A PRIORI ERROR ANALYSIS OF RESIDUAL-FREE BUBBLES FOR ADVECTION-DIFFUSION PROBLEMS*

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Abstract. We develop an *a priori* error analysis of a finite element approximation to the elliptic advection-diffusion equation $-\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u = f$ subject to a homogeneous Dirichlet boundary condition, based on the use of residual-free bubble functions. An optimal order error bound is derived in the so-called stability-norm

$$\left(\varepsilon\|\nabla v\|^2_{L_2(\Omega)} + \sum_T h_T\|\boldsymbol{a}\cdot\nabla v\|^2_{L_2(T)}\right)^{1/2},$$

where h_T denotes the diameter of element T in the subdivision of the computational domain.

1. Introduction. Suppose that Ω is a bounded polygonal domain in the plane and assume, for simplicity, that $\boldsymbol{a} = (a_1, a_2)$ is a two-component vector function whose entries are constant on Ω. Assume further that f is a piecewise constant function defined on Ω. We note that our results are valid under more general hypotheses on the data (which will be discussed in the final section) and in any number of space dimensions. Given that ε is a positive constant, we consider the elliptic boundary-value problem

$$\begin{cases} -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

This is a fundamental model problem in computational fluid dynamics and one that exposes the weaknesses of classical numerical approaches, such as central and upwind finite difference methods, as well as Galerkin finite element methods (see [6] for examples and elaboration). To simultaneously achieve stability and accuracy, uniformly in advection- and diffusion-dominated limits, a new finite element method was introduced by Hughes and colleagues in a series of papers (see [6] and references therein for earlier works). This method was referred to as SUPG and is now viewed as falling within the general class of techniques referred to as *stabilised methods*, which have been further developed and studied by a number of authors (see, e.g., [8], [9], [17], [18], [20]). The basic idea is this: starting with the Galerkin finite element method, add terms depending on the residual which enhance stability. This can be done in such a way that accuracy is retained simultaneously with achieving better stability behaviour, and thus the method represents a solution to a long-standing and fundamental problem of computational fluid dynamics. The original instantiation of the method was developed intuitively and corroborated with Fourier analysis of simple

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cases and numerical verification. Johnson and collaborators soon after discerned the mathematical structure of the method and developed error estimates in Sobolev norms exhibiting uniform behaviour over the full range of advective-diffusive phenomena [21]. The mathematical analysis of stabilised methods is by now a mature topic and many practically important cases have been investigated.

In recent years, attempts have been made to derive stabilised methods from fundamental principles and thereby attain at once a deeper understanding and a road map to more precise generalisations. This goal has led to the development of two methodologies: *residual-free bubbles* (see, [1], [2], [5], [11], [12], [13], [14], [23], [24], [10]) and *the variational multiscale method* (see, [15], [16], [19]). Both methods view the numerical solution to be composed of a standard finite element approximation and additional functions which are constructed to improve resolution of scales which are unresolvable by conventional finite element approximations. This can take a variety of forms, and only the simplest incarnations of the idea have been extensively studied so far, namely, the case in which the additional functions are the so called *residual-free bubbles* which vanish on element boundaries and are chosen to satisfy the advection-diffusion equation strongly on each element. Remarkably, this idea provides a complete framework for deriving classical stabilised methods.

It needs to be mentioned that many stabilised methods, such as SUPG, GLS (Galerkin/Least-Squares), GGLS (Galerkin/Gradient Least-Squares), etc., usually do not fit exactly within the framework of residual-free bubbles. Nevertheless, these methods are closely related to the one derived from residual-free bubbles. Likewise, the variational multiscale method suggests a more general structure for stabilised methods (see [15], [16], [19]), but these newer ideas still remain in an initiatory and untested state.

Our current understanding of the mathematical behaviour of methods derived from residual-free bubbles and variational multiscale concepts emanates from their identification with stabilised methods, which, as mentioned previously, is mathematically well-developed. It has occurred to us that it should be possible, and may be enlightening, to directly perform a mathematical analysis of these newer methods. We embark upon this path in the current work in which we investigate the residual-free bubbles method assuming underlying piecewise linear, C^0-continuous triangular finite elements. In this case, the classical SUPG and GLS methods coincide with what Johnson refers to as the *streamline diffusion method*. We caution the reader that in more general situations there is a lot more to stabilised methods than a simple addition of a streamline diffusion operator. Even in the present case there is an absolutely crucial alteration to the source term f, which cannot be omitted without serious degeneration of accuracy (see [6]).

Several noteworthy aspects of the present study emerge:

- We are able to recapture the standard error estimates for the streamline diffusion method with piecewise linear finite element approximation.
- We are also able to estimate the error in the entire solution consisting of the finite element approximation supplemented with residual-free bubbles.
- The mesh-dependent norm employed in the analysis of stabilised methods, referred to herein as the *stability norm*, emerges naturally from the present analysis as well as a precise formula for the so-called stabilisation parameter.

In addition, we view the present analysis as more fundamental and revealing than the usual analysis of the related stabilised methods. However, the downside is that it is considerably more involved. Hopefully, it will represent the first step towards a complete analysis of a broader class of residual-free bubbles/variational multiscale methods which we believe will be useful for the development of improved methodology for computational fluid dynamics and other important physical problems.

The paper is structured as follows. In Section 2, we study the properties of the bubble function, and derive an upper bound on its maximum norm and a lower bound on its L_1 norm. These preliminary results will play a crucial role in the error analysis. In Section 3, we formulate the discrete problem and show that the use of the bubble function induces a natural norm, the so called stability-norm, on $H_0^1(\Omega)$. In particular, we show that the stability norm is similar to the norm that arises in the analysis of the streamline diffusion method. In Section 4, we embark on the error analysis of the method and derive optimal bounds on the error in the stability norm. We conclude, in Section 5, by commenting on various extensions of our theoretical results.

2. Basic properties of the bubble b_1^T. A fundamental ingredient of the numerical method and its error analysis is the bubble function b_1^T, which is defined, in every triangle T, as the solution of the local boundary value problem

$$\begin{cases} -\varepsilon\Delta b_1^T + \boldsymbol{a}\cdot\nabla b_1^T = 1 & \text{in } T, \\ b_1^T = 0 & \text{on } \partial T. \end{cases} \tag{2.1}$$

Multiplying (2.1) by b_1^T and integrating over T, we have

$$\varepsilon\|\nabla b_1^T\|_{0,T}^2 = \int_T b_1^T \,\mathrm{d}x, \tag{2.2}$$

an equality that will be used frequently throughout the paper.

Lemma 1.1 *For every $\varepsilon > 0$ and every triangle T, we have*

$$0 \le b_1^T \le \boldsymbol{a}\cdot(x - x_{\boldsymbol{a}})/|\boldsymbol{a}|^2 \qquad \forall x \in T, \tag{2.3}$$

where $x_{\boldsymbol{a}}$ is the "upwind-most" point in $\overline{T}$, defined by the inequality

$$\boldsymbol{a}\cdot(x - x_{\boldsymbol{a}}) \ge 0 \qquad \forall x \in T. \tag{2.4}$$

Proof. The first inequality in (2.3) follows directly from the maximum principle. Indeed,

$$\begin{cases} \mathcal{L}b_1^T := -\varepsilon\Delta b_1^T + \boldsymbol{a}\cdot\nabla b_1^T = 1 > 0 & \text{in } T, \\ b_1^T \ge 0 & \text{on } \partial T. \end{cases} \tag{2.5}$$

The second inequality in (2.3) follows by considering the auxiliary function

$$w(x) := \boldsymbol{a}\cdot(x - x_{\boldsymbol{a}})/|\boldsymbol{a}|^2 - b_1^T. \tag{2.6}$$

Since

$$\begin{cases} \mathcal{L}w = 0 & \text{in } T, \\ w \ge 0 & \text{on } \partial T, \end{cases} \tag{2.7}$$

we have $w(x) \ge 0$ in all T, again from the maximum principle, and the result follows.
∎

Lemma 1.1 gives rise to some useful inequalities. Upon defining

$$h_T^{\boldsymbol{a}} := \max_{x\in\overline{T}} \boldsymbol{a}\cdot(x - x_{\boldsymbol{a}})/|\boldsymbol{a}|^2, \tag{2.8}$$

we have from (2.3) and (2.8) that

$$0 \le b_1^T \le h_T^{\boldsymbol{a}} \qquad \forall x \in T, \tag{2.9}$$

which implies

$$\int_T b_1^T \,\mathrm{d}x \le h_T^{\boldsymbol{a}}|T|, \qquad \int_T (b_1^T)^2 \,\mathrm{d}x \le h_T^{\boldsymbol{a}} \int_T b_1^T \,\mathrm{d}x. \tag{2.10}$$

In what follows, it will be convenient to set

$$\widetilde{h}_T := \frac{1}{|T|}\int_T b_1^T \,\mathrm{d}x, \qquad h_T^* := \int_T (b_1^T)^2 \,\mathrm{d}x \Big/ \int_T b_1^T \,\mathrm{d}x. \tag{2.11}$$

Note that, from (2.10), we have

$$\widetilde{h}_T \le h_T^{\boldsymbol{a}} \le h_T/|\boldsymbol{a}|, \qquad h_T^* \le h_T^{\boldsymbol{a}} \le h_T/|\boldsymbol{a}|, \qquad h_T = \text{diameter of } T. \tag{2.12}$$

N.B.: In several applications (e.g. in fluid dynamics), u represents a velocity, and the right-hand side f is a force per unit mass (i.e., it has dimension of an acceleration). In that case, the quantities b_1^T, $\widetilde{h}_T$, h_T^* and $h_T^{\boldsymbol{a}}$ have dimensions of time, whereas h_T has dimension of length. Hughes and collaborators refer to $\widetilde{h}_T$ as the "intrinsic time scale" and denote it by τ (see, e.g., [15]). It plays a fundamental role in the definition of the stability norm and in the error analysis.

The following formulae will be used in the sequel: from (2.11) we have, for any constant function c_T on T,

$$c_T^2 \int_T b_1^T \,\mathrm{d}x = \widetilde{h}_T ||c_T||_{0,T}^2; \tag{2.13}$$

from (2.2) we also have

$$||b_1^T||_{0,T}^2 = h_T^* \varepsilon ||\nabla b_1^T||_{0,T}^2. \tag{2.14}$$

In our analysis we shall also need a *lower* bound on $\widetilde{h}_T$; this is provided by the following Lemma.

Lemma 1.2 *Assume that the minimum angle of T is bounded below by a fixed positive constant $\theta_0 > 0$. Then, there exists a constant K, independent of T, $\boldsymbol{a}$ and ε, such that*

$$\widetilde{h}_T \ge K\frac{h_T}{|\boldsymbol{a}|}\min\left\{\frac{h_T|\boldsymbol{a}|}{\varepsilon}, 1\right\}. \tag{2.15}$$

Proof. Let $\widehat{T}$ be a triangle having one vertex at the origin of the $(\widehat{x}_1, \widehat{x}_2)$ coordinate system, and another vertex at $(1,0)$, such that T can be mapped onto $\widehat{T}$ by means of rigid movements (translation and rotation) and dilation by a factor $1/h_T$, so that the longest edge of T is mapped into a subset of the $\widehat{x}_1$ axis. The image $\widehat{b}_1$ of b_1^T defined on $\widehat{T}$ satisfies

$$\begin{cases} -\dfrac{\varepsilon}{h_T^2}\Delta\widehat{b}_1 + \dfrac{\widehat{\boldsymbol{a}}}{h_T}\cdot\nabla\widehat{b}_1 = 1 & \text{in } \widehat{T}, \\ \widehat{b}_1 = 0 & \text{on } \partial\widehat{T}, \end{cases} \tag{2.16}$$

where $\widehat{\boldsymbol{a}}$ is such that $|\widehat{\boldsymbol{a}}| = |\boldsymbol{a}|$. Now let $\widehat{\lambda}_1,\ \widehat{\lambda}_2,\ \widehat{\lambda}_3$ be the barycentric coordinates on $\widehat{T}$, and put

$$\widehat{b}_3 := \widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3. \tag{2.17}$$

We set

$$M_\Delta := \max_{\widehat{T}} |\Delta\widehat{b}_3|, \qquad M_g := \max_{\widehat{T}} |\nabla\widehat{b}_3|, \tag{2.18}$$

and remark that these quantities depend continuously on the coordinates of the third vertex of $\widehat{T}$. Finally, we define

$$\gamma := \frac{1}{M_\Delta + M_g}\min\left\{\frac{h_T}{\varepsilon}, \frac{1}{|\widehat{\boldsymbol{a}}|}\right\}, \tag{2.19}$$

$$\widehat{w} := \gamma\widehat{b}_3, \tag{2.20}$$

$$\widehat{v} := h_T^{-1}\widehat{b}_1, \tag{2.21}$$

and introduce the operator

$$\widehat{\mathcal{L}}\varphi := -\varepsilon h_T^{-1}\Delta\varphi + \widehat{\boldsymbol{a}}\cdot\nabla\varphi. \tag{2.22}$$

Now, using (2.20), (2.18)), and (2.19)) we have

$$|\widehat{\mathcal{L}}\widehat{w}| = |-\varepsilon h_T^{-1}\Delta\widehat{w} + \widehat{\boldsymbol{a}}\cdot\nabla\widehat{w}| \le \gamma(\varepsilon h_T^{-1}M_\Delta + |\widehat{\boldsymbol{a}}|M_g) \le 1 \tag{2.23}$$

so that, from (2.21), (2.16) and (2.23) we deduce

$$\widehat{\mathcal{L}}(\widehat{v} - \widehat{w}) = \widehat{\mathcal{L}}\widehat{v} - \widehat{\mathcal{L}}\widehat{w} = 1 - \widehat{\mathcal{L}}\widehat{w} \ge 0. \tag{2.24}$$

As both $\widehat{v}$ and $\widehat{w}$ vanish on $\partial\widehat{T}$, the maximum principle gives

$$\widehat{v} \ge \widehat{w} \qquad \forall x \in \widehat{T}. \tag{2.25}$$

Thus, using (2.21), (2.25), and (2.19)-(2.20),

$$\begin{aligned} \int_{\widehat{T}}\widehat{b}_1\,\mathrm{d}x &= h_T\int_{\widehat{T}}\widehat{v}\,\mathrm{d}x \ge h_T\int_{\widehat{T}}\widehat{w}\,\mathrm{d}x \\ &\ge h_T\tfrac{1}{M_\Delta+M_g}\min\left\{\tfrac{h_T}{\varepsilon}, \tfrac{1}{|\widehat{\boldsymbol{a}}|}\right\}\int_{\widehat{T}}\widehat{b}_3\,\mathrm{d}x \\ &=: h_T\min\left\{\tfrac{h_T}{\varepsilon}, \tfrac{1}{|\widehat{\boldsymbol{a}}|}\right\}\widehat{K}(\widehat{T}). \end{aligned} \tag{2.26}$$

Clearly $\widehat{K}(\widehat{T})$ depends continuously on the coordinates of the third vertex of $\widehat{T}$. The assumption on the minimum angle forces this vertex to remain inside a compact set, so that $\widehat{K}(\widehat{T})$ has a positive minimum $\widehat{K}$ depending only on θ_0. Finally, from (2.26) we obtain

$$\widetilde{h}_T := \int_T b_1^T\,\mathrm{d}x/|T| = \int_{\widehat{T}}\widehat{b}_1\,\mathrm{d}x \ge h_T\min\left\{\frac{h_T}{\varepsilon}, \frac{1}{|\widehat{\boldsymbol{a}}|}\right\}\widehat{K}, \tag{2.27}$$

which is equivalent to (2.15). ∎

3. The discrete problem and the "stability norm". Let $\mathcal{T}$ be a subdivision of Ω into triangles (satisfying the usual minimum angle condition [7], and the additional requirement that no edge of any triangle crosses a line of discontinuity of f) and let V_L be the space of piecewise linear functions on $\mathcal{T}$ that are in $H_0^1(\Omega)$. For every triangle T in $\mathcal{T}$ we consider the one-dimensional space B_T spanned by the function b_1^T defined in (2.1), and we set

$$V_B := \bigoplus_{T\in\mathcal{T}} B_T. \tag{3.1}$$

Finally, we define

$$V_h = V_L + V_B. \tag{3.2}$$

The discrete problem now reads

$$\begin{cases} \text{find } u_h = u_L + u_B \text{ in } V_h \text{ such that} \\ \varepsilon \int_\Omega \nabla u_h \cdot \nabla v_h \,\mathrm{d}x + \int_\Omega (\boldsymbol{a}\cdot\nabla u_h)\, v_h \,\mathrm{d}x = \int_\Omega f v_h \,\mathrm{d}x \quad \forall v_h \in V_h. \end{cases} \tag{3.3}$$

It is easy to verify, using the Lax-Milgram lemma, that problem (3.3) has a unique solution. Moreover, for every triangle T, taking $v_h = b_1^T$ gives

$$\varepsilon \int_T \nabla u_h \cdot \nabla b_1^T \,\mathrm{d}x + \int_T (\boldsymbol{a}\cdot\nabla u_h)\, b_1^T \,\mathrm{d}x = \int_T f b_1^T \,\mathrm{d}x. \tag{3.4}$$

Inserting $u_h = u_L + u_B$ into (3.4) and observing that

$$\int_T \nabla v_L \cdot \nabla b_1^T \,\mathrm{d}x = 0 \qquad \forall v_L \in V_L, \tag{3.5}$$

we obtain

$$\varepsilon \int_T \nabla u_B \cdot \nabla b_1^T \,\mathrm{d}x + \int_T (\boldsymbol{a}\cdot\nabla u_L)\, b_1^T \,\mathrm{d}x + \int_T (\boldsymbol{a}\cdot\nabla u_B)\, b_1^T \,\mathrm{d}x = \int_T f b_1^T \,\mathrm{d}x; \tag{3.6}$$

rearranging terms and recalling that f and $\boldsymbol{a}$ are piecewise constant and globally constant, respectively, we have from (3.6)

$$\varepsilon \int_T \nabla u_B \cdot \nabla b_1^T \,\mathrm{d}x + \int_T (\boldsymbol{a}\cdot\nabla u_B)\, b_1^T \,\mathrm{d}x = (f - \boldsymbol{a}\cdot\nabla u_L)|_T \int_T b_1^T \,\mathrm{d}x. \tag{3.7}$$

We remark now that $u_B|_T \in B_T$ and therefore it must be a constant multiple of b_1^T. Using again the definition (2.1) of b_1^T we obtain from (3.7)

$$u_B|_T = (f - \boldsymbol{a}\cdot\nabla u_L) b_1^T \qquad \text{in each } T. \tag{3.8}$$

In order to derive error estimates, we shall compare u_h with a suitable function $\widetilde{u} \in V_h$ that is "close to u". As an element of V_h, the function $\widetilde{u}$ will have the form

$$\widetilde{u} = \widetilde{u}_L + \widetilde{u}_B, \qquad \widetilde{u}_L \in V_L,\ \widetilde{u}_B \in V_B. \tag{3.9}$$

We now choose

$$\widetilde{u}_L = \text{(usual) piecewise linear interpolant of } u \text{ at the vertices of } \mathcal{T}, \tag{3.10}$$

and select $\widetilde{u}_B$ such that

$$\varepsilon \int_T \nabla(u-\widetilde{u}) \cdot \nabla b_1^T \,\mathrm{d}x + \int_T (\boldsymbol{a} \cdot \nabla(u-\widetilde{u}))\, b_1^T \,\mathrm{d}x = 0 \qquad \text{for each } T. \tag{3.11}$$

Let us see how (3.11) defines $\widetilde{u}_B$; we begin by rewriting (3.11) in the form

$$\varepsilon \int_T \nabla u \cdot \nabla b_1^T \,\mathrm{d}x + \int_T (\boldsymbol{a} \cdot \nabla u)\, b_1^T \,\mathrm{d}x = \varepsilon \int_T \nabla \widetilde{u} \cdot \nabla b_1^T \,\mathrm{d}x + \int_T (\boldsymbol{a} \cdot \nabla \widetilde{u})\, b_1^T \,\mathrm{d}x. \tag{3.12}$$

Now inserting (3.9) into (3.12) and using (3.5) and the fact that u is the solution of (1.1), we obtain

$$\int_T f b_1^T \,\mathrm{d}x = \varepsilon \int_T \nabla \widetilde{u}_B \cdot \nabla b_1^T \,\mathrm{d}x + \int_T (\boldsymbol{a} \cdot \nabla \widetilde{u}_L)\, b_1^T \,\mathrm{d}x \qquad \text{for each } T, \tag{3.13}$$

Rearranging terms,

$$\varepsilon \int_T \nabla \widetilde{u}_B \cdot \nabla b_1^T \,\mathrm{d}x = \int_T (f - \boldsymbol{a} \cdot \nabla \widetilde{u}_L)\, b_1^T \,\mathrm{d}x \qquad \text{for each } T. \tag{3.14}$$

Since $f - \boldsymbol{a} \cdot \nabla \widetilde{u}_L$ is constant in each T, and since, in each T, $\widetilde{u}_B$ must be a constant multiple of b_1^T, we immediately have from (2.2) that

$$\widetilde{u}_B|_T = (f - \boldsymbol{a} \cdot \nabla \widetilde{u}_L) b_1^T \qquad \text{in each } T. \tag{3.15}$$

Now, by setting

$$e = u_h - \widetilde{u} \in V_h, \tag{3.16}$$

we have naturally

$$e = e_L + e_B, \qquad \text{with } e_L \in V_L \text{ and } e_B \in V_B, \tag{3.17}$$

and from (3.8) and (3.15) we get

$$e_B|_T = -(\boldsymbol{a} \cdot \nabla e_L|_T) b_1^T. \tag{3.18}$$

Let us introduce the bilinear form

$$a(w,v) := \varepsilon \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\boldsymbol{a} \cdot \nabla w) v \,\mathrm{d}x; \tag{3.19}$$

then, it follows that

$$a(e,e) = \varepsilon ||\nabla e||_{0,\Omega}^2. \tag{3.20}$$

Now we shall show that the norm appearing in (3.20) is not as weak as it might seem at the first sight. Indeed, we have the following result.

THEOREM 3.1. *Suppose that u_h is the solution of (3.3) and let $\widetilde{u}$ be defined by (3.9), (3.10) and (3.15). Suppose further that $e_h = u_h - \widetilde{u}$, as in (3.16), with the natural decomposition (3.17). Then,*

$$\varepsilon^{1/2} ||\nabla e||_{0,\Omega} = \left(\varepsilon ||\nabla e_L||_{0,\Omega}^2 + \sum_T \widetilde{h}_T ||\boldsymbol{a} \cdot \nabla e_L||_{0,T}^2 \right)^{1/2}. \tag{3.21}$$

Proof. Let us rewrite (3.20) in a different form. In order to do so, we remark first that, due to (3.5), we have, for every $v = v_L + v_B$ in V_h,

$$\varepsilon||\nabla v||^2_{0,\Omega} = \varepsilon(\nabla v_L + \nabla v_B, \nabla v_L + \nabla v_B) = \varepsilon||\nabla v_L||^2_{0,\Omega} + \varepsilon||\nabla v_B||^2_{0,\Omega}. \tag{3.22}$$

On the other hand, by (3.18), using (2.2) and (2.13) we have

$$\begin{aligned} \varepsilon||\nabla e_B||^2_{0,T} &= \textstyle\sum_T (\boldsymbol{a}\cdot\nabla e_L)^2_T\, \varepsilon||\nabla b_1^T||^2_{0,T} \\ &= \textstyle\sum_T (\boldsymbol{a}\cdot\nabla e_L)^2_T \int_T b_1^T \,\mathrm{d}x = \sum_T \widetilde{h}_T ||\boldsymbol{a}\cdot\nabla e_L||^2_{0,T} \end{aligned} \tag{3.23}$$

Collecting (3.22) and (3.23) we obtain (3.21). ∎

The norm which appears on the right-hand side of (3.21) also arises in the analysis of the streamline diffusion finite element method for advection-dominated problems, and will be referred to in the remainder of the paper as the *stability-norm*. In fact, it is in this norm that we shall derive our error bounds.

4. Error estimates. As usual, we begin by observing that the exact solution u satisfies

$$a(u, v) = \int_\Omega f v \,\mathrm{d}x \qquad \forall v \in H^1_0(\Omega) \tag{4.1}$$

and that (3.3) can be rewritten as

$$a(u_h, v_h) = \int_\Omega f v_h \,\mathrm{d}x \qquad \forall v_h \in V_h, \tag{4.2}$$

so that, as $V_h \subset H^1_0(\Omega)$, we have the usual Galerkin property

$$a(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h. \tag{4.3}$$

In order to estimate the error $u - u_h$, we begin by seeking a bound on $e = u_h - \widetilde{u}$. For this purpose, we restart from (3.20). Upon adding and subtracting u, and using (4.3) we obtain

$$\varepsilon||\nabla e||^2_{0,\Omega} = a(e, e) = a(u_h - \widetilde{u}, e) = a(u_h - u, e) + a(u - \widetilde{u}, e) = a(u - \widetilde{u}, e). \tag{4.4}$$

Recalling the definition (3.19) of $a(\cdot,\cdot)$ we have

$$a(u - \widetilde{u}, e) = \varepsilon \int_\Omega \nabla(u - \widetilde{u}) \cdot \nabla e \,\mathrm{d}x + \int_\Omega (\boldsymbol{a}\cdot\nabla(u - \widetilde{u}))\, e \,\mathrm{d}x. \tag{4.5}$$

By (3.11), we have

$$a(u - \widetilde{u}, b_1^T) = 0 \qquad \forall T \in \mathcal{T}, \tag{4.6}$$

and, since e_B belongs to V_B, we have from (4.6) that

$$a(u - \widetilde{u}, e_B) = 0, \tag{4.7}$$

so that we must only evaluate

$$a(u - \widetilde{u}, e_L) = \varepsilon \int_\Omega \nabla(u - \widetilde{u}) \cdot \nabla e_L \,\mathrm{d}x + \int_\Omega (\boldsymbol{a}\cdot\nabla(u - \widetilde{u}))\, e_L \,\mathrm{d}x. \tag{4.8}$$

Thus (4.4) becomes

$$\varepsilon\|\nabla e\|_{0,\Omega}^2 = \varepsilon\int_\Omega \nabla(u-\widetilde{u})\cdot\nabla e_L\,\mathrm{d}x + \int_\Omega (\boldsymbol{a}\cdot\nabla(u-\widetilde{u}))\,e_L\,\mathrm{d}x \equiv I + II. \tag{4.9}$$

Thanks to (3.9), (3.5), and Cauchy-Schwarz inequality, the first term in (4.9) is easily bounded as follows:

$$\begin{aligned} I &= \varepsilon\textstyle\int_\Omega \nabla(u-\widetilde{u}_L-\widetilde{u}_B)\cdot\nabla e_L\,\mathrm{d}x = \varepsilon\int_\Omega \nabla(u-\widetilde{u}_L)\cdot\nabla e_L\,\mathrm{d}x \\ &\le \varepsilon\|\nabla(u-\widetilde{u}_L)\|_{0,\Omega}\,\|\nabla e_L\|_{0,\Omega}. \end{aligned} \tag{4.10}$$

Now, we deal with the second term in (4.9). Integration by parts and (3.9) give

$$\begin{aligned} II &= -\textstyle\int_\Omega (u-\widetilde{u})(\boldsymbol{a}\cdot\nabla e_L)\,\mathrm{d}x \\ &= -\textstyle\int_\Omega (u-\widetilde{u}_L)(\boldsymbol{a}\cdot\nabla e_L)\,\mathrm{d}x + \int_\Omega \widetilde{u}_B(\boldsymbol{a}\cdot\nabla e_L)\,\mathrm{d}x \equiv III + IV. \end{aligned} \tag{4.11}$$

The first term in (4.11) can be bounded via Cauchy-Schwarz inequality and (3.23)

$$\begin{aligned} III &= \sum_T \int_T (\widetilde{u}_L-u)(\boldsymbol{a}\cdot\nabla e_L)\,\mathrm{d}x \le \sum_T \|\widetilde{u}_L-u\|_{0,T}\,\|\boldsymbol{a}\cdot\nabla e_L\|_{0,T} \\ &= \sum_T \widetilde{h}_T^{-1/2}\|\widetilde{u}_L-u\|_{0,T}\,\widetilde{h}_T^{1/2}\|\boldsymbol{a}\cdot\nabla e_L\|_{0,T} \\ &\le \left(\sum_T \widetilde{h}_T^{-1}\|\widetilde{u}_L-u\|_{0,T}^2\right)^{1/2}\left(\sum_T \widetilde{h}_T\|\boldsymbol{a}\cdot\nabla e_L\|_{0,T}^2\right)^{1/2} \\ &= \left(\sum_T \widetilde{h}_T^{-1}\|\widetilde{u}_L-u\|_{0,T}^2\right)^{1/2}\left(\varepsilon^{1/2}\|\nabla e_B\|_{0,\Omega}\right). \end{aligned} \tag{4.12}$$

For the second term in (4.11), from (3.15), (3.18), (1.1), and integration by parts we deduce

$$\begin{aligned} IV &= \textstyle\sum_T\int_T (f-\boldsymbol{a}\cdot\nabla\widetilde{u}_L)\,b_1^T(\boldsymbol{a}\cdot\nabla e_L)\,\mathrm{d}x = \int_\Omega (\boldsymbol{a}\cdot\nabla\widetilde{u}_L - f)e_B\,\mathrm{d}x \\ &= \textstyle\int_\Omega (\varepsilon\Delta u - \boldsymbol{a}\cdot\nabla u + \boldsymbol{a}\cdot\nabla\widetilde{u}_L)e_B\,\mathrm{d}x \\ &= -\textstyle\int_\Omega \varepsilon\nabla u\cdot\nabla e_B\,\mathrm{d}x - \int_\Omega (\boldsymbol{a}\cdot\nabla(u-\widetilde{u}_L))e_B\,\mathrm{d}x \equiv V + VI. \end{aligned} \tag{4.13}$$

Now, from (3.5), Cauchy-Schwarz, and (2.14),

$$V = -\int_\Omega \varepsilon\nabla(u-\widetilde{u}_L)\cdot\nabla e_B\,\mathrm{d}x \le \varepsilon\|\nabla(u-\widetilde{u}_L)\|_{0,\Omega}\|\nabla e_B\|_{0,\Omega}, \tag{4.14}$$

and

$$\begin{aligned} VI &= -\textstyle\sum_T\int_T (\boldsymbol{a}\cdot\nabla(u-\widetilde{u}_L))\,e_B\,\mathrm{d}x \le \sum_T \|\boldsymbol{a}\cdot\nabla(u-\widetilde{u}_L)\|_{0,T}\,\|e_B\|_{0,T} \\ &= \textstyle\sum_T \|\boldsymbol{a}\cdot\nabla(u-\widetilde{u}_L)\|_{0,T}(h_T^*)^{1/2}\varepsilon^{1/2}\|\nabla e_B\|_{0,T} \\ &\le \left(\textstyle\sum_T h_T^*\|\boldsymbol{a}\cdot\nabla(u-\widetilde{u}_L)\|_{0,T}^2\right)^{1/2}\left(\varepsilon^{1/2}\|\nabla e_B\|_{0,\Omega}\right). \end{aligned} \tag{4.15}$$

Inserting (4.14) and (4.15) into (4.13) gives

$$
\begin{aligned}
& IV = V + VI \\
& \leq \varepsilon^{1/2} \|\nabla e_B\|_{0,\Omega} \left(\varepsilon^{1/2} \|\nabla (u - \widetilde{u}_L)\|_{0,\Omega} + \left(\textstyle\sum_T h_T^* \|\boldsymbol{a} \cdot \nabla (u - \widetilde{u}_L)\|_{0,T}^2 \right)^{1/2} \right).
\end{aligned} \tag{4.16}
$$

Now substituting (4.12) and (4.16) into (4.11) yields

$$
\begin{aligned}
II &= III + IV \leq \varepsilon^{1/2} \|\nabla e_B\|_{0,\Omega} \times \left\{ \left(\textstyle\sum_T \widetilde{h}_T^{-1} \|\widetilde{u}_L - u\|_{0,T}^2 \right)^{1/2} \right. \\
& \qquad \left. + \, \varepsilon^{1/2} \|\nabla (u - \widetilde{u}_L)\|_{0,\Omega} + \left(\textstyle\sum_T h_T^* \|\boldsymbol{a} \cdot \nabla (u - \widetilde{u}_L)\|_{0,T}^2 \right)^{1/2} \right\} \\
&\equiv \varepsilon^{1/2} \|\nabla e_B\|_{0,\Omega} \{A + B + C\}.
\end{aligned} \tag{4.17}
$$

Upon inserting (4.10) and (4.17) into (4.9), we have

$$
\begin{aligned}
\varepsilon \|\nabla e\|_{0,\Omega}^2 &= I + II \leq \varepsilon \|\nabla (u - \widetilde{u}_L)\|_{0,\Omega} \|\nabla e_L\|_{0,\Omega} + \varepsilon^{1/2} \|\nabla e_B\|_{0,\Omega} (A + B + C) \\
&= \varepsilon^{1/2} \|\nabla e_L\|_{0,\Omega} B + \varepsilon^{1/2} \|\nabla e_B\|_{0,\Omega} (A + B + C) \\
&\leq \frac{1}{2} \left(\varepsilon \|\nabla e_L\|_{0,\Omega}^2 + \varepsilon \|\nabla e_B\|_{0,\Omega}^2 \right) + \frac{1}{2} \left(B^2 + (A + B + C)^2 \right) \\
&= \frac{1}{2} \varepsilon \|\nabla e\|_{0,\Omega}^2 + \frac{1}{2} \left(B^2 + (A + B + C)^2 \right).
\end{aligned}
$$

Noting that $B^2 + (A + B + C)^2 \leq 4(A^2 + B^2 + C^2)$, we arrive at the following result.

THEOREM 4.1. *Suppose that u_h is the solution of (3.3) and let $\widetilde{u}$ be defined by (3.9), (3.10) and (3.15). Assume further that $e = u_h - \widetilde{u}$, and let $\widetilde{h}_T$ and h_T^* be defined by (2.11). Then,*

$$
\varepsilon \|\nabla e\|_{0,\Omega}^2 \leq 4(A^2 + B^2 + C^2), \tag{4.18}
$$

where

$$
A^2 = \sum_T \widetilde{h}_T^{-1} \|u - \widetilde{u}_L\|_{0,T}^2, \; B^2 = \varepsilon \sum_T \|\nabla (u - \widetilde{u}_L)\|_{0,T}^2, \; C^2 = \sum_T h_T^* \|\boldsymbol{a} \cdot \nabla (u - \widetilde{u}_L)\|_{0,T}^2.
$$

We proceed by assuming that $u \in H^s(\Omega) \cap H_0^1(\Omega)$, $1 < s \leq 2$, to further bound the terms A^2, B^2, C^2 which appear on the right-hand side of (4.18). In what follows k will denote a generic positive constant which *only* depends on the minimum angle in the triangulation $\mathcal{T}$, and we define

$$
\gamma_T = |\boldsymbol{a}| \max \left\{ 1, \frac{\varepsilon}{h_T |\boldsymbol{a}|} \right\}. \tag{4.19}
$$

Recalling (2.15), we have that

$$
A^2 \leq k \sum_T \gamma_T h_T^{2s-1} |u|_{s,T}^2. \tag{4.20}
$$

Further, on writing

$$
\varepsilon = \frac{\varepsilon}{h_T |\boldsymbol{a}|} h_T |\boldsymbol{a}| \leq \max \left\{ 1, \frac{\varepsilon}{h_T |\boldsymbol{a}|} \right\} h_T |\boldsymbol{a}| = \gamma_T h_T,
$$

it follows that

$$B^2 \leq k \sum_T \gamma_T h_T^{2s-1} |u|^2_{s,T}. \tag{4.21}$$

Finally, by (2.12) and (4.19),

$$h_T^* |\boldsymbol{a}|^2 \leq h_T |\boldsymbol{a}| \leq h_T \gamma_T,$$

so that

$$C^2 \leq k \sum_T \gamma_T h_T^{2s-1} |u|^2_{s,T}. \tag{4.22}$$

After substituting the bounds (4.20), (4.21) and (4.22) into (4.18) and recalling (3.21), we deduce the following result.

THEOREM 4.2. *Suppose that u_h is the solution of (3.3) and let $\widetilde{u}$ be defined by (3.9), (3.10) and (3.15). Assume further that $e = u_h - \widetilde{u}$, and let $\widetilde{h}_T$ and γ_T be defined by (2.11) and (4.19), respectively. Then, supposing $u \in H^s(\Omega) \cap H^1_0(\Omega)$, $1 < s \leq 2$, we have that*

$$\varepsilon \|\nabla e\|^2_{0,\Omega} \leq k \sum_T \gamma_T h_T^{2s-1} |u|^2_{s,T}, \tag{4.23}$$

$$\varepsilon \|\nabla e_L\|^2_{0,\Omega} + \sum_T \widetilde{h}_T \|\boldsymbol{a} \cdot \nabla e_L\|^2_{0,T} \leq k \sum_T \gamma_T h_T^{2s-1} |u|^2_{s,T}, \tag{4.24}$$

where k is a positive constant which only depends on the minimum angle in the triangulation $\mathcal{T}$.

In order to complete the error analysis, it remains to bound

$$\varepsilon^{1/2} \|\nabla(u - \widetilde{u})\|_{0,\Omega} \qquad \text{and} \qquad \left(\varepsilon \|\nabla(u - \widetilde{u}_L)\|^2_{0,\Omega} + \sum_T \widetilde{h}_T \|\boldsymbol{a} \cdot \nabla(u - \widetilde{u}_L)\|^2_{0,T} \right)^{1/2};$$

note that, in contrast with (3.21), these two expressions are not equal. Once we have obtained bounds on these, the final error estimates, stated in (4.27) and (4.28) below, will follow from (4.23) and (4.24) by the triangle inequality.

First,

$$\varepsilon^{1/2} \|\nabla(u - \widetilde{u})\|_{0,\Omega} \leq \varepsilon^{1/2} \|\nabla(u - \widetilde{u}_L)\|_{0,\Omega} + \varepsilon^{1/2} \|\nabla \widetilde{u}_B\|_{0,\Omega}. \tag{4.25}$$

Further, by the identity (4.6), and performing an argument analogous to that which led to (4.16), starting in (4.14) with e_B replaced by $\widetilde{u}_B$, we find that

$$\begin{aligned} \varepsilon \|\nabla \widetilde{u}_B\|^2_{0,\Omega} &= a(u - \widetilde{u}_L, \widetilde{u}_B) \\ &\leq \varepsilon^{1/2} \|\nabla \widetilde{u}_B\|_{0,\Omega} \left\{ \varepsilon^{1/2} \|\nabla(u - \widetilde{u}_L)\|_{0,\Omega} + \left(\sum_T h_T^* \|\boldsymbol{a} \cdot \nabla(u - \widetilde{u}_L)\|^2_{0,T} \right)^{1/2} \right\}. \end{aligned}$$

Hence, after simplification,

$$\varepsilon^{1/2} \|\nabla \widetilde{u}_B\|_{0,\Omega} \leq \varepsilon^{1/2} \|\nabla(u - \widetilde{u}_L)\|_{0,\Omega} + \left(\sum_T h_T^* \|\boldsymbol{a} \cdot \nabla(u - \widetilde{u}_L)\|^2_{0,T} \right)^{1/2}. \tag{4.26}$$

Upon substituting (4.26) into (4.25) and recalling (4.21) and (4.22), we deduce that

$$\varepsilon^{1/2}\|\nabla(u-\widetilde{u})\|_{0,\Omega} \le 2B + C \le \left(k\sum_T \gamma_T h_T^{2s-1}|u|_{s,T}^2\right)^{1/2}, \qquad \text{for } 1 < s \le 2.$$

Thus, by (4.23) and the triangle inequality,

$$\varepsilon^{1/2}\|\nabla(u-u_h)\|_{0,\Omega} \le \left(k\sum_T \gamma_T h_T^{2s-1}|u|_{s,T}^2\right)^{1/2}, \qquad \text{for } 1 < s \le 2. \tag{4.27}$$

In order to obtain our second error bound, we write $u - u_L = (u - \widetilde{u}_L) - e_L$ and apply the triangle inequality for the stability norm to conclude, by (4.24), (4.21) and (4.22), that

$$\left(\varepsilon\|\nabla(u-u_L)\|_{0,\Omega}^2 + \sum_T \widetilde{h}_T\|\boldsymbol{a}\cdot\nabla(u-u_L)\|_{0,T}^2\right)^{1/2} \le \left(k\sum_T \gamma_T h_T^{2s-1}|u|_{s,T}^2\right)^{1/2} \tag{4.28}$$

for $1 < s \le 2$.

The next theorem summarises the final error bounds.

THEOREM 4.3. *Let u_h denote the solution of (3.3), and suppose that $u \in H^s(\Omega) \cap H_0^1(\Omega)$, $1 < s \le 2$. Then,*

$$\varepsilon^{1/2}\|\nabla(u-u_h)\|_{0,\Omega} \le \left(k\sum_T \gamma_T h_T^{2s-1}|u|_{s,T}^2\right)^{1/2},$$

$$\left(\varepsilon\|\nabla(u-u_L)\|_{0,\Omega}^2 + \sum_T \widetilde{h}_T\|\boldsymbol{a}\cdot\nabla(u-u_L)\|_{0,T}^2\right)^{1/2} \le \left(k\sum_T \gamma_T h_T^{2s-1}|u|_{s,T}^2\right)^{1/2},$$

where k is a positive constant which only depends on the minimum angle in the triangulation $\mathcal{T}$, and $\widetilde{h}_T$ and γ_T are defined by (2.11) and (4.19), respectively.

The results of the last two theorems are essentially known in the context of the streamline diffusion method, in particular regarding the estimates on $u - u_L$ (see, e.g., [21], [22].) A good feature of the streamline derivative in the norm appearing on the left-hand side in the last bound is that it provides a natural measure (based on the residual) of the approximation error to the reduced problem which arises by taking the limit $\varepsilon \to 0$. Another good feature is that error control in that norm guarantees that the method will not develop excessive oscillations in the streamline direction, such as those that typically arise in standard Galerkin methods. It is also noteworthy that for the complete approximation which includes the bubble part of the solution the estimate in the norm $\varepsilon^{1/2}|\cdot|_{1,\Omega}$, alone, contains so much information. Still, one might argue that such a norm is too strong to be included in the left-hand side. Indeed, for the exact solution u of a generic problem with smooth data, $\varepsilon^{1/2}|u|_{1,\Omega}$ remains bounded independent of ε, but it is, in some sense, the strongest norm with this property, leaving no margin for extracting positive powers of h in the error analysis, given that $\varepsilon^{1/2}|u|_{s,\Omega}$ blows up as $\varepsilon \to 0$ when $s > 1$. An estimate of the ideal type would instead involve, say, the L_1-norm of the discretization error in the left-hand side, and the W_1^1-norm of the solution in the right-hand side, with a multiplicative constant independent of ε. Unfortunately, estimates of this type do not look very easy to derive in the multi-dimensional case.

5. Extensions, conclusions and implementational aspects. In this final section, we comment on certain extensions of our results. Denoting by n the number of space dimensions, we observe that our results trivially extend to the case of $n = 1$. Concerning the extension to $n = 3$, Theorems 3.1 and 4.1 still hold, while Theorems 4.2 and 4.3 require $3/2 < s \leq 2$ to ensure that the usual continuous piecewise linear interpolant is well defined. If a linear quasi-interpolant of u is used as $\widetilde{u}_L$ instead of a linear interpolant, then Theorems 3.1 to 4.1 remain unaffected and Theorems 4.2 and 4.3 still hold for $1 < s \leq 2$; in fact, with this altered definition of $\widetilde{u}_L$, all of our results extend to the case of $n \geq 3$.

Our second comment concerns the smoothness hypotheses on $\boldsymbol{a}$ and f. Recall that in our analysis the components of $\boldsymbol{a}$ were taken to be constant and f was assumed to be a piecewise constant function on $\overline{\Omega}$. We now indicate a simple, although nonoptimal, way of dealing with more general cases within the framework of the present paper. Suppose that $f \in L_2(\Omega)$. Assume further that $\boldsymbol{a}$ is a divergence-free vector function defined on $\overline{\Omega}$ whose entries are in $C^1(\overline{\Omega})$, $|\boldsymbol{a}| > 0$ on $\overline{\Omega}$, and for every element T in the partition $\mathcal{T}$ there exists an "upwind-most" point $x_{\boldsymbol{a}} \in T$ such that $\boldsymbol{a}(x_{\boldsymbol{a}})\cdot(x - x_{\boldsymbol{a}}) \geq 0$. Under these hypotheses, all of our proofs can be completed in the same manner as before, only with some small changes. For example, instead of (3.8) and (3.15), we now have, respectively,

$$u_B|_T = (\widehat{f}_T - \widehat{\boldsymbol{a}}_T \cdot \nabla u_L) b_1^T, \quad \widetilde{u}_B|_T = (\widehat{f}_T - \widehat{\boldsymbol{a}}_T \cdot \nabla \widetilde{u}_L) b_1^T \qquad \text{in each } T,$$

where we have used the notation

$$\widehat{w}_T := \int_T w b_1^T \,\mathrm{d}x \Big/ \int_T b_1^T \,\mathrm{d}x.$$

The associated stability norm is defined by

$$\left(\varepsilon \|\nabla v\|_{0,\Omega}^2 + \sum_T \widetilde{h}_T \|\widehat{\boldsymbol{a}}_T \cdot \nabla v\|_{0,T}^2 \right)^{1/2},$$

and, subject to this minor alteration, Theorem 3.1 still holds. Due to the mismatch between $\boldsymbol{a}$ and $\widehat{\boldsymbol{a}}$, the error analysis leading to Theorem 4.1 will contain (in the estimation of III) an additional term, denoted D^2 below, thus giving rise to the bound $4(A^2 + B^2 + C^2 + D^2)$ on the right-hand side of (4.18), with A and B as before,

$$C^2 = \sum_T h_T^* \|\widehat{\boldsymbol{a}}_T \cdot \nabla (u - \widetilde{u}_L)\|_{0,T}^2 \quad \text{and} \quad D^2 = \sum_T \varepsilon^{-1} \|\boldsymbol{a} - \widehat{\boldsymbol{a}}_T\|_{L_\infty(T)}^2 \|u - \widetilde{u}_L\|_{0,T}^2.$$

Consequently, the right-hand sides in Theorems 4.2 and 4.3 will contain the added expression

$$k \sum_T \varepsilon^{-1} \|\nabla \boldsymbol{a}\|_{L_\infty(T)}^2 h_T^{2s+2} |u|_{s,T}^2 \tag{5.1}$$

and its square-root, respectively, with $1 < s \leq 2$; nevertheless, as long as

$$\varepsilon^{-1} \|\nabla \boldsymbol{a}\|_{L_\infty(T)}^2 h_T^3 \leq K_0 \gamma_T \equiv K_0 \|\boldsymbol{a}\|_{L_\infty(T)} \max\left\{1, \frac{\varepsilon}{h_T \min_T |\boldsymbol{a}|}\right\}, \tag{5.2}$$

where K_0 is a fixed positive constant, the extra term displayed in (5.1) can be absorbed into the bound on $4(A^2 + B^2 + C^2)$, leading to the same error estimate as in the constant-coefficient case. When the problem is convection-dominated, (5.2) demands that

$$h_T \leq \left(K_0 \, \varepsilon \, \|\boldsymbol{a}\|_{L_\infty(T)} / \|\nabla \boldsymbol{a}\|^2_{L_\infty(T)} \right)^{1/3} .$$

Whether this restriction on the mesh-size is acceptable from the practical point of view depends on the nature of the problem. On the other hand, a different and more general analysis [3] shows that optimal estimates can indeed be obtained for the residual-free-bubble approach, in the variable coefficient case, without the assumption (5.2).

We conclude this section with some remarks on computational aspects. It is known that the introduction and the elimination of one bubble function (of any shape) per element leads to a streamline diffusion method where the stabilising term

$$\sum_T \tau_T \int_T (\boldsymbol{a} \cdot \nabla u_L - f) \boldsymbol{a} \cdot \nabla v_L \, \mathrm{d}x$$

is added to the continuous piecewise linear Galerkin approximation. In particular, the value of τ_T depends on the shape of the bubble function through the formula

$$\tau_T = \frac{1}{|T|} \frac{\left(\int_T b_T \, \mathrm{d}x \right)^2}{\varepsilon \|\nabla b_T\|^2_{0,T}}$$

where b_T is the chosen bubble function in triangle T. It is clear that any non-negative function contained in $H^1_0(T)$ can be scaled so that (2.2) holds. Hence the actual computation of the coefficient τ_T can be reduced to the computation of the integral of the *scaled* bubble function over T. If, as in the present paper, the bubble function on T is defined as the solution of the local problem (2.1) (which has already been scaled so as to satisfy (2.2)), then the computation of the exact value of the integral of the bubble over T may be difficult to perform. We make two remarks in this respect.

First, as in [5], we note that for large values of the Péclet number the solution b_1^T of (2.1) is close to the solution of the reduced problem

$$\left\{ \begin{array}{rcll} \boldsymbol{a} \cdot \nabla b_0^T & = & 1 & \text{in } T, \\ b_0^T & = & 0 & \text{on the inflow part of } \partial T. \end{array} \right. \tag{5.3}$$

Thus, in practice, the computation of the integral of b_1^T can be replaced by calculating the integral of b_0^T and this, in turn, is equivalent to finding the volume of a certain pyramid with base T (see, [5], for details).

Second, in order to deal with problems where there is a substantial variation of the Péclet number over the computational domain, one can consider choosing as bubble an approximation b_T to b_1^T from within $H^1_0(T)$ (see [4]), always scaled in such a way that (2.2) holds. Once this is done, we keep the formulae which define h_T^* and $\widetilde{h}_T$ as in (2.11), with b_1^T replaced by b_T. We also have to assume that, in doing so, uniform bounds for h_T^* and $\widetilde{h}_T$ similar to (2.12) and (2.15) hold true. Then, the error analysis presented here immediately extends to this new case. We can then say that any bubble b_T, scaled via (2.2), will work in our analysis, provided that the corresponding h_T^* and $\widetilde{h}_T$ are suitably bounded. This should not be surprising, as it matches perfectly the situation encountered in SUPG, where one can change the stabilization parameter τ_T by a multiplicative constant without affecting the proof of the asymptotic error bounds. However, the choice (2.1) of b_1^T has the merit of supplying a precise value for τ_T. See the discussion in [2] for further details on this point.

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