\,\mathbf{G}]_E\ \le S^*|E|\sum_{e\in\partial E}|\mathbf{G}_E^e|^2. \tag{3.15}$$

The *third* step of the MFD method is to derive an approximation to the divergence operator. The discrete divergence operator, $\mathcal{DIV}^d : X^d \to Q^d$, arises naturally from the Gauss divergence theorem as

$$(\mathcal{DIV}^d\ \mathbf{F})_E \ \stackrel{def}{=}\ \frac{1}{|E|}\sum_{e\in\partial E}\mathbf{F}_E^e\cdot\tilde{\mathbf{n}}_E^e\,|e| \equiv \frac{1}{|E|}\sum_{e\in\partial E}\int_e \mathbf{F}_E^e\cdot\mathbf{n}_E^e\,\mathrm{d}S. \tag{3.16}$$

We point out that our interpolation operators, in some sense, *commute* with the divergence operator. Indeed, for every sufficiently smooth vector field $\vec{G}$, we can use (3.16), (3.2), (3.10), the Gauss divergence theorem, and (3.6) to obtain

$$\begin{aligned}(\mathcal{DIV}^d\,\mathbf{G}^I)_E &= \frac{1}{|E|}\sum_{e\in\partial E}(\mathbf{G}_E^I)^e\cdot\tilde{\mathbf{n}}_E^e\,|e|\ = \frac{1}{|E|}\sum_{e\in\partial E}\int_e(\mathbf{G}_E^I)^e\cdot\mathbf{n}_E^e\\ &= \frac{1}{|E|}\int_{\partial E}\vec{G}\cdot\mathbf{n}_E\,\mathrm{d}S\ = \frac{1}{|E|}\int_E \operatorname{div}\vec{G}\,\mathrm{d}V\ = (\operatorname{div}\vec{G})_E^I\end{aligned} \tag{3.17}$$

for every element E in Ω_h.

The *fourth* step of the MFD method is to define the discrete velocity operator, $\mathcal{G}^d : Q^d \to X^d$, as the adjoint to the discrete divergence operator, $\mathcal{DIV}^d$, with respect to inner products (3.12) and (3.14), i.e.

$$[\mathbf{F},\,\mathcal{G}^d\ \mathbf{p}]_{X^d} = [\mathbf{p},\,\mathcal{DIV}^d\ \mathbf{F}]_{Q^d} \qquad \forall\mathbf{p}\in Q^d \quad \forall\mathbf{F}\in X^d. \tag{3.18}$$

Using the discrete velocity and divergence operators, the continuous problem (2.1) is discretized as follows:

$$\mathcal{DIV}^d\ \mathbf{F}_d = \mathbf{b}, \qquad \mathbf{F}_d = \mathcal{G}^d\,\mathbf{p}_d, \tag{3.19}$$

where

$$\mathbf{b}\equiv\mathbf{b}^I \tag{3.20}$$

is the vector of mean values of the source function b.

4 Stability analysis

In this section we analyze the stability of the mimetic finite difference discretization (3.19) following the well-established theory of saddle-point problems [7]. More precisely, we prove the coercivity condition (4.3) and the inf-sup condition (4.4).

Using the discrete Green formula (3.18), we rewrite equations (3.19) in a form more suitable for analysis:

$$[\mathbf{F}_d,\, \mathbf{G}]_{X^d} - [\mathbf{p}_d,\, \mathcal{DIV}^d\, \mathbf{G}]_{Q^d} = 0 \qquad \forall\, \mathbf{G} \in X^d \tag{4.1}$$

$$[\mathcal{DIV}^d\, \mathbf{F}_d,\, \mathbf{q}]_{Q^d} = [\mathbf{b},\, \mathbf{q}]_{Q^d} \qquad \forall\, \mathbf{q} \in Q^d. \tag{4.2}$$

For future analysis, we need the following L^2-type norms:

$$|||\mathbf{p}|||^2_{Q^d} = [\mathbf{p},\, \mathbf{p}]_{Q^d} \qquad \text{and} \qquad |||\mathbf{F}|||^2_{X^d} = [\mathbf{F},\, \mathbf{F}]_{X^d},$$

together with the mesh dependent H_{div} norms:

$$|||\mathbf{F}|||^2_{div,E} = [\mathbf{F}, \mathbf{F}]_E + h_E^2\, \|\mathcal{DIV}^d\, \mathbf{F}\|^2_{L^2(E)}, \qquad |||\mathbf{F}|||^2_{div} = \sum_{E \in \Omega_h} |||\mathbf{F}|||^2_{div,E},$$

and the mesh dependent H^1 norms:

$$\|\vec{F}\|^2_{1,h,E} = \|\vec{F}\|^2_{(L^2(E))^3} + h_E^2 |\vec{F}|^2_{(H^1(E))^3} \qquad \text{and} \qquad \|\vec{F}\|^2_{1,h} = \sum_{E \in \Omega_h} \|\vec{F}\|^2_{1,h,E}.$$

Let V^d be the space of divergence-free discrete velocities:

$$V^d = \{\mathbf{F} \in X^d : \quad \mathcal{DIV}^d\, \mathbf{F} = 0\}.$$

We begin the stability analysis by noticing that the inner product (3.14) is continuous. It is also obvious that the inner product satisfies the V^d-ellipticity condition:

$$[\mathbf{F},\, \mathbf{F}]_{X^d} \geq |||\mathbf{F}|||^2_{div} \qquad \forall\, \mathbf{F} \in V^d. \tag{4.3}$$

The analysis of the inf-sup condition is more involved. Following [7], for every $\mathbf{q} \in Q^d$, we have to find a non-zero vector $\mathbf{G} \in X^d$ such that

$$[\mathcal{DIV}^d\, \mathbf{G},\, \mathbf{q}]_{Q^d} \geq \beta_* |||\mathbf{G}|||_{div}\, |||\mathbf{q}|||_{Q^d} \tag{4.4}$$

where β_* is a positive constant independent of $\mathbf{q}$, $\mathbf{G}$, and h.

The next result is well known for smooth domains and has been extended to Lipschitz domains by Bramble (see [5] and the references therein).

Proposition 4.1 *Let Ω be a connected bounded Lipschitz domain in $\mathbb{R}^3$. There exists a positive constant $\tilde{\beta} = \tilde{\beta}(\Omega)$ such that: for every $q \in L^2(\Omega)$ with zero mean value in Ω there exists a vector-valued function $\vec{G} \in (H^1_0(\Omega))^3$ such that*

$$\mathrm{div}\vec{G} = q \qquad \textit{and} \qquad \tilde{\beta}\, \|\vec{G}\|_{(H^1_0(\Omega))^3} \;\leq\; \|q\|_{L^2(\Omega)}. \tag{4.5}$$

From this we immediately get the following result.

Proposition 4.2 *Let Ω be a connected bounded Lipschitz domain in $\mathbb{R}^3$. There exists a positive constant $\beta = \beta(\Omega)$ such that: for every $q \in L^2(\Omega)$ there exists a vector-valued function $\vec{G} \in (H^1(\Omega))^3$ such that*

$$\mathrm{div}\vec{G} = q \qquad \textit{and} \qquad \beta\, \|\vec{G}\|_{(H^1(\Omega))^3} \,\leq\, \|q\|_{L^2(\Omega)}. \tag{4.6}$$

<u>Proof.</u> First, for every $q \in L^2(\Omega)$, we define $\overline{q}$ by

$$\overline{q} = \frac{1}{|\Omega|} \int_\Omega q \,\mathrm{d}V.$$

Then, we consider the function $\psi = (x^2 + y^2 + z^2)\overline{q}/6$ and set $\vec{G}^1 = \nabla\psi$. Thus,

$$\mathrm{div}\,\vec{G}^1 = \overline{q} \qquad \text{and} \qquad c_1(\Omega)\,\|\vec{G}^1\|_{(H^1(\Omega))^3} \leq \|\overline{q}\|_{L^2(\Omega)}$$

for some constant $c_1(\Omega)$ depending only on Ω. Since the mean value of $q - \overline{q}$ is zero, we can use Proposition 4.1 to find a vector-valued function $\vec{G}^0$ such that

$$\mathrm{div}\,\vec{G}^0 = q - \overline{q} \qquad \text{and} \qquad \tilde{\beta}\,\|\vec{G}^0\|_{(H^1_0(\Omega))^3} \,\leq\, \|q - \overline{q}\|_{L^2(\Omega)}.$$

Setting now $\vec{G} := \vec{G}^0 + \vec{G}^1$ and using the L_2-orthogonality of $\overline{q}$ and $q - \overline{q}$, we have easily the desired result with $1/\beta = \sqrt{2}\max\{1/\tilde{\beta},\, 1/c_1(\Omega)\}$. □

Let now E be an element in Ω_h, and e be one of its faces. According to Assumption **(M2)** there exists a pseudo-pyramid $\mathcal{P}_E^e$ having e as base. Let $\mathcal{P}_0$ be the pyramid used in Definition 2.1 (together with the map Φ) to construct the pseudo-pyramid $\mathcal{P}_E^e$, i.e $\mathcal{P}_E^e = \Phi(\mathcal{P}_0)$ and $e = \Phi(b_0)$. We recall a result due to Agmon, made popular in the numerical analysis community by D.N. Arnold [2]. Applied to our case, it says that there exists a constant C_{agm}, depending only on the shape constant γ appearing in Definition 2.1, such that for every function $\chi \in H^1(\mathcal{P}_0)$, we have

$$\|\chi\|^2_{L^2(b_0)} \leq\, C_{agm}\left({h_0}^{-1}\|\chi\|^2_{L^2(\mathcal{P}_0)} + h_0\,|\chi|^2_{H^1(\mathcal{P}_0)}\right). \tag{4.7}$$

From this we easily deduce (mapping χ back and forth from $\mathcal{P}$ to $\mathcal{P}_0$ and using (2.3)) that there exists a constant C^*_{agm}, depending only on the shape constants γ_* and τ_* appearing in Assumption **(M2)**, such that for every function $\chi \in H^1(E)$, we have

$$\|\chi\|^2_{L^2(e)} \leq\, C^*_{agm}\left(h_E^{-1}\|\chi\|^2_{L^2(\mathcal{P}_E^e)} + h_E\,|\chi|^2_{H^1(\mathcal{P}_E^e)}\right). \tag{4.8}$$

At this point we need a technical lemma.

Lemma 4.1 *Under assumptions* **(M1)**, **(M2)** *and* **(S1)**, *there exists a positive constant β^*_s such that, for every $\vec{G} \in (H^1(\Omega))^3$, we have*

$$\beta^*_s|||\mathbf{G}^I|||_{div} \leq \|\vec{G}\|_{1,h} \tag{4.9}$$

where $\mathbf{G}^I$ is defined in (3.8).

Proof. Using (3.11) and applying (4.8) to each component of $\vec{G}$, we get:

$$
\begin{aligned}
|(\mathbf{G}_E^I)^e| &\leq \frac{\nu_*}{|e|^{1/2}}\Big(\int_e |\vec{G}|^2 \,\mathrm{d}S\Big)^{1/2} \\
&\leq \frac{\nu_*}{|e|^{1/2}}\left(C^*_{agm}(h_E^{-1}\|\vec{G}\|^2_{L^2(\mathcal{P}^e_E)} + h_E|\vec{G}|^2_{H^1(\mathcal{P}^e_E)})\right)^{1/2} \\
&\leq \nu_*\left(\tfrac{C^*_{agm}}{a_*}(h_E^{-3}\|\vec{G}\|^2_{L^2(\mathcal{P}^e_E)} + h_E^{-1}|\vec{G}|^2_{H^1(\mathcal{P}^e_E)})\right)^{1/2}.
\end{aligned}
$$

Recalling Assumption **(S1)**, we have

$$
\begin{aligned}
[\mathbf{G}^I, \mathbf{G}^I]_E &\leq S^* \sum_{e\in\partial E} h_E^3 |(\mathbf{G}_E^I)^e|^2 \\
&\leq S^* \sum_{e\in\partial E} h_E^3 \nu_*^2\left(\frac{C^*_{agm}}{a_*}(h_E^{-3}\|\vec{G}\|^2_{L^2(\mathcal{P}^e_E)} + h_E^{-1}|\vec{G}|^2_{H^1(\mathcal{P}^e_E)})\right) \qquad (4.10)\\
&\leq \frac{S^*\nu_*^2 C^*_{agm}}{a_*}\|\vec{G}\|^2_{1,h,E}.
\end{aligned}
$$

Further, from (3.17) and (3.7), we get

$$
\|\mathcal{DIV}^d\,\mathbf{G}_E^I\|^2_{L^2(E)} = \|(\mathrm{div}\,\vec{G})^I_E\|^2_{L^2(E)} \leq \|\mathrm{div}\,\vec{G}\|^2_{L^2(E)} \leq 3|\vec{G}|^2_{H^1(E)}.
$$

Using this and (4.10), we get (4.9) with $1/\beta_s^* = \max\{3, S^*\nu_*^2 C^*_{agm}/a_*\}$. This proves the assertion of the lemma. □

Combining (4.9) with (4.6) and using once more (3.17), we get

$$
\mathcal{DIV}^d\,\mathbf{G}^I = q \qquad \text{and} \qquad \beta_s^*\beta\,|||\mathbf{G}^I|||_{div} \leq \|q\|_{L^2(\Omega)}
$$

that gives immediately the *inf-sup* condition (4.4) with $\beta_* = \beta_s^*\beta$.

5 Convergence analysis

In this section, we prove optimal convergence estimates for both primary variables. Some of the proofs follow the pattern established in [8] where we proved the optimal convergence estimates for meshes with planar polygonal faces. Therefore, we shall omit some technical details which can be found there and focus more on the careful treatment of curved faces.

For the sake of simplicity, we assume that our solution p is in $H^2(\Omega)$. Note that with a little additional effort we could use a weaker regularity, and get a lower order of convergence.

We begin by introducing the second (and the final) assumption on the inner product (3.14), and more precisely on its relationship with the continuous inner product.

(S2) *(Consistency)*. For every element E, every linear function q^1 on E and every $\mathbf{G} \in X^d$, we have

$$
[(\tilde{\mathbf{K}}\,\nabla q^1)^I, \mathbf{G}]_E = \int_{\partial E} q^1\,\mathbf{G}_E\cdot\mathbf{n}_E\,\mathrm{d}S - \int_E q^1\,(\mathcal{DIV}^d\,\mathbf{G})_E\,\mathrm{d}V \qquad (5.1)
$$

where $(\cdot)^I$ is the interpolation operator (3.8) and $\tilde{\mathbf{K}}$ is a constant tensor on E such that

$$\sup_{x\in E}\ \sup_{1\le i,j\le 3} |\{\mathbf{K}(\mathbf{x})\}_{i,j} - \{\tilde{\mathbf{K}}\}_{i,j}| \le C_K^*\, h_E \tag{5.2}$$

where C_K^* is a constant independent of E.

Note that $\tilde{\mathbf{K}}$ may be any reasonable piecewise constant approximation of $\mathbf{K}$. In practice, we use either the mean value of $\mathbf{K}$ or the value at the polyhedron's center of mass.

Taking $q^1 = 1$ in (5.1), we recover the definition of the discrete divergence operator. Therefore, up to a rigid displacement, we can assume that the center of mass of polyhedron E is in the origin. For such a polyhedron, instead of (5.1), it is sufficient to consider

$$[(\tilde{\mathbf{K}}\,\nabla q^1)^I,\, \mathbf{G}]_E = \int_{\partial E} q^1\, \mathbf{G}_E \cdot \mathbf{n}_E \,\mathrm{d}S$$

where $q^1(0) = 0$, which shows the remarkable property of characterising the inner product using only *boundary integrals*. This property has been used in [9] to build a family of symmetric positive definite matrices M_E for a polyhedron with planar faces.

5.1 Error estimate for the vector variable

Let $(p,\, \vec{F})$ be the exact solution of (2.1), $(\mathbf{p}_d, \mathbf{F}_d)$ be the discrete solution (see (3.19)), and $(\mathbf{p}^I,\, \mathbf{F}^I)$ be the interpolants of the exact solution. Finally, for every element E, we denote by p_E^1 a suitable polynomial of degree ≤ 1 that approximates p, and that will be decided later on. We notice first that from (2.1), (3.17), (3.19), and (3.20) we easily have:

$$\mathcal{DIV}^d\,(\mathbf{F}^I - \mathbf{F}_d) = \mathbf{b} - \mathbf{b} = 0. \tag{5.3}$$

Using the second equations of (2.1) and (3.19), then the discrete Green formula (3.18), and finally (5.3), we get

$$|||\mathbf{F}^I - \mathbf{F}_d|||^2_{X^d} = [(-\mathbf{K}\,\nabla p)^I,\, \mathbf{F}^I - \mathbf{F}_d]_{X^d} - [\mathcal{G}^{\,d}\mathbf{p}_d, \mathbf{F}^I - \mathbf{F}_d]_{X^d} = [(-\mathbf{K}\,\nabla p)^I,\, \mathbf{F}^I - \mathbf{F}_d]_{X^d}.$$

Then, adding and subtracting terms, we have

$$\begin{aligned}
|||\mathbf{F}^I - \mathbf{F}_d|||^2_{X^d} &= [(-\mathbf{K}\,\nabla p)^I + (\mathbf{K}\,\nabla p^1)^I,\, \mathbf{F}^I - \mathbf{F}_d]_{X^d} + [(-\mathbf{K}\nabla p^1)^I,\, \mathbf{F}^I - \mathbf{F}_d]_{X^d} \\
&= \mathbf{I}_1 + [(-\mathbf{K}\nabla p^1 + \tilde{\mathbf{K}}\nabla p^1)^I,\, \mathbf{F}^I - \mathbf{F}_d]_{X^d} + [(-\tilde{\mathbf{K}}\nabla p^1)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d} \\
&= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.
\end{aligned} \tag{5.4}$$

On the other hand, using (5.1) and (5.3), the third term reads:

$$\begin{aligned}
\mathbf{I}_3 &= \sum_{E\in\Omega_h} \left\{ \int_{\partial E} p_E^1\,(\mathbf{F}^I - \mathbf{F}_d)_E \cdot \mathbf{n}_E \,\mathrm{d}S - \int_E p_E^1\,(\mathcal{DIV}^d\,(\mathbf{F}^I - \mathbf{F}_d))_E \,\mathrm{d}V \right\} \\
&= \sum_{E\in\Omega_h} \int_{\partial E} p_E^1\,(\mathbf{F}^I - \mathbf{F}_d)_E \cdot \mathbf{n}_E \,\mathrm{d}S.
\end{aligned} \tag{5.5}$$

Before estimating $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ we have to make precise the choice of p^1. To do that, we recall some known results of approximation theory. As we mentioned in Section 2, every element E is star-shaped with respect to a sphere of radius $\rho_* h_E$. Hence, it is possible to find a constant C^*_{app}, depending only on ρ_*, such that, for every $p \in H^2(E)$, there exist a constant function p^0_E and a polynomial p^1_E of degree ≤ 1 such that

$$\|p - p^0_E\|_{L^2(E)} \leq \, C^*_{app}\, h_E \,\|p\|_{H^1(E)}, \tag{5.6}$$

$$\|p - p^1_E\|_{L^2(E)} \leq \, C^*_{app}\, h^2_E \,\|p\|_{H^2(E)}, \qquad \|p - p^1_E\|_{H^1(E)} \leq \, C^*_{app}\, h_E \,\|p\|_{H^2(E)} \tag{5.7}$$

(see [6, Lemma 4.3.8]). This also implies that

$$\|\nabla(p - p^1)\|_{1,h,E} \leq \, ((C^*_{app})^2 + 1)^{1/2}\, h_E \,\|p\|_{H^2(E)}. \tag{5.8}$$

Concerning the error on faces, we can immediately derive from (4.8) that

$$\|\nabla \chi\|^2_{L^2(e)} \leq \, C^*_{agm} \left(h^{-1}_E \,\|\chi\|^2_{H^1(\mathcal{P}^e_E)} + h_E \,\|\chi\|^2_{H^2(\mathcal{P}^e_E)} \right)$$

for every $\chi \in H^2(E)$. Applying this to the difference $p - p^1_E$, and using (5.7), we get:

$$\|p - p^1_E\|^2_{L^2(e)} + h^2_E \|\nabla(p - p^1_E)\|^2_{L^2(e)} \leq \, C^*_{face}\, h^3_E \,\|p\|^2_{H^2(E)} \tag{5.9}$$

where C^*_{face} is a constant depending only on C^*_{app} and C^*_{agm}.

We can now go back and estimate $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$. The estimate of $\mathbf{I}_1$ follows immediately from Lemma 4.1, ellipticity property **(P1)**, and the approximation result (5.8):

$$\begin{aligned} \mathbf{I}_1 \;&\leq\; |||(-\mathbf{K}\,\nabla p)^I + (\mathbf{K}\,\nabla p^1)^I|||_{X^d}\; |||\mathbf{F}^I - \mathbf{F}_d|||_{X^d} \\ &\leq\; \frac{1}{\beta^*_s} \|\mathbf{K}(\nabla p^1 - \nabla p)\|_{1,h} |||\mathbf{F}^I - \mathbf{F}_d|||_{X^d} \leq C^*_{I_1}\, h\,\|p\|_{H^2(\Omega)} |||\mathbf{F}^I - \mathbf{F}_d|||_{X^d} \end{aligned} \tag{5.10}$$

where the constant $C^*_{I_1}$ is equal to $\kappa^*((C^*_{app})^2 + 1)^{1/2}/\beta^*_s$.

The estimate for $\mathbf{I}_2$ is also quick. From (5.2), we have immediately that

$$\mathbf{I}_2 \equiv [(-\mathbf{K}\nabla p^1 + \tilde{\mathbf{K}}\nabla p^1)^I,\, \mathbf{F}^I - \mathbf{F}_d]_{X^d} \leq \, C^*_K h\, |||(\nabla p^1)^I|||_{X^d}\, |||\mathbf{F}^I - \mathbf{F}_d|||_{X^d}. \tag{5.11}$$

Using (4.9), the triangle inequality, and (5.7) we have:

$$\beta^*_s |||(\nabla p^1_E)^I|||_{X^d} \leq \|\nabla p^1_E\|_{1,h,E} \leq \|\nabla p\|_{1,h,E} + \|\nabla(p - p^1_E)\|_{1,h,E} \leq (1 + h_E\, C^*_{app})\|p\|_{H^2(E)}.$$

Using this in (5.11), we get

$$\mathbf{I}_2 \leq \, C^*_{I_2}\, h\,\|p\|_{H^2(\Omega)}\, |||\mathbf{F}^I - \mathbf{F}_d|||_{X^d} \tag{5.12}$$

where the constant $C^*_{I_2}$ is equal to $C^*_K(1 + h_E\, C^*_{app})/\beta^*_s$.

The following Lemma gives the estimate for $\mathbf{I}_3$.

Lemma 5.1 *Let $p \in H^2(\Omega) \cap H^1_0(\Omega)$, let p^1 satisfy (5.7) in every element E, and let $\mathbf{G} \in X^d$. Then, under Assumptions* **(M2)**, **(M3)** *and* **(C1)**, *we have*

$$\sum_{E \in \Omega_h} \int_{\partial E} p^1\, \mathbf{G}_E \cdot \mathbf{n}_E \,\mathrm{d}S \leq C^*_{I_3}\, h\,\|p\|_{H^2(\Omega)}\, |||\mathbf{G}|||_{X^d} \tag{5.13}$$

where the constant $C^*_{I_3}$ is independent of p, $\mathbf{G}$ and h.

Proof. In this proof we have to distinguish between boundary faces, strongly curved and moderately curved faces.

First, we consider the case of boundary faces. Let e be a boundary face, and E be the only element containing e. The homogeneous Dirichlet boundary condition implies that $p = 0$ on e. Therefore, the contribution of e to the sum in (5.13) can be estimated using (5.9), then (2.7), and finally (3.15):

$$\begin{aligned}
\int_e p_E^1\, \mathbf{G}_E \cdot \mathbf{n}_E \,\mathrm{d}S &= \int_e (p_E^1 - p)\, \mathbf{G}_E \cdot \mathbf{n}_E \,\mathrm{d}S \\
&\leq \|p - p_E^1\|_{L^2(e)}\, \|\mathbf{G}_E^e\|_{(L^2(e))^3} = \|p - p_E^1\|_{L^2(e)}\, |\mathbf{G}_E^e|\, |e|^{1/2} \\
&\leq v_*^{-1/2} (C_{face}^*)^{1/2} h_E \|p\|_{H^2(E)} |\mathbf{G}_E^e|\, |E|^{1/2} \\
&\leq C_{3,b}^*\, h \|p\|_{H^2(E)}\, |||\mathbf{G}_E|||_E
\end{aligned} \tag{5.14}$$

where the constant $C_{3,b}^*$ is equal to $(C_{face}^*)^{1/2}\, (s_* v_*)^{-1/2}$ and $|||\mathbf{G}|||_E \equiv [\mathbf{G}_E,\, \mathbf{G}_E]_E^{1/2}$.

Second, we consider the case of strongly curved faces. Let e be a strongly curved face, and E_1 and E_2 be two elements having e in common. Due to Assumption **(C1)**, all three components of $\mathbf{G}$ are continuous across e, so that *at every point* of e we have

$$\mathbf{G}_{E_1}^e \cdot \mathbf{n}_{E_1}^e + \mathbf{G}_{E_2}^e \cdot \mathbf{n}_{E_2}^e = 0.$$

Using the continuity of p, we can estimate the contribution of the face e to the sum in (5.13):

$$\begin{aligned}
\sum_{i=1}^{2} \int_e p_{E_i}^1\, \mathbf{G}_{E_i} \cdot \mathbf{n}_{E_i} \,\mathrm{d}S &= \sum_{i=1}^{2} \int_e (p_{E_i}^1 - p)\, \mathbf{G}_{E_i} \cdot \mathbf{n}_{E_i} \,\mathrm{d}S \\
&\leq \sum_{i=1}^{2} \|p - p_{E_i}^1\|_{L^2(e)}\, \|\mathbf{G}_{E_i}^e\|_{(L^2(e))^3} \\
&\leq \sum_{i=1}^{2} v_*^{-1/2} (C_{face}^*)^{1/2} h_{E_i} \|p\|_{H_2(E_i)} |\mathbf{G}_{E_i}^e|\, |E_i|^{1/2} \\
&\leq \sum_{i=1}^{2} C_{3,b}^*\, h \|p\|_{H^2(E_i)}\, |||\mathbf{G}_E|||_{E_i}.
\end{aligned} \tag{5.15}$$

Third, we consider the case of a moderately curved face e shared by two elements E_1 and E_2. Due to Assumption **(C1)**, only the component of $\mathbf{G}_E$ in the direction of $\tilde{\mathbf{n}}_E^e$ will be continuous across e. However, we have obviously from (2.4) that

$$\int_e q_0 (\mathbf{n}_{E_i}^e - \tilde{\mathbf{n}}_{E_i}^e) \,\mathrm{d}S = 0 \tag{5.16}$$

for $i = 1, 2$ and every constant q_0. Adding and subtracting $\tilde{\mathbf{n}}_{E_i}$, and then using (5.16) in the first term and the continuity of p and $\mathbf{G}_E \cdot \tilde{\mathbf{n}}_E$ in the second term, we get

$$\begin{aligned}
\sum_{i=1}^{2} \int_e p_{E_i}^1\, \mathbf{G}_{E_i} \cdot \mathbf{n}_{E_i} \,\mathrm{d}S &= \sum_{i=1}^{2} \int_e (p_{E_i}^1 - p_{E_i}^0)\, \mathbf{G}_{E_i} \cdot (\mathbf{n}_{E_i} - \tilde{\mathbf{n}}_{E_i}) \,\mathrm{d}S \\
&\quad + \sum_{i=1}^{2} \int_e (p_{E_i}^1 - p)\, \mathbf{G}_{E_i} \cdot \tilde{\mathbf{n}}_{E_i} \,\mathrm{d}S.
\end{aligned} \tag{5.17}$$

The second term in (5.17) can be estimated exactly as in (5.15):

$$\sum_{i=1}^{2} \int_e (p^1_{E_i} - p)\, \mathbf{G}_{E_i} \cdot \mathbf{n}_{E_i}\, \mathrm{d}S \leq \sum_{i=1}^{2} C^*_{3,b}\, h \|p\|_{H^2(E_i)}\, |||\mathbf{G}|||_{E_i}. \tag{5.18}$$

To estimate the first term, we finally use the fact that e is moderately curved, and in particular inequality (2.6):

$$\begin{aligned} \int_e (p^1_{E_i} - p^0_{E_i})\, \mathbf{G}_{E_i} \cdot (\mathbf{n}_{E_i} - \tilde{\mathbf{n}}_{E_i})\, \mathrm{d}S \;&\leq\; \sigma_* |e|^{1/2} \|p^1_{E_i} - p^0_{E_i}\|_{L^2(e)}\, \|\mathbf{G}^e_{E_i}\|_{(L^2(e))^3} \\ &\leq\; \sigma_* v_*^{-1/2} C^{**}_{face} h_{E_i} \|p\|_{H^1(E_i)} |\mathbf{G}^e_{E_i}|\, |E_i|^{1/2} \\ &\leq\; C^*_{3,m}\, h \|p\|_{H^2(E)}\, |||\mathbf{G}|||_E \end{aligned} \tag{5.19}$$

where C^{**}_{face} depends only on C^*_{app} and C^*_{agm} while $C^*_{3,m}$ also depends on the constant v_* appearing in (2.7) and the constant σ_* appearing in (2.6).

Collecting (5.14), (5.15), (5.17), (5.18), and (5.19) and noting that every element appears only as many times as the number of its faces, we prove the assertion of the lemma. $\square$

Combining (5.10), (5.12) and (5.13) with (5.4), we get the main convergence result.

Theorem 5.1 *Let $(p,\, \vec{F})$ be the solution of (2.1) and $(\mathbf{p}_d,\, \mathbf{F}_d)$ be the solution of (3.19). Moreover, let $\mathbf{F}^I$ be the interpolant of $\vec{F}$ introduced in (3.8). Then, under assumptions* **(P1)**, **(M1)–(M3)** *and* **(S1)–(S2)**, *we have*

$$|||\mathbf{F}^I - \mathbf{F}_d|||_{X^d} \leq\; C^*_F\, h\, \|p\|_{H^2(\Omega)} \tag{5.20}$$

where the constant C^*_F is independent of h and p.

We note that the constant C^*_F in (5.20) grows linearly with the parameter σ_* defining moderately curved faces.

5.2 Error estimates for the scalar variable

The estimates for the scalar variable mimic closely (but not exactly) the corresponding results for flat faces obtained in [8]. We report them for the convenience of the reader.

As is [8], the main estimate is based on a sort of duality estimate, and to get a full $O(h)$ order of convergence we assume that Ω is convex. Lower order of convergence could clearly be obtained under less restrictive assumptions.

Theorem 5.2 *Under assumptions of Theorem 5.1, plus the convexity of Ω, we have*

$$|||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d} \leq\; C^*_{pr,1}\, h\, \left(\|p\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)}\right) \tag{5.21}$$

where the constant $C^*_{pr,1}$ is independent of h, p and b.

Proof. Let ψ be the solution of

$$\begin{aligned} -\mathrm{div}(\mathbf{K}\nabla\psi) &= \mathbf{p}^I - \mathbf{p}_d && \text{in } \Omega \\ \psi &= 0 && \text{on } \partial\Omega \end{aligned} \tag{5.22}$$

where, for simplicity, we identified $\mathbf{p}_d - \mathbf{p}^I$ with the corresponding piecewise constant function. The convexity of Ω implies that there exists a constant C^*_Ω, depending only on Ω, such that

$$\|\psi\|_{H^2(\Omega)} \le C^*_\Omega\, |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}. \tag{5.23}$$

We set $\vec{G} = \mathbf{K}\nabla\psi$ and denote by $\mathbf{G}^I$ its interpolant. Then, using (3.17) and (5.22) we have

$$\mathcal{DIV}^d\, \mathbf{G}^I = (\mathrm{div}(\mathbf{K}\nabla\psi))^I = \mathrm{div}(\mathbf{K}\nabla\psi) = \mathbf{p}_d - \mathbf{p}^I. \tag{5.24}$$

Finally, we denote by ψ^1 a piecewise linear approximation of ψ that satisfies (5.7) for each E in Ω_h. Using (5.24), then (4.1), then (3.6) and (5.24), then integrating by parts, and finally integrating once more by parts and using (2.1), we get

$$\begin{aligned} |||\mathbf{p}_d - \mathbf{p}^I|||^2_{Q^d} &= [\mathcal{DIV}^d\, \mathbf{G}^I,\, \mathbf{p}_d - \mathbf{p}^I]_{Q^d} \\ &= [\mathbf{F}_d,\, \mathbf{G}^I]_{X^d} - [\mathcal{DIV}^d\, \mathbf{G}^I,\, \mathbf{p}^I]_{Q^d} = [\mathbf{F}_d,\, \mathbf{G}^I]_{X^d} - \int_\Omega p\, \mathrm{div}(\mathbf{K}\nabla\psi)\, \mathrm{d}V \\ &= [\mathbf{F}_d,\, \mathbf{G}^I]_{X^d} + \int_\Omega \mathbf{K}\nabla\, p \cdot \nabla\psi\, \mathrm{d}V = [\mathbf{F}_d,\, \mathbf{G}^I]_{X^d} + \int_\Omega b\, \psi\, \mathrm{d}V. \end{aligned}$$

Now, using the above equation the definition of $\mathbf{G}^I$ and adding and subtracting terms, we have

$$\begin{aligned} |||\mathbf{p}_d - \mathbf{p}^I|||^2_{Q^d} &= [\mathbf{F}_d,\, (\mathbf{K}\nabla\psi)^I - (\mathbf{K}\nabla\psi^1)^I]_{X^d} + [\mathbf{F}_d,\, (\mathbf{K}\nabla\psi^1)^I]_{X^d} + \int_\Omega b\,\psi\, \mathrm{d}V \\ &= J_1 + [\mathbf{F}_d,\, ((\mathbf{K} - \tilde{\mathbf{K}})\nabla\psi^1)^I]_{X^d} + [\mathbf{F}_d,\, (\tilde{\mathbf{K}}\nabla\psi^1)^I]_{X^d} + \int_\Omega b\,\psi\, \mathrm{d}V \\ &= J_1 + J_2 + [\mathbf{F}_d,\, (\tilde{\mathbf{K}}\nabla\psi^1)^I]_{X^d} + \int_\Omega b\,\psi\, \mathrm{d}V. \end{aligned} \tag{5.25}$$

As in (5.10), the term J_1 can be easily bounded by

$$J_1 \equiv\ [\mathbf{F}_d,\, (\mathbf{K}\nabla\psi)^I - (\mathbf{K}\nabla\psi^1)^I]_{X^d} \le\ C^*_{I_1}\, h\, |||\mathbf{F}_d|||_{X^d}\, \|\psi\|_{H^2(\Omega)}. \tag{5.26}$$

The term J_2 is bounded as in (5.11), (5.12) by

$$J_2 \equiv\ [\mathbf{F}_d,\, ((\mathbf{K} - \tilde{\mathbf{K}})\nabla\psi^1)^I]_{X^d} \le\ C^*_{I_2}\, h\, |||\mathbf{F}_d|||_{X^d}\, \|\psi\|_{H^2(\Omega)}. \tag{5.27}$$

For the third term in the last line of (5.25) we can first use (5.1) to obtain

$$[\mathbf{F}_d,\, (\tilde{\mathbf{K}}\nabla\psi^1)^I]_{X^d} = \sum_{E\in\Omega_h} \int_{\partial E} \psi^1 (\mathbf{F}_d)_E \cdot \mathbf{n}_E\, \mathrm{d}S - \int_\Omega \mathbf{b}\, \psi^1\, \mathrm{d}V.$$

With the help of (5.13), we get then

$$\left| [\mathbf{F}_d,\, (\tilde{\mathbf{K}}\nabla\psi^1)^I]_{X^d} + \int_\Omega b\,\psi\, \mathrm{d}V \right| \le\ C^*_{I_3}\, h\, |||\mathbf{F}_d|||_{X^d}\, \|\psi\|_{H^2(\Omega)} + \left| \int_\Omega (b\,\psi - \mathbf{b}\psi^1)\, \mathrm{d}V \right| \tag{5.28}$$

where the last term is easily bounded by $2\,C^*_{app}\,h\,\|b\|_{H^1(\Omega)}\,\|\psi\|_{H^1(\Omega)}$. Collecting the above inequalities (5.25) - (5.28), we obtain

$$|||\mathbf{p}_d - \mathbf{p}^I|||^2_{Q^d} \le\, C^*\,h\,\left\{|||\mathbf{F}_d|||_{X^d} + \|b\|_{H^1(\Omega)}\right\}\|\psi\|_{H^2(\Omega)}$$

that combined with estimates (5.23), Theorem 5.1 and Lemma 4.1 completes the proof of the theorem. □

It is interesting to note that, if we had, in each element E, a suitable lifting operator R_E from ∂E to the interior of E, a better estimate for the scalar variable would be obtained. But first, we recall the following result which is proved in [8].

Lemma 5.2 *Assume that for every element E in Ω_h we have a lifting operator R_E acting on X^d_E (the restriction of X^d to E) and with values in $(L^2(E))^3$ such that*

$$\begin{aligned} R_E(\mathbf{G}_E)\big|_e &\equiv \mathbf{G}^e_E \qquad && \textit{on} \quad \partial E \\ \operatorname{div} R_E(\mathbf{G}_E) &\equiv (\mathcal{DIV}^d\;\mathbf{G})_E && \textit{in} \quad E \end{aligned} \tag{5.29}$$

for all $\mathbf{G} \in X^d$, and

$$R_E(\mathbf{G}^I_E) = \vec{G} \tag{5.30}$$

for all $\vec{G}$ constant on E. Then, the choices

$$\{\tilde{\mathbf{K}}\}_{i,j} := \frac{1}{|E|}\int_E \{\mathbf{K}\}_{i,j}\,\mathrm{d}V$$

and

$$[\mathbf{F},\,\mathbf{G}]_E := \int_E \tilde{\mathbf{K}}^{-1} R_E(\mathbf{F}_E)\cdot R_E(\mathbf{G}_E)\,\mathrm{d}V$$

will automatically satisfy (5.2) and (5.1). If moreover there exist two positive constants c^*_R and C^*_R, independent of E, such that

$$c^*_R h_E\,\|R_E(\mathbf{G})\|^2_{(L^2(\partial E))^3} \,\le\, \|R_E(\mathbf{G})\|^2_{(L^2(E))^3} \,\le\, C^*_R h_E\,\|R_E(\mathbf{G})\|^2_{(L^2(\partial E))^3} \tag{5.31}$$

for all $\mathbf{G} \in X^d$, then (3.15) will also hold with constants s_* and S^* depending only on c^*_R, C^*_R and on the constants κ_*, κ^* appearing in (2.2).

The next result shows the superconvergence of the scalar variable in the mesh dependent L_2 norm.

Theorem 5.3 *In addition to the assumptions of Theorem 5.2, we assume that for each element E there exists a lifting operator R_E with the properties (5.29), (5.30) and (5.31). Then, the choice*

$$[\mathbf{F},\,\mathbf{G}]_E := \int_E \mathbf{K}^{-1} R_E(\mathbf{F}_E)\cdot R_E(\mathbf{G}_E)\,\mathrm{d}V \tag{5.32}$$

will give

$$|||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d} \le\, C^*_{pr,2}\,h^2\,\left(\|p\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)}\right) \tag{5.33}$$

where the constant $C^*_{pr,2}$ is independent of h, p and b.

Proof. Let $\vec{G}$ be a vector-valued function in $(H^1(E))^3$ and $\mathbf{G}^I$ be its interpolant. Using properties (5.31) and (4.10), we get

$$\|R_E(\mathbf{G}^I)\|_{(L^2(E))^3} \leq \frac{C_R^*}{\beta_s^*}\|\vec{G}\|_{1,h,E}. \tag{5.34}$$

We denote now by $\vec{G}_0$ the integral average (component-wise) of $\vec{G}$ over E. Using property (5.30), estimate (5.34) and the approximation result (5.6) we have then

$$\begin{aligned} \|R_E(\mathbf{G}^I) - \vec{G}\|_{(L^2(E))^3} &\leq \|R_E(\mathbf{G}^I - \mathbf{G}_0^I)\|_{(L^2(E))^3} + \|\vec{G}_0 - \vec{G}\|_{(L_2(E))^3} \\ &\leq \frac{C_R^*}{\beta_s^*}\|\vec{G} - \vec{G}_0\|_{1,h,E} + \|\vec{G}_0 - \vec{G}\|_{(L^2(E))^3} \\ &\leq C_{Ra}^* h_E \|\vec{G}\|_{(H^1(E))^3} \end{aligned} \tag{5.35}$$

where the constant C_{Ra} depends only on C_R^*, β_s^*, and C_{app}^*.

Now, we set $\vec{G} = \mathbf{K}\nabla\psi$ where ψ is the solution of (5.22). Let ψ^I be the piecewise constant interpolant of ψ as in (3.6), and let $R(\mathbf{G}^I)$ be such that $R(\mathbf{G}^I)|_E = R_E(\mathbf{G}_E^I)$ for all elements E. Following essentially [11] and using (5.24), then (4.1), then (3.6) and (5.24) (as in the previous proof) with (5.29), then integrating by parts, and finally using (2.1) and (5.32), we get

$$\begin{aligned} |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}^2 &= [\mathcal{DIV}^d\, \mathbf{G}^I,\, \mathbf{p}_d - \mathbf{p}^I]_{Q^d} = [\mathbf{F}_d,\, \mathbf{G}^I]_{X^d} - \int_\Omega p \operatorname{div} R(\mathbf{G}^I)\,\mathrm{d}V \\ &= [\mathbf{F}_d,\, \mathbf{G}^I]_{X^d} + \int_\Omega \nabla p \cdot R(\mathbf{G}^I)\,\mathrm{d}V = [\mathbf{F}_d,\, \mathbf{G}^I]_{X^d} + \int_\Omega \mathbf{K}^{-1}\mathbf{K}\nabla p \cdot R(\mathbf{G}^I)\,\mathrm{d}V \\ &= \int_\Omega \mathbf{K}^{-1}(R(\mathbf{F}_d) - \vec{F})\, R(\mathbf{G}^I)\,\mathrm{d}V. \end{aligned}$$

Adding and subtracting $\vec{G}$, we get

$$\begin{aligned} |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}^2 &= \int_\Omega \mathbf{K}^{-1}(R(\mathbf{F}_d) - \vec{F})\,(R(\mathbf{G}) - \vec{G})\,\mathrm{d}V + \int_\Omega \mathbf{K}^{-1}(R(\mathbf{F}_d) - \vec{F})\,\vec{G}\,\mathrm{d}V \\ &= J_3 + \int_\Omega (R(\mathbf{F}_d) - \vec{F})\,\nabla\psi\,\mathrm{d}V = J_3 - \int_\Omega \psi \operatorname{div}(R(\mathbf{F}_d) - \vec{F})\,\mathrm{d}V \\ &= J_3 - \int_\Omega (\mathbf{b}^I - b)\psi\,\mathrm{d}V \\ &= J_3 - \int_\Omega (\mathbf{b}^I - b)(\psi - \boldsymbol{\psi}^I)\,\mathrm{d}V = J_3 + J_4. \end{aligned} \tag{5.36}$$

The terms J_3 and J_4 can be easily bounded using the previous estimates and usual arguments. Indeed, the triangle inequality, then (5.31), and finally (5.20) and (5.35) imply that

$$\begin{aligned} \|R(\mathbf{F}_d) - \vec{F}\|_{(L^2(\Omega))^3} &\leq \|R(\mathbf{F}_d - \mathbf{F}^I)\|_{(L^2(\Omega))^3} + \|R(\mathbf{F}^I) - \vec{F}\|_{(L^2(\Omega))^3} \\ &\leq C_R^*|||\mathbf{F}_d - \mathbf{F}^I|||_{X^d} + \|R(\mathbf{F}^I) - \vec{F}\|_{(L^2(\Omega))^3} \\ &\leq C\,h\,\|p\|_{H^2(\Omega)}. \end{aligned} \tag{5.37}$$

Using (5.35) and (5.23), we get

$$\|R(\mathbf{G}^I) - \vec{G}\|_{(L^2(\Omega))^3} \leq C^*_{Ra} h \|\vec{G}\|_{(H^1(\Omega))^3} \leq C^*_{Ra} C^*_{\Omega} h |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}. \tag{5.38}$$

The approximation property (5.6) gives the following estimates:

$$|||\mathbf{b}^I - b\|_{L^2(\Omega)} \leq C^*_{app} h \|b\|_{H^1(\Omega)} \tag{5.39}$$

and

$$\|\psi - \boldsymbol{\psi}^I\|_{L^2(\Omega)} \leq C^*_{app} h \|\psi\|_{H^1(\Omega)} \leq C^*_{app} C^*_{\Omega} h |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}. \tag{5.40}$$

Inserting estimates (5.37)-(5.40) into (5.36), we prove the assertion of the theorem. □

6 Conclusion

We have developed a new mimetic finite difference method for the diffusion problem on unstructured polyhedral meshes with moderately and strongly curved faces. We have proved the optimal convergence rates for both the scalar and vector variables.

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Appendix A

Proof of Lemma 2.1. Definition 2.1 implies that there exists a bijective mapping $\varphi\colon b_0 \to \mathbb{R}$ such that the restriction of Φ to b_0 can be written as

$$x' = x\varphi(x,y), \qquad y' = y\varphi(x,y), \qquad z' = h_0\varphi(x,y). \tag{A.1}$$

Using assumption (2.3), it is not difficult to check that for every pair of points $\mathbf{P}_1$ and $\mathbf{P}_2$ on b_0, and their images $\mathbf{P}'_1 = \Phi(\mathbf{P}_1)$ and $\mathbf{P}'_2 = \Phi(\mathbf{P}_2)$ on e, we have

$$|\mathbf{P'}_1 - \mathbf{P}'_2| \leq \tau|\mathbf{P}_1 - \mathbf{P}_2| \qquad \text{and} \qquad |\mathbf{P}_1 - \mathbf{P}_2| \leq \tau|\mathbf{P}'_1 - \mathbf{P}'_2|. \tag{A.2}$$

By basic vector calculus, we have

$$\int_e \mathbf{n}\,\mathrm{d}S = \int_{b_0} \frac{\partial \mathbf{P}'}{\partial x} \wedge \frac{\partial \mathbf{P}'}{\partial y}\,\mathrm{d}x\,\mathrm{d}y. \tag{A.3}$$

Differentiating (A.1), we get

$$\frac{\partial \mathbf{P}'}{\partial x} = (\varphi + x\varphi_x,\, y\varphi_x,\, h_0\varphi_x) \qquad \text{and} \qquad \frac{\partial \mathbf{P}'}{\partial y} = (x\varphi_y,\, \varphi + y\varphi_y,\, h_0\varphi_y).$$

A lengthy but easy calculation gives

$$\frac{\partial \mathbf{P}'}{\partial x} \wedge \frac{\partial \mathbf{P}'}{\partial y} = \left(-h_0 \varphi\, \varphi_x,\, -h_0 \varphi\, \varphi_y,\, \varphi^2 + \varphi(x\varphi_x + y\varphi_y)\right). \tag{A.4}$$

Now, let $\boldsymbol{\xi} \equiv (\xi_1,\, \xi_2,\, h_0)$ be a point in b_0 and $g = \varphi^2/2$. Using (A.4) and (A.3) in (2.4), and then integrating by parts, we get

$$\begin{aligned}
\tilde{\mathbf{n}} \cdot \boldsymbol{\xi} &= \frac{1}{|e|} \int_{b_0} h_0 (2g + (x - \xi_1) g_x + (y - \xi_2) g_y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{h_0}{|e|} \left(\int_{b_0} (2g - g - g)\, \mathrm{d}x\, \mathrm{d}y + \int_{\partial b_0} g \left\{ (x - \xi_1)\nu_x + (y - \xi_2)\nu_y \right\} \mathrm{d}\ell \right)
\end{aligned}$$

where $(\nu_x,\, \nu_y)$ is the outward unit normal to ∂b_0 lying in the plane $z = h_0$. Let g_{min} be the minimum value of g on ∂b_0. Since $\boldsymbol{\xi}$ is internal to b_0 and b_0 is convex, we have

$$(x - \xi_1)\nu_x + (y - \xi_2)\nu_y \geq 0.$$

Using the mean value theorem for integrals (since g is also nonnegative) and then using the divergence theorem on b_0 (in the plane $z = h_0$) we have then

$$\tilde{\mathbf{n}} \cdot \boldsymbol{\xi} \geq g_{min} \frac{h_0}{|e|} \int_{\partial b_0} \left\{ (x - \xi_1)\nu_x + (y - \xi_2)\nu_y \right\}\, \mathrm{d}\ell = (\varphi^2)_{min} h_0 \frac{|b_0|}{|e|}.$$

Thus, the Cauchy-Schwarz inequality implies that

$$|\tilde{\mathbf{n}}| \geq\, (\varphi^2)_{min} \frac{h_0}{|\boldsymbol{\xi}|} \frac{|b_0|}{|e|}. \tag{A.5}$$

To complete the proof, we have to estimate three factors in the right hand side of (A.5). From (A.2), we have easily that

$$|e| \leq \tau^2\, |b_0|. \tag{A.6}$$

Next, using (2.3) and taking any point $\mathbf{P}$ on ∂b_0, its image point $\mathbf{P}' = \Phi(\mathbf{P})$ on ∂e, and the vertex $\mathbf{V} = \mathbf{V}'$ (the origin), we have

$$\frac{|\mathbf{P}|}{|\mathbf{P}'|} \equiv \frac{|\mathbf{P} - \mathbf{V}|}{|\mathbf{P}' - \mathbf{V}'|} \leq \tau.$$

Thus, (A.1) implies that

$$(\varphi^2)_{min} \geq \frac{1}{\tau^2}. \tag{A.7}$$

Finally, we recall that the pyramid $\mathcal{P}_0$ contains a sphere of radius $r \geq \gamma d_0$, where d_0 is the diameter of $\mathcal{P}_0$. Since $|\boldsymbol{\xi}| = |\boldsymbol{\xi} - \mathbf{V}| \leq d_0$ and $2r \leq h_0$, we deduce that

$$|\boldsymbol{\xi}| \leq d_0 \leq \frac{r}{\gamma} \leq \frac{h_0}{2\gamma}. \tag{A.8}$$

The result follows from estimates (A.5), (A.6), (A.7), and (A.8). $\square$

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