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A Simple Preconditioner for a Discontinuous Galerkin Method for the Stokes Problem

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Abstract In this paper we construct Discontinuous Galerkin approximations of the Stokes problem where the velocity field is $H(\text{div}, \Omega)$-conforming. This implies that the velocity solution is divergence-free in the whole domain. This property can be exploited to design a simple and effective preconditioner for the final linear system.

Keywords Discontinuous Galerkin · Stokes equation · Auxiliary space

1 Introduction

In this paper we present a preconditioning strategy for a family of discontinuous Galerkin discretizations of the Stokes problem in a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$:

$$\begin{cases} -\mathbf{div}(2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})) + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \text{div}\, \boldsymbol{u} &= 0 \quad \text{in } \Omega \end{cases} \tag{1.1}$$

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Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

where, with the usual notation, $\boldsymbol{u}$ is the velocity field, p the pressure, ν the viscosity of the fluid, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}}$ is the symmetric (linearized) strain rate tensor defined by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$.

The methods considered here were introduced in [40] for the Stokes problem and in [21] for the Navier-Stokes equations when pure Dirichlet boundary conditions are prescribed. In both works, the authors showed that the approximate velocity field is exactly divergence-free, namely it is $H(\mathrm{div};\Omega)$-conforming and divergence-free almost everywhere. These same methods were also used in [25].

Numerical methods that preserve divergence free condition exactly are important from both practical and theoretical points of view. First of all, it means that the numerical method conserves the mass everywhere, namely, for any $D \subset \Omega$ we have

$$\int_{\partial D} \boldsymbol{u} \cdot \boldsymbol{n} = 0.$$

As an example of its theoretical importance, the exact divergence free condition plays a crucial view for the stability of the mathematical models (see [30]) and their numerical discretizations (see [28]) for complex fluids.

The focus of this paper is to develop new solvers for the resulting algebraic systems for this type of discretization by exploring the divergence-free property. In general, the numerical discretization of the Stokes problem produces algebraic linear systems of equations of the saddle-point type. Solving such algebraic linear systems has been the subject of considerable attention from various communities and many different approaches can be used to solve them efficiently (see [22] and references cited therein). One popular approach is to use a block diagonal preconditioner with two blocks: one containing the inverse or a preconditioner of the stiffness matrix of a vector Poisson discretization, and one containing the inverse of a lumped mass matrix for the pressure. This preconditioner when used in conjunction with MINRES (MINimal RESidual) leads to a solver which is uniformly convergent with respect to the mesh size.

While the existing solvers such as this diagonal preconditioner can also be used for these DG methods, in this paper, we would like to explore an alternative approach by taking the advantage of the divergence-free property. Our new approach reduces the solution of the Stokes systems (which is indefinite) to the solution of several Poisson equations (which are symmetric positive definite) by using auxiliary space preconditioning techniques, which we hope would open new doors for the design of algebraic solvers for PDE systems that involve subsystems that are related to the Stokes operator.

In [21,40] the classical Stokes operator is considered for the special case of purely homogeneous Dirichlet boundary conditions (no-slip Dirichlet's condition). While this special case is theoretically important, it does not model well most of the cases that occur in the engineering applications (for instance, it is not realistic in applications in immiscible two-phase flows, aeronautics, in weather forecasts or in hemodynamics). For the pure homogenous no-slip Dirichlet boundary conditions, we have the following identity

$$\int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_{\Omega} \nabla \boldsymbol{u} : \nabla \boldsymbol{v}.$$

when $\boldsymbol{u}$ and $\boldsymbol{v}$ vanish on the boundary of Ω. This identity can be used when deriving the variational formulation, thus leading to simplifications of the analysis in the details related to the Korn's inequality on the discrete level.

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

To extend the results in [21,40] to this different boundary condition we provide detailed analysis showing that the resulting DG-$\boldsymbol{H}(\text{div};\Omega)$-conforming methods are stable and converge with optimal order. Furthermore, a key feature of the DG-$\boldsymbol{H}(\text{div};\Omega)$-conforming schemes of providing a divergence-free velocity approximation is satisfied as in [21,40], by the appropriate choice of the discretization spaces. This property is fully exploited in designing and constructing efficient preconditioners and we reduce the solution of the Stokes problem to the solution of a "second-order" problem in the space $\mathbf{curl}\, H_0^1(\Omega)$.

We propose then a preconditioner for the solution of the corresponding problem in $\mathbf{curl}\, H_0^1(\Omega)$. This is done by means of the fictitious space [33,34] (or auxiliary space [35,41]) framework. The proposed preconditioner amounts to the solution of one vector and two scalar Laplacians. The solution of such systems can then be *efficiently* computed with classical approaches, for instance the Geometric Multigrid (GMG) or Algebraic Multigrid (AMG) methods.

Throughout the paper, we use the standard notation for Sobolev spaces [1]. For a bounded domain $D \subset \mathbb{R}^d$, we denote by $H^m(D)$ the L^2-Sobolev space of order $m \geq 0$ and by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ the usual Sobolev norm and seminorm, respectively. For $m = 0$, we write $L^2(D)$ instead of $H^0(D)$. For a general summability index p, we also denote by $W^{m,p}(D)$ the usual L^p-Sobolev spaces of order $m \geq 0$ with norm $\|\cdot\|_{m,p,D}$ and seminorm $|\cdot|_{m,p,D}$. By convention, we use boldface type for the vector-valued analogues: $\boldsymbol{H}^m(D) = [H^m(D)]^d$, likewise, we use boldface italics for the symmetric-tensor-valued analogues: $\boldsymbol{\mathcal{H}}^m(D) := [H^m(D)]^{d\times d}_{\text{sym}}$. $H^m(D)/\mathbb{R}$ denotes the quotient space consisting of equivalence classes of elements of $H^m(D)$ that differ by a constant; for $m = 0$ the quotient space is denoted by $L^2(D)/\mathbb{R}$. We indicate by $L_0^2(D)$ the space of the $L^2(D)$ functions with zero average over D (which is obviously isomorphic to $L^2(D)/\mathbb{R}$). We use $(\cdot,\cdot)_D$ to denote the inner product in the spaces $L^2(D)$, $\boldsymbol{L}^2(D)$, and $\boldsymbol{\mathcal{L}}^2(D)$.

2 Continuous Problem

In this section, we discuss the well posedness of the Stokes problem which is of interest. We remark that the results in the paper are valid in two and three dimensions, although to make the presentation more transparent we focus on the two dimensional case, discussing only briefly the main changes (if any) needed to carry over the results to three dimensions.

We begin by restating (for reader's convenience) the equations already given in (1.1) with a bit more detail regarding the boundary conditions. For a simply connected polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with boundary $\Gamma = \partial\Omega$, we consider the Stokes equations for a viscous incompressible fluid:

$$\begin{cases} -\mathbf{div}(2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})) + \nabla p = \mathbf{f} & \text{in } \Omega \\ \text{div}\, \boldsymbol{u} = 0 & \text{in } \Omega \end{cases} \tag{2.1}$$

On the boundary Γ we impose kinematic boundary condition

$$\boldsymbol{u}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma, \tag{2.2}$$

together with the natural condition on the tangential component of the normal stresses

$$((2\,\nu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbf{I})\mathbf{n})\cdot\boldsymbol{t} = 0 \quad \text{on } \Gamma, \tag{2.3}$$

where $\mathbf{I}$ is the identity tensor. Note that as $\mathbf{n}\cdot\boldsymbol{t} \equiv 0$ then (2.3) is reduced to

$$(\boldsymbol{\varepsilon}(\boldsymbol{u})\mathbf{n})\cdot\boldsymbol{t} = 0 \quad \text{on } \Gamma. \tag{2.4}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

When the space

$$\boldsymbol{H}^1_{0,n}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \;\; : \;\; \boldsymbol{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\} \tag{2.5}$$

is introduced, the variational formulation of the Stokes problem reads: *Find* $(\boldsymbol{u}, p) \in \boldsymbol{H}^1_{0,n}(\Omega) \times L^2(\Omega)/\mathbb{R}$ *as the solution of:*

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) & \forall\, \boldsymbol{v} \in \boldsymbol{H}^1_{0,n}(\Omega) \\ b(\boldsymbol{u}, q) = 0 & \forall\, q \in L^2(\Omega)/\mathbb{R} \end{cases} \tag{2.6}$$

where for all $\boldsymbol{u} \in \boldsymbol{H}^1_{0,n}(\Omega)$, $\boldsymbol{v} \in \boldsymbol{H}^1_{0,n}(\Omega)$ and $q \in L^2(\Omega)/\mathbb{R}$ the (bi)linear forms are defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) := 2\nu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x, \quad b(\boldsymbol{v}, q) := -\int_\Omega q \operatorname{div} \boldsymbol{v}\, \mathrm{d}x, \quad (\boldsymbol{f}, \boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x.$$

For the classical mathematical treatment of the Stokes problem (where the Laplace operator is used instead of the divergence of the stress tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$) existence and uniqueness of the solution $(\boldsymbol{u}, p)$ are very well known and have been reported with different boundary conditions in many places (see for instance [23,24,27,38]). The Stokes problem considered here (2.1), (2.2), (2.3) has been derived and used in different applications [26,39,42].

For the Stokes problem with the slip boundary conditions (2.2), (2.3), existence, uniqueness and interior regularity was first established in [37] (for even the more general linearized Navier-Stokes). The study of well-posedness and regularity up to the boundary for the solutions of this problem has received substantial attention only in very recent years. For example, analysis can be found in [3,10] for weak and strong solutions in the $H^1(\Omega) \times L^2(\Omega)$ and $W^{1,p}(\Omega) \times L^p(\Omega)$, $1 < p < \infty$. In these works it is assumed that the boundary of Ω is at least of class $\mathcal{C}^{1,1}(\Omega)$ and the more general boundary condition of Navier slip-type is studied. In [4], the authors provide the analysis in the $W^{1,p}(\Omega) \times L^p(\Omega)$, $1 < p < \infty$ for less regular domains.

Here, for the sake of completeness, we provide a very brief outline of the proof of well-posedness of the problem, in the case Ω is a polygonal or polyhedral domain (which is the relevant case for the numerical approximation we have in mind). By introducing the operator $D_0 = -\text{div} : \boldsymbol{H}^1_{0,n}(\Omega) \longrightarrow L^2_0(\Omega)$, it can be shown [14,38] that D_0 is surjective, i.e., Range $(D_0) = L^2_0(\Omega)$. Therefore, the operator D_0 has a continuous lifting which implies that the continuous inf-sup condition is satisfied. Hence, from the classical theory follows that to guarantee the well-posedness of the Stokes problem (2.1), (2.2), it is enough to show that the bilinear form $a(\cdot, \cdot)$ is coercive; ie., there exists $\gamma_0 > 0$ such that

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq \gamma_0 |\boldsymbol{v}|^2_{1,\Omega} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1_{0,n}(\Omega). \tag{2.7}$$

Once continuity is established, existence, uniqueness and a-priori estimates follow in a standard way. The proof of (2.7) requires a Korn inequality, that in general imposes some restrictions on the domain (see Remark 2.3). For the case considered in this work the needed result is contained in next Lemma:

Lemma 2.1 *Let* $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ *be a polygonal or polyhedral domain. Then, there exists a constant* $C_{Kn} > 0$ *(depending on the domain through its diameter and shape) such that*

$$|\boldsymbol{v}|^2_{1,\Omega} \leq C_{Kn} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\Omega}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1_{0,n}(\Omega). \tag{2.8}$$

To prove the above Lemma, we first need the following auxiliary result

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Lemma 2.2 *For every polygonal or polyhedral domain Ω there exists a positive constant $\kappa(\Omega)$ such that*

$$\kappa(\Omega)\|\boldsymbol{\eta}\|^2_{0,\Omega} \le \|\boldsymbol{\eta}\cdot\boldsymbol{n}\|^2_{0,\partial\Omega} \qquad \forall\boldsymbol{\eta}\in\boldsymbol{RM}(\Omega) \tag{2.9}$$

where $\boldsymbol{RM}(\Omega)$ is the space of rigid motions on Ω defined by

$$\boldsymbol{RM}(\Omega) = \left\{\boldsymbol{a}+\boldsymbol{b}\boldsymbol{x} \;:\; \boldsymbol{a}\in\mathbb{R}^d \quad \boldsymbol{b}\in so(d)\right\}$$

with $so(d)$ denoting the set of skew-symmetric $d\times d$ matrices, $d=2,3$.

Proof To ease the presentation we provide the proof only in two dimensions. The extension to three dimensions involve only notational changes and therefeore it is ommitted. To show the lemma we observe that a polygon contains always at least two edges not belonging to the same straight line. A rigid movement whose normal component vanishes identically on those two edges is easily seen to be identically zero. This implies that for $\boldsymbol{c}\equiv(c_1,c_2,c_3)\in\mathbb{R}^3$ on the (compact) manyfold

$$\int_\Omega |(c_1-c_3x_2, c_2+c_3x_1)|^2\,\mathrm{d}x = 1$$

the function

$$\boldsymbol{c}\to\int_{\partial\Omega}|(c_1-c_3x_2, c_2+c_3x_1)\cdot\boldsymbol{n}|^2\,\mathrm{d}s \tag{2.10}$$

(which is obviously continuous) is never equal to zero. Hence it has a positive minimum, that equals the required $\kappa(\Omega)$. □

As a direct consequence of last Lemma, we can now provide the proof of the desired Korn inequality given in Lemma 2.1.

Proof (*Proof of Lemma* 2.1.)
For every $\boldsymbol{v}\in\boldsymbol{H}^1_{0,n}(\Omega)$ we consider first its L^2 projection $\boldsymbol{v}_R$ on the space $\boldsymbol{RM}(\Omega)$ of rigid motions and the projection $\boldsymbol{v}_\perp := \boldsymbol{v}-\boldsymbol{v}_R$ on the orthogonal subspace. As $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on $\partial\Omega$ we obviously have

$$\boldsymbol{v}_R\cdot\boldsymbol{n} = -\boldsymbol{v}_\perp\cdot\boldsymbol{n}. \tag{2.11}$$

Moreover, as $\boldsymbol{v}_\perp$ is orthogonal to rigid motions we have

$$|\boldsymbol{v}_\perp|^2_{1,\Omega} \le C_K\|\boldsymbol{\varepsilon}(\boldsymbol{v})_\perp\|^2_{0,\Omega} \tag{2.12}$$

for some constant C_K (note that the rigid motions include the constants, so that Poincaré inequality also holds for $\boldsymbol{v}_\perp$). On the other hand, since $\boldsymbol{RM}(\Omega)$ is finite dimensional we have obviously

$$|\boldsymbol{v}_R|^2_{1,\Omega} \le C_P\|\boldsymbol{v}_R\|^2_{0,\Omega} \tag{2.13}$$

that using (2.9) gives

$$|\boldsymbol{v}_R|^2_{1,\Omega} \le \frac{C_P}{\kappa(\Omega)}\|\boldsymbol{v}_R\cdot\boldsymbol{n}\|^2_{0,\partial\Omega} \tag{2.14}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

and using also (2.11) and (2.12)

$$\begin{aligned}\frac{1}{2}|\boldsymbol{v}|^2_{1,\Omega} &\le |\boldsymbol{v}_R|^2_{1,\Omega} + |\boldsymbol{v}_\perp|^2_{1,\Omega} \le \frac{C_P}{\kappa(\Omega)}\|\boldsymbol{v}_R\cdot\boldsymbol{n}\|^2_{0,\partial\Omega} + |\boldsymbol{v}_\perp|^2_{1,\Omega}\\ &= \frac{C_P}{\kappa(\Omega)}\|\boldsymbol{v}_\perp\cdot\boldsymbol{n}\|^2_{0,\partial\Omega} + |\boldsymbol{v}_\perp|^2_{1,\Omega} \le \frac{C_T\,C_P}{\kappa(\Omega)}|\boldsymbol{v}_\perp|^2_{1,\Omega}\\ &\le \frac{C_T\,C_P\,C_K}{\kappa(\Omega)}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_\perp)\|^2_{0,\Omega} = \frac{C_T\,C_P\,C_K}{\kappa(\Omega)}\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\Omega}\end{aligned} \tag{2.15}$$

where the constant C_T depends on the trace inequality on Ω. Defining now $C_{Kn} = \frac{2C_T\,C_P\,C_K}{\kappa(\Omega)}$ we conclude the proof. □

Remark 2.3 The proof of Lemma 2.1 relies on the assumption that the domain is polygonal or polyhedral. For more general smooth bounded domains, the Korn inequality (2.8) is still true, as long as the domain is assumed to be not rotationally symmetric. Otherwise a Korn inequality can be established by restricting the solution space (see [[29], Appendix] for further details).

3 Abstract Setting and Basic Notations

Let $\mathcal{T}_h$ be a shape-regular family of partitions of Ω into triangles T in $d = 2$ or tetrahedra in $d = 3$. We denote by h_T the diameter of T, and we set $h = \max_{T\in\mathcal{T}_h} h_T$. We also assume that the decomposition $\mathcal{T}_h$ is conforming in the sense that it does not contain any hanging nodes.

We denote by $\mathcal{E}_h$ the set of all edges/faces and by $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$ the collection of all interior and boundary edges, respectively.

For $s \ge 1$, we define

$$H^s(\mathcal{T}_h) = \left\{\phi \in L^2(\Omega)\ \text{, such that}\ \phi\big|_T \in H^s(T),\quad \forall\, T \in \mathcal{T}_h\right\},$$

and their vector $\boldsymbol{H}^s(\mathcal{T}_h)$ and tensor $\boldsymbol{\mathcal{H}}^s(\mathcal{T}_h)$ analogues, respectively. For scalar, vector-valued, and tensor functions, we use $(\cdot\,,\cdot)_{\mathcal{T}_h}$ to denote the $L^2(\mathcal{T}_h)$-inner product and $\langle\cdot\,,\cdot\rangle_{\mathcal{E}_h}$ to denote the $L^2(\mathcal{E}_h)$-inner product elementwise.

The vector functions are represented column-wise. We recall the definitions of the following operators acting on vectors $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and on scalar functions $\phi \in H^1(\Omega)$ as

$$\begin{aligned}&\operatorname{div}\boldsymbol{v} = \sum_{i=1}^{d}\frac{\partial v^i}{\partial x_i}\\ &\operatorname{curl}\boldsymbol{v} = \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} \qquad \mathbf{curl}\,\phi = \nabla^\perp\phi := \left[\frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial x_1}\right]^T \qquad (d=2)\\ &\mathbf{curl}\,\boldsymbol{v} = \nabla\times\boldsymbol{v} = \left[\frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3}, \frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1}, \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}\right]^T \qquad (d=3)\end{aligned}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

and, we recall the definitions of the spaces to be used herein:

$$
\begin{aligned}
\boldsymbol{H}(\mathrm{div};\Omega) &:= \{\boldsymbol{v}\in \boldsymbol{L}^2(\Omega) \;:\; \mathrm{div}\,\boldsymbol{v}\in L^2(\Omega)\,\}, \quad d=2,3,\\
\boldsymbol{H}(\mathrm{curl};\Omega) &:= \{\boldsymbol{v}\in \boldsymbol{L}^2(\Omega) \;:\; \mathrm{curl}\,\boldsymbol{v}\in L^2(\Omega)\,\} \quad d=2,\\
\boldsymbol{H}(\mathbf{curl};\Omega) &:= \{\boldsymbol{v}\in \boldsymbol{L}^2(\Omega) \;:\; \mathbf{curl}\,\boldsymbol{v}\in \boldsymbol{L}^2(\Omega)\,\} \quad d=3\,.\\
\boldsymbol{H}_{0,n}(\mathrm{div};\Omega) &:= \{\boldsymbol{v}\in \boldsymbol{H}(\mathrm{div};\Omega) \;:\; \boldsymbol{v}\cdot\mathbf{n}=0 \;\text{ on } \Gamma\,\},\\
\boldsymbol{H}_{0,t}(\mathbf{curl};\Omega) &:= \{\boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl};\Omega) \;:\; \boldsymbol{v}\times\mathbf{n}=0 \;\text{ on } \Gamma\,\},\\
\boldsymbol{H}_{0,n}(\mathrm{div}^0;\Omega) &:= \{\boldsymbol{v}\in \boldsymbol{H}_{0,n}(\mathrm{div};\Omega) \;:\; \mathrm{div}\,\boldsymbol{v}=0 \;\text{ in } \Omega\,\}.
\end{aligned}
$$

The above spaces are Hilbert spaces with the norms

$$
\begin{aligned}
\|\boldsymbol{v}\|^2_{H(\mathrm{div},\Omega)} &:= \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{0,\Omega} \qquad \forall\,\boldsymbol{v}\in \boldsymbol{H}(\mathrm{div};\Omega)\;,\\
\|\boldsymbol{v}\|^2_{H(\mathrm{curl},\Omega)} &:= \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{curl}\,\boldsymbol{v}\|^2_{0,\Omega} \qquad \forall\,\boldsymbol{v}\in \boldsymbol{H}(\mathrm{curl};\Omega).\\
\|\boldsymbol{v}\|^2_{H(\mathbf{curl},\Omega)} &:= \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega} \qquad \forall\,\boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl};\Omega)\;.
\end{aligned}
$$

Remark 3.1 It is worth noting that if we restrict our analysis to vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)\cap \boldsymbol{H}_{0,n}(\mathrm{div}^0;\Omega)$ then problem (2.6) becomes: *Find* $\boldsymbol{u}\in \boldsymbol{H}^1(\Omega)\cap\boldsymbol{H}_{0,n}(\mathrm{div}^0;\Omega)$ *as the solution of:*

$$
a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in \boldsymbol{H}^1(\Omega)\cap \boldsymbol{H}_{0,n}(\mathrm{div}^0;\Omega). \tag{3.1}
$$

As is usual in the DG approach, we now define some trace operators. Let $e\in\mathcal{E}_h^o$ be an internal edge/face of $\mathcal{T}_h$ shared by two elements T^1 and T^2, and let $\mathbf{n}^1$ ($\mathbf{n}^2$) denote the unit normal on e pointing outwards from T^1 (T^2). For a scalar function $\varphi\in H^1(\mathcal{T}_h)$, a vector field $\boldsymbol{\tau}\in \boldsymbol{H}^1(\mathcal{T}_h)$, or a tensor field $\boldsymbol{\tau}\in \boldsymbol{\mathcal{H}}^1(\mathcal{T}_h)$ we define the average operator in the usual way (see for instance [5]), that is, on internal edges/faces

$$
\{\varphi\} = \frac{1}{2}(\varphi^1+\varphi^2), \quad \{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}^1+\boldsymbol{v}^2), \quad \{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^1+\boldsymbol{\tau}^2).
$$

However, on a boundary edge/face, we take $\{\varphi\}$, $\{\boldsymbol{v}\}$, and $\{\boldsymbol{\tau}\}$ as the trace of φ, $\boldsymbol{v}$, and $\boldsymbol{\tau}$,respectively, on that edge.

For a scalar function $\varphi\in H^1(\mathcal{T}_h)$, the jump operator is defined as

$$
[\![\,\varphi\,]\!] = \varphi^1\mathbf{n}^1+\varphi^2\mathbf{n}^2 \quad \text{on } e\in\mathcal{E}_h^o, \;\text{ and } [\![\,\varphi\,]\!] = \varphi\mathbf{n} \quad \text{on } e\in\mathcal{E}_h^\partial
$$

(where obviously $\mathbf{n}$ is the outward unit normal), so that the jump of a scalar function is a vector in the normal direction.

For a vector field $\boldsymbol{v}\in \boldsymbol{H}^1(\mathcal{T}_h)$, following, for example, [8], the jump is the symmetric matrix-valued function given on e by

$$
[\![\boldsymbol{v}]\!] = \boldsymbol{v}^1\odot\mathbf{n}^1+\boldsymbol{v}^2\odot\mathbf{n}^2 \quad \text{on } e\in\mathcal{E}_h^o, \;\text{ and } [\![\boldsymbol{v}]\!] = \boldsymbol{v}\odot\mathbf{n} \quad \text{on } e\in\mathcal{E}_h^\partial,
$$

where $\boldsymbol{v}\odot\mathbf{n} = (\boldsymbol{v}\mathbf{n}^T+\mathbf{n}\boldsymbol{v}^T)/2$ is the symmetric part of the tensor product of $\boldsymbol{v}$ and $\mathbf{n}$. Hence, the jump of a vector-valued function is a symmetric tensor.

If we denote by $\mathbf{n}_T$ the outward unit normal to ∂T, it is easy to check that

$$
\sum_{T\in\mathcal{T}_h}\int_{\partial T} \boldsymbol{v}\cdot\mathbf{n}_T\, q \;\mathrm{d}s = \sum_{e\in\mathcal{E}_h}\int_e \{\boldsymbol{v}\}\cdot[\![\,q\,]\!]\;\mathrm{d}s \quad \forall\,\boldsymbol{v}\in\boldsymbol{H}^1(\mathcal{T}_h)\;,\;\; \forall\, q\in H^1(\mathcal{T}_h). \tag{3.2}
$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Also for $\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}^1(\Omega)$ and for all $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h)$, we have

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\tau} \mathbf{n}_T) \cdot \boldsymbol{v} \, \mathrm{d}s = \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\tau}\} : [\![\boldsymbol{v}]\!] \, \mathrm{d}s. \tag{3.3}$$

3.1 Discrete Spaces: General Framework

We present three choices for each of the finite element spaces $\boldsymbol{V}_h$ and $\mathcal{Q}_h$ to approximate velocity and pressure, respectively. For each choice, we also need an additional space $\mathcal{N}_h$ (resp. $\boldsymbol{\mathcal{N}}_h$ in $d = 3$) made of piecewise polynomial scalars and of piecewise polynomial vectors in three dimensions, to be used as a sort of *potentials* or *vector potentials*. We will explain the reason for doing this and the way in which to do this later on. Note, too, that we will use this space more heavily in the construction of our preconditioner. The different choices for the spaces $\boldsymbol{V}_h$, $\mathcal{Q}_h$, and $\mathcal{N}_h$ or $\boldsymbol{\mathcal{N}}_h$ rely on different choices of the local polynomial spaces $\boldsymbol{\mathcal{R}}(T)$, $\mathcal{S}(T)$, and $\mathcal{M}(T)$ or $\boldsymbol{\mathcal{M}}(T)$, respectively, made for each element T. Specifically, we have

$$\boldsymbol{V}_h \;:\;= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \;:\; \boldsymbol{v}|_T \in \boldsymbol{\mathcal{R}}(T) \; \forall\, T \in \mathcal{T}_h, \;\; \boldsymbol{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}, \tag{3.4}$$

$$\mathcal{Q}_h := \left\{q \in L^2(\Omega)/\mathbb{R} \;:\; q|_T \in \mathcal{S}(T) \; \forall\, T \in \mathcal{T}_h\right\}, \tag{3.5}$$

and

$$\mathcal{N}_h := \left\{\varphi \in H^1_0(\Omega) \;:\; \varphi|_T \in \mathcal{M}(T) \; \forall\, T \in \mathcal{T}_h\right\} \text{ for } d = 2, \text{ and} \tag{3.6}$$

$$\boldsymbol{\mathcal{N}}_h := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega) \;:\; \boldsymbol{v}|_T \in \boldsymbol{\mathcal{M}}(T) \; \forall\, T \in \mathcal{T}_h \;\; \boldsymbol{v} \times \mathbf{n} = 0 \text{ on } \Gamma\} \text{ for } d = 3. \tag{3.7}$$

The three spaces $\boldsymbol{V}_h$, $\mathcal{Q}_h$, and $\mathcal{N}_h$ (or $\boldsymbol{\mathcal{N}}_h$) will always be related by this exact sequences:

$$0 \longrightarrow \mathcal{N}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} \mathcal{Q}_h \longrightarrow 0. \tag{3.8}$$

in two dimensions, and

$$0 \longrightarrow \boldsymbol{\mathcal{N}}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} \mathcal{Q}_h \tag{3.9}$$

in three dimensions. It is also necessary for each operator in (3.8) and (3.9) to have a continuous right inverse whose norm is uniformly bounded in h. For instance, it is necessary that

$$\exists\, \beta > 0 \text{ s.t. } \forall h, \forall\, q \in \mathcal{Q}_h \,\exists\, \boldsymbol{v} \in \boldsymbol{V}_h \text{ with: } \mathrm{div}\, \boldsymbol{v} = q \quad \text{and} \quad \|\boldsymbol{v}\|_{0,\Omega} \le \frac{1}{\beta}\|q\|_{0,\Omega}. \tag{3.10}$$

Obviously, for the **curl** operator (in 2 and 3 dimensions) these bounded right inverses will be defined only on $\boldsymbol{V}_h \cap \boldsymbol{H}_{0,n}(\mathrm{div}^0, \Omega)$.

Remark 3.2 In all our examples, the pair $(\boldsymbol{V}_h, \mathcal{Q}_h)$ is among the classical (and very old) finite element spaces specially tailored for the approximation of the Poisson equation in mixed form. In particular, properties (3.8) and (3.10) always hold.

3.2 Examples

We now present three examples of finite element spaces that can be used in the above framework. For each example, we specify the corresponding polynomial spaces used on each element and describe the corresponding sets of degrees of freedom. We restrict our analysis to the case of triangles or tetrahedra; more general cases can also be considered when corresponding changes are made (see [19]).

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Let us first fix the notation concerning the *spaces of polynomials*. For $m \geq 0$, we denote by $\mathbb{P}^m(T)$ the space of polynomials defined on T of degree of at most m; the corresponding vector space is denoted by $\boldsymbol{P}^m(T) = (\mathbb{P}^m(T))^2$. A polynomial of degree $m \geq 3$ that vanishes throughout ∂T (hence it belongs to $H^1_0(T)$) is called *a bubble (or an H-bubble) of degree m over T*. The space of bubbles of degree m over T is denoted by $HB^m(T)$. and its vector-valued analogue by $\boldsymbol{HB}^m(T)$. We denote by $\mathbb{P}^m_{hom}(T)$ the space of *homogeneous* polynomials of degree m, and we denote by $\mathbf{x}^\perp$ the vector $(-x_2, x_1)$.

For $m \geq 2$,

$$\mathbb{P}^+_m(T) := \mathbb{P}^m(T) + HB^{m+1}(T) \quad \boldsymbol{P}^+_m(T) := \boldsymbol{P}^m(T) + \boldsymbol{HB}^{m+1}(T). \tag{3.11}$$

and, for $m \geq 1$, we set

$$\mathbf{BDM}_m(T) := \boldsymbol{P}^m(T), \quad \mathbf{RT}_m(T) := \boldsymbol{P}^m(T) \oplus \mathbf{x}\,\mathbb{P}^m_{hom}(T). \tag{3.12}$$

Moreover we set, for $d = 2$ and $m \geq 0$

$$\mathbf{TR}_m(T) := \boldsymbol{P}^m(T) \oplus \mathbf{x}^\perp \mathbb{P}^m_{hom}(T). \tag{3.13}$$

and for $d = 3$ and $m \geq 0$ (see [31])

$$\mathbf{ND}_m(T) := \boldsymbol{P}^m(T) \oplus \mathbf{x} \wedge \boldsymbol{P}^m_{hom}(T). \tag{3.14}$$

We also consider some generalized bubbles: a vector-valued polynomial of degree $m \geq 2$ that belongs to $\boldsymbol{H}_{0,n}(\mathrm{div}, T)$ (hence whose normal component vanishes throughout ∂T) is called *a D-bubble of degree m over T*. The space of D-bubbles of degree m over T is denoted by $\boldsymbol{DB}^m(T)$. Similarly a vector valued polynomial of degree $m \geq d$ that belongs to $\boldsymbol{H}_{0,t}(\mathbf{curl}, T)$ (hence whose tangential components vanish all over ∂T) is called *a C-bubble of degree m over T*. The space of C-bubbles of degree m over T will be denoted by $\boldsymbol{CB}^m(T)$.

All the spaces used herein are well known and widely used. They are usually referred to as *Brezzi-Douglas-Marini, Raviart-Thomas, and Rotated Raviart-Thomas* spaces, respectively.

The first example follows.

1. *Raviart-Thomas* For $k \geq 1$, we take in each T, $\mathcal{S}(T) = \mathbb{P}^k(T)$, and $\boldsymbol{\mathcal{R}}(T) := \mathbf{RT}_k(T)$. The degrees of freedom in $\mathbf{RT}_k(T)$ are

$$\begin{aligned} &\int_e \boldsymbol{u}\cdot\mathbf{n}_e\, q\, ds \qquad \forall e \in \partial T,\ \forall q \in \mathbb{P}^k(e), \\ &\int_T \boldsymbol{u}\cdot\mathbf{p}\, dx \qquad\qquad \forall \mathbf{p} \in \mathbb{P}^{k-1}(T). \end{aligned} \tag{3.15}$$

As $\mathcal{Q}_h$ is made of discontinuous piecewise polynomials, here and in the following examples the degrees of freedom in $\mathcal{S}(T)$ can be taken in an almost arbitrary way. The corresponding pair of spaces $(\boldsymbol{V}_h, \mathcal{Q}_h)$ gives the classical Raviart-Thomas finite element approximation for second-order elliptic equations in mixed form, as introduced in [36]. It is well known and easy to check that the pair $(\boldsymbol{V}_h, \mathcal{Q}_h)$ satisfies

$$\mathrm{div}(\boldsymbol{V}_h) = \mathcal{Q}_h \tag{3.16}$$

and that the property (3.10) is verified. We then take $\mathcal{M}(T) := \mathbb{P}^{k+1}(T)$ and $\boldsymbol{\mathcal{M}}(T) := \mathbf{ND}_k(T)$ and note that

$$\mathbf{curl}(\mathcal{N}_h) \subseteq \boldsymbol{V}_h \qquad \mathbf{curl}(\overset{o}{\boldsymbol{\mathcal{N}}}_h) \subseteq \boldsymbol{V}_h \tag{3.17}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

and that the operator **curl** (for $d = 2$ and $d = 3$) has a continuous right inverse uniformly bounded from $\boldsymbol{V}_h \cap \boldsymbol{H}_{0,n}(\mathrm{div}^0, \Omega)$ to $\mathcal{N}_h$ and $\overset{o}{\boldsymbol{\mathcal{N}}}_h$ respectively; that is,

$$\exists\, C > 0 \text{ such that } \forall h,\ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \cap \boldsymbol{H}_{0,n}(\mathrm{div}^0, \Omega)\ \exists \varphi \in \mathcal{N}_h, \text{ such that}$$
$$\mathbf{curl}\,\varphi = \boldsymbol{v}_h \quad \text{and } \|\varphi\|_{1,\Omega} \le C\,\|\boldsymbol{v}_h\|_{0,\Omega}. \tag{3.18}$$

2. *Brezzi-Douglas-Marini:* For $k \ge 1$, we take $\mathcal{S}(T) = \mathbb{P}^{k-1}(T)$, and $\boldsymbol{\mathcal{R}}(T) = \mathbf{BDM}_k(T)$. The degrees of freedom for $\mathbf{BDM}_k(T)$ are (see [9]):

$$\begin{aligned}
&\int_e \boldsymbol{u}\cdot\mathbf{n}_e\, q\, ds && \forall e \in \partial T,\ \forall q \in \mathbb{P}^k(e);\\
&\int_T \boldsymbol{u}\cdot\boldsymbol{v}\, dx && \forall \boldsymbol{v} \in \mathbf{TR}_{k-2}(T)\quad k \ge 2 \text{ and } d = 2,\\
&\int_T \boldsymbol{u}\cdot\boldsymbol{v}\, dx && \forall \boldsymbol{v} \in \mathbf{ND}_{k-2}(T)\quad k \ge 2 \text{ and } d = 3.
\end{aligned} \tag{3.19}$$

The resulting finite element pair $(\boldsymbol{V}_h, \mathcal{Q}_h)$ is also commonly used for the approximation of second-order elliptic equations in mixed form introduced in [15] for $d = 2$ and in [17,32] for $d = 3$. Also in this case it has been established that the pair $(\boldsymbol{V}_h, \mathcal{Q}_h)$ verifies the properties of (3.16) and (3.10). We then take $\mathcal{M}(T) := \mathbb{P}^{k+1}(T)$, and $\boldsymbol{\mathcal{M}}(T) := \mathbf{ND}_{k+1}(T)$ and note that (3.17) and (3.18) are also satisfied.

3. *Brezzi-Douglas-Fortin-Marini:* For $k \ge 1$, we take $\mathcal{S}(T) = \mathbb{P}^k(T)$ and $\boldsymbol{\mathcal{R}}(T) = \mathbf{BDFM}_{k+1}(T)$, which can be written as $\mathbf{BDFM}_{k+1} = \mathbf{BDM}_k(T) + \boldsymbol{DB}_{k+1}(T)$. The degrees of freedom for $\mathbf{BDFM}_{k+1}(T)$, though similar to the previous ones, are given here:

$$\begin{aligned}
&\int_e \boldsymbol{u}\cdot\mathbf{n}_e\, q\, ds && \forall e \in \partial T,\ \forall q \in \mathbb{P}^k(e);\\
&\int_T \boldsymbol{u}\cdot\boldsymbol{v}\, dx && \forall \boldsymbol{v} \in \mathbf{TR}_{k-1}(T)\quad d = 2,\\
&\int_T \boldsymbol{u}\cdot\boldsymbol{v}\, dx && \forall \boldsymbol{v} \in \mathbf{ND}_{k-1}(T)\quad d = 3.
\end{aligned} \tag{3.20}$$

The resulting finite element pair $(\boldsymbol{V}_h, \mathcal{Q}_h)$ gives the triangular analogue of the element BDFM_k introduced in [18] for the approximation of second-order elliptic equations in mixed form. It is easy to check that the pair $(\boldsymbol{V}_h, \mathcal{Q}_h)$ verifies (3.16) and (3.10). We then take $\mathcal{M}(T) := \mathbb{P}^+_{k+1}(T)$ and $\boldsymbol{\mathcal{M}}(T) := \mathbf{ND}_k(T) + \boldsymbol{CB}_{k+1}(T) \cap \mathbf{ND}_{k+1}(T)$ and note that (3.17) and (3.18) hold.

The three choices above are quite similar to each other, and the best choice among them generally depends on the problem and the way in which the discrete solution is to be used. We also use basic approximation properties: for instance, we recall that a constant C exists such that for all $T \in \mathcal{T}_h$ and for all $\boldsymbol{v}$, e.s. in $\boldsymbol{H}^s(T)$, an interpolant $\boldsymbol{v}^I \in \boldsymbol{\mathcal{R}}(T)$ exists such that

$$\|\boldsymbol{v} - \boldsymbol{v}^I\|_{0,T} + h_T|\boldsymbol{v}^I|_{1,T} \le Ch_T^s|\boldsymbol{v}|_{s,T}, \quad s \le k+1. \tag{3.21}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

4 The Discontinuous Galerkin $H(\mathrm{div};\Omega)$-Conforming Method

To introduce our DG-approximation, we start by defining, for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^2(\mathcal{T}_h)$ and any $p, q \in L^2(\Omega)/\mathbb{R}$, the bilinear forms

$$\begin{aligned} A_h(\boldsymbol{u},\boldsymbol{v}) &= 2\nu\left[(\boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{T}_h} - \langle\{\boldsymbol{\varepsilon}(\boldsymbol{u})\}:[\![\boldsymbol{v}]\!]\rangle_{\mathcal{E}_h^o} - \langle[\![\boldsymbol{u}]\!]:\{\boldsymbol{\varepsilon}(\boldsymbol{v})\}\rangle_{\mathcal{E}_h^o}\right] \\ &\quad - 2\nu\left[\langle\boldsymbol{\varepsilon}(\boldsymbol{u})\mathbf{n}, (\boldsymbol{v}\cdot\mathbf{n})\mathbf{n}\rangle_{\mathcal{E}_h^\partial} + \langle(\boldsymbol{u}\cdot\mathbf{n})\mathbf{n}, \boldsymbol{\varepsilon}(\boldsymbol{v})\mathbf{n}\rangle_{\mathcal{E}_h^\partial}\right] \\ &\quad + 2\nu\left[\sum_{e\in\mathcal{E}_h^o}\alpha h_e^{-1}\int_e [\![\boldsymbol{u}]\!]:[\![\boldsymbol{v}]\!]\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h^\partial}\alpha h_e^{-1}\int_e(\boldsymbol{u}\cdot\mathbf{n})(\boldsymbol{v}\cdot\mathbf{n})\,\mathrm{d}s\right] \\ B_h(\boldsymbol{v},q) &= -(q,\operatorname{div}\boldsymbol{v})_{\mathcal{T}_h} \qquad \forall\,\boldsymbol{v}\in\boldsymbol{H}^2(\mathcal{T}_h), \forall\, q\in L^2(\Omega)/\mathbb{R} \end{aligned} \tag{4.1}$$

where as usual α is the penalty parameter that we assume to be positive and large enough.

It is easy to check that the solution $(\boldsymbol{u}, p)$ of (2.6) verifies:

$$\begin{cases} A_h(\boldsymbol{u},\boldsymbol{v}) + B_h(\boldsymbol{v},p) = (\boldsymbol{f},\boldsymbol{v}) & \forall\,\boldsymbol{v}\in\boldsymbol{H}^2(\mathcal{T}_h) \\ B_h(\boldsymbol{u},q) = 0 & \forall\, q\in L^2(\Omega)/\mathbb{R}. \end{cases} \tag{4.2}$$

For a general DG approximation, we now replace the spaces $\boldsymbol{H}^2(\mathcal{T}_h)$ and $L^2(\Omega)/\mathbb{R}$ with the discrete ones $\boldsymbol{\mathcal{X}}_h$ and $\mathcal{Q}_h$, respectively. Following [21], we choose for $(\boldsymbol{\mathcal{X}}_h, \mathcal{Q}_h)$ one of the pairs $(\boldsymbol{V}_h, \mathcal{Q}_h)$ of the previous examples in order to get a global divergence-free approximation.

More generally, we can choose a pair $(\boldsymbol{V}_h, \mathcal{Q}_h)$ in order to find a third space $\mathcal{N}_h$ in such a way that (3.8), (3.16), (3.10), (3.17), and (3.18) are satisfied. This set of assumptions will come out several times in the sequel and, therefore, it is helpful to give it a special name.

Definition 4.1 In the above setting, we say that the three spaces $(\boldsymbol{V}_h, \mathcal{Q}_h, \mathcal{N}_h)$ (resp. $(\boldsymbol{V}_h, \mathcal{Q}_h, \boldsymbol{\mathcal{N}}_h)$) satisfy Assumption **H0** if (3.8) (resp. (3.9)), (3.16), (3.10), (3.17) and (3.18) are satisfied.

We note that, according to the definition of $\boldsymbol{V}_h$, the normal component of any $\boldsymbol{v}\in\boldsymbol{V}_h$ is continuous on the internal edges and vanishes on the boundary edges. Therefore, by splitting a vector $\boldsymbol{v}\in\boldsymbol{V}_h$ into its tangential and normal components $\boldsymbol{v}_n$ and $\boldsymbol{v}_t$

$$\boldsymbol{v}_n := (\boldsymbol{v}\cdot\mathbf{n})\mathbf{n}, \quad \boldsymbol{v}_t := (\boldsymbol{v}\cdot\mathbf{t})\mathbf{t} \equiv \boldsymbol{v} - \boldsymbol{v}_n, \tag{4.3}$$

we have

$$\forall\, e\in\mathcal{E}_h \quad \int_e [\![\boldsymbol{v}_n]\!]:\boldsymbol{\tau}\,\mathrm{d}s = 0 \quad \forall\,\boldsymbol{\tau}\in\boldsymbol{\mathcal{H}}^1(\mathcal{T}_h), \tag{4.4}$$

implying that

$$\forall\, e\in\mathcal{E}_h \quad \int_e [\![\boldsymbol{v}]\!]:\boldsymbol{\tau}\,\mathrm{d}s = \int_e [\![\boldsymbol{v}_t]\!]:\boldsymbol{\tau}\,\mathrm{d}s \quad \forall\,\boldsymbol{\tau}\in\boldsymbol{\mathcal{H}}^1(\mathcal{T}_h). \tag{4.5}$$

The resulting approximation to (2.6), therefore, becomes: *Find* $(\boldsymbol{u}_h, p_h)$ *in* $\boldsymbol{V}_h\times\mathcal{Q}_h$ *such that*

$$\begin{cases} a_h(\boldsymbol{u}_h,\boldsymbol{v}) + b(\boldsymbol{v},p_h) = (\boldsymbol{f},\boldsymbol{v}) & \forall\,\boldsymbol{v}\in\boldsymbol{V}_h \\ b(\boldsymbol{u}_h,q) = 0 & \forall\, q\in\mathcal{Q}_h, \end{cases} \tag{4.6}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

where

$$
\begin{aligned}
a_h(\boldsymbol{u},\boldsymbol{v}) &:= 2\nu\Big[(\boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{T}_h} - \langle\{\boldsymbol{\varepsilon}(\boldsymbol{u})\}:[\![\, \boldsymbol{v}_t \,]\!]\rangle_{\mathcal{E}_h^o} - \langle[\![\, \boldsymbol{u}_t \,]\!]:\{\boldsymbol{\varepsilon}(\boldsymbol{v})\}\rangle_{\mathcal{E}_h^o}\Big] \\
&\quad + 2\nu\alpha \sum_{e\in\mathcal{E}_h^o} h_e^{-1}\int_e [\![\, \boldsymbol{u}_t \,]\!]:[\![\, \boldsymbol{v}_t \,]\!]\,\mathrm{d}s \qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}_h, \qquad (4.7)\\
b(\boldsymbol{v},q) &:= -(q,\operatorname{div}\boldsymbol{v})_\Omega \qquad \forall\,\boldsymbol{v}\in\boldsymbol{V}_h,\ \forall q\in\mathcal{Q}_h.
\end{aligned}
$$

Consistency The consistency of the formulation (4.6) can be checked by means of the usual DG-machinery. In this case, it is sufficient to compare (4.1) and (4.7) and to observe that if $(\boldsymbol{u},p)$ is the solution of (2.6), then

$$
A_h(\boldsymbol{u},\boldsymbol{v}_h)\equiv a_h(\boldsymbol{u},\boldsymbol{v}_h),\quad B_h(\boldsymbol{v}_h,p)\equiv b(\boldsymbol{v}_h,p),\quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h\subseteq\boldsymbol{H}_{0,n}(\operatorname{div};\Omega),
$$

Further, it is evident that, $B_h(\boldsymbol{u},q_h)\equiv b(\boldsymbol{u},q_h)$ for all $q_h\in\mathcal{Q}_h$. Hence, as $(\boldsymbol{u},p)$ verifies (4.2), it also verifies (4.6); that is,

$$
\begin{cases}
a_h(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)=(\boldsymbol{f},\boldsymbol{v}) & \forall\,\boldsymbol{v}\in\boldsymbol{V}_h\\
b(\boldsymbol{u},q)=0 & \forall\,q\in\mathcal{Q}_h.
\end{cases}
\qquad (4.8)
$$

Thus, consistency is proved.

To prove the existence and uniqueness of the solution of (4.6) and to obtain the optimal error bounds, we need to define suitable norms. We define the following semi-norms

$$
|\boldsymbol{v}|_{1,h}^2=\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{v}\|_{0,T}^2,\quad |[\![\, \boldsymbol{v} \,]\!]|_*^2:=\sum_{e\in\mathcal{E}_h^o}h_e^{-1}\|[\![\, \boldsymbol{v} \,]\!]\|_{0,e}^2,\qquad \forall\,\boldsymbol{v}\in\boldsymbol{H}^1(\mathcal{T}_h),
$$

and norms

$$
\begin{aligned}
\|\boldsymbol{v}\|_{DG}^2 &:= 2\nu\,|\boldsymbol{v}|_{1,h}^2+2\nu\,|[\![\, \boldsymbol{v}_t \,]\!]|_*^2 && \boldsymbol{v}\in\boldsymbol{H}^1(\mathcal{T}_h),\\
|\!|\!|\boldsymbol{v}|\!|\!|^2 &:= \|\boldsymbol{v}\|_{DG}^2+\sum_{T\in\mathcal{T}_h}2\nu\,h_T^2|\boldsymbol{\varepsilon}(\boldsymbol{v})|_{1,T}^2 && \boldsymbol{v}\in\boldsymbol{H}^2(\mathcal{T}_h).
\end{aligned}
\qquad (4.9)
$$

We also remark that the seminorms defined in (4.9) are actually norms with the additional requirement that $\boldsymbol{v}\in\boldsymbol{H}_{0,n}(\operatorname{div};\Omega)$. We also observe that when restricted to discrete functions $\boldsymbol{v}\in\boldsymbol{V}_h$, the $\|\cdot\|_{DG}$-norm and the $|\!|\!|\cdot|\!|\!|$ are equivalent (using inverse inequality). Continuity can easily be shown for both bilinear forms:

$$
\begin{aligned}
|a_h(\boldsymbol{u},\boldsymbol{v})| &\le |\!|\!|\boldsymbol{u}|\!|\!|\,|\!|\!|\boldsymbol{v}|\!|\!| && \forall\,\boldsymbol{u},\ \boldsymbol{v}\in\boldsymbol{H}^2(\mathcal{T}_h),\\
|b(\boldsymbol{v},q)| &\le \|\boldsymbol{v}\|_{1,h}\|q\|_{0,\Omega} && \forall\,\boldsymbol{v}\in\boldsymbol{H}^1(\mathcal{T}_h),\ \ q\in L^2(\Omega)/\mathbb{R}\,.
\end{aligned}
$$

Following [19], the existence and uniqueness of the approximate solution and optimal error bounds are guaranteed if the following two conditions are satisfied:

(H1): *coercivity*: $\exists\,\gamma>0$ independent of the mesh size h such that

$$
a_h(\boldsymbol{v},\boldsymbol{v})\ge\gamma\|\boldsymbol{v}\|_{DG}^2\qquad\forall\,\boldsymbol{v}\in\boldsymbol{V}_h. \qquad (4.10)
$$

(H2): *inf-sup condition*: $\exists\,\beta>0$ independent of the mesh size h such that

$$
\sup_{\boldsymbol{v}\in\boldsymbol{V}_h}\frac{(\operatorname{div}\boldsymbol{v},q_h)_\Omega}{\|\boldsymbol{v}\|_{DG}}\ge\beta\|q_h\|_{0,\Omega}\quad\forall q_h\in\mathcal{Q}_h. \qquad (4.11)
$$

Condition **(H2)** is a consequence of the *inf-sup* condition that holds for the continuous problem (2.6):

$$\exists\,\beta > 0 \text{ s.t. } \forall h,\ \forall q_h \in \mathcal{Q}_h \quad \exists\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega):\ \operatorname{div} \boldsymbol{v} = q_h \text{ and } \|\boldsymbol{v}\|_{1,\Omega} \leq \frac{1}{\beta}\|q_h\|_{0,\Omega}.$$

It is well known that for all the families considered here an interpolation operator $\boldsymbol{v} \to \boldsymbol{v}^I \in \boldsymbol{V}_h$ exists that verifies (3.21) (in particular for $s = 1$), and

$$\operatorname{div} \boldsymbol{v}^I = \operatorname{div} \boldsymbol{v}\ (= q_h).$$

By observing that $[\![\, \boldsymbol{v} \,]\!] = 0$ on the internal edges as $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, and by using the Agmon trace inequality [2] and (3.21) (for $s = 1$), we have

$$|[\![\, \boldsymbol{v}^I \,]\!]|_*^2 := \sum_{e\in\mathcal{E}_h^o} h_e^{-1} \|[\![\, \boldsymbol{v}_t^I \,]\!]\|_{0,e}^2 = \sum_{e\in\mathcal{E}_h^o} h_e^{-1} \|[\![\, (\boldsymbol{v}^I - \boldsymbol{v})_t \,]\!]\|_{0,e}^2 \leq C\, |\boldsymbol{v}|_{1,\Omega}^2. \tag{4.12}$$

Hence, again using (3.21), we deduce that

$$\|\boldsymbol{v}^I\|_{DG} \leq C\, |\boldsymbol{v}|_{1,\Omega}.$$

Thus (4.11) is proved.

In order to prove (4.10) we need to extend (2.8) from Lemma 2.1 to spaces of discontinuous vectors. We have therefore the following result. Also see Appendix for further comments on the validity of the result in three dimensions.

Lemma 4.2 *Let $\boldsymbol{V}_h$ be a piecewise polynomial subspace of $\boldsymbol{H}_{0,n}(\operatorname{div};\Omega)$. Then, $\exists\, C_K > 0$ independent of h such that*

$$|\boldsymbol{v}|_{1,h}^2 \leq C_K \left(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 + \sum_{e\in\mathcal{E}_h^o} h_e^{-1} \|[\![\, \boldsymbol{v}_t \,]\!]\|_{0,e}^2 \right), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h. \tag{4.13}$$

Proof To show (4.13), a direct application of [13, Inequality (1.14)] to $\boldsymbol{v} \in \boldsymbol{V}_h$ gives

$$|\boldsymbol{v}|_{1,h}^2 \leq C_K \left(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 + \sum_{e\in\mathcal{E}_h^o} h_e^{-1} \|[\![\, \boldsymbol{v}_t \,]\!]\|_{0,e}^2 + \sup_{\substack{\boldsymbol{\eta}\in \boldsymbol{L}^2(\Omega)\\ \|\boldsymbol{\eta}\|_{0,\Omega}=1,\ \int_\Omega \boldsymbol{\eta}=0}} \left(\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\eta}\, dx \right)^2 \right), \tag{4.14}$$

We now show that the last term in (4.14) can be bounded by the first two. We claim that

$$\sup_{\substack{\boldsymbol{\eta}\in \boldsymbol{L}^2(\Omega)\\ \|\boldsymbol{\eta}\|_{0,\Omega}=1,\ \int_\Omega \boldsymbol{\eta}=0}} \left(\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\eta}\, dx \right)^2 \leq C\Big(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 + \sum_{e\in\mathcal{E}_h^o} h_e^{-1} \|[\![\, \boldsymbol{v}_t \,]\!]\|_{0,e}^2 \Big). \tag{4.15}$$

There are surely many ways of checking (4.15). Here, we propose one. For $\boldsymbol{v} \in \boldsymbol{V}_h$ and $\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega)$ with $\int_\Omega \boldsymbol{\eta}\,\mathrm{d}x = 0$, we set

$$\mathcal{I}(\boldsymbol{v},\boldsymbol{\eta}) := \int_\Omega \boldsymbol{v}\cdot\boldsymbol{\eta}\,\mathrm{d}x,$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

and we want to prove that

$$\mathcal{I}(\boldsymbol{v},\boldsymbol{\eta}) \le C\Big(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\mathcal{T}_h} + \sum_{e\in\mathcal{E}_h^o} h_e^{-1}\|[\![v_t]\!]\|^2_{0,e}\Big)^{1/2}\|\boldsymbol{\eta}\|_{0,\Omega} \tag{4.16}$$

that will easily give (4.15) taking the supremum with respect to $\boldsymbol{\eta}$ with $\|\boldsymbol{\eta}\|_{0,\Omega} = 1$. To prove (4.16) for every $\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega)$ with $\int_\Omega \boldsymbol{\eta}\,\mathrm{d}x = 0$, we consider the following auxiliary elasticity problem: *Find* $\boldsymbol{\chi} \in \boldsymbol{H}^1_{0,n}$ *such that:*

$$(\boldsymbol{\varepsilon}(\boldsymbol{\chi}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{0,\Omega} = (\boldsymbol{\eta},\boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v}\in\boldsymbol{H}^1_{0,n}. \tag{4.17}$$

Thanks to (2.8) problem (4.17) has a unique solution, and we set

$$\boldsymbol{\tau} := \boldsymbol{\varepsilon}(\boldsymbol{\chi}). \tag{4.18}$$

We note that as natural boundary condition for (4.17) we easily have

$$(\boldsymbol{\tau})_{nt} \equiv (\boldsymbol{\varepsilon}(\boldsymbol{\chi})\cdot\boldsymbol{n})\cdot\boldsymbol{t} = 0 \qquad \text{on } \Gamma, \tag{4.19}$$

where $\boldsymbol{t}$ is any tangent unit vector to Γ.

Due to well-known results on the regularity of the solutions of PDE systems on polygons, the solution $\boldsymbol{\tau}$ of (4.17), (4.18) (which, a priori, on a totally general domain would only be in $(L^2(\Omega))^{2\times2}_{sym}$) satisfies the following a priori estimate: *there exists a* $p > 2$ (*depending on the geometry of* Ω) *and a constant* C_p *such that for all* $\boldsymbol{\eta}\in\boldsymbol{L}^2(\Omega)$ *the corresponding* $\boldsymbol{\tau}$ *satisfies*

$$\|\boldsymbol{\tau}\|_{(L^p(\Omega))^{2\times2}_{sym}} + \|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega} \le C_p\|\boldsymbol{\eta}\|_{0,\Omega}. \tag{4.20}$$

The proof of the following proposition (actually, in two or three dimensions) is given in Appendix. □

Proposition 4.3 *Let T be a triangle with minimum angle $\theta > 0$, and let e be an edge of T. Then for every $p > 2$ and for every integer k_{max}, a constant $C_{p,\theta,kmax}$ exists such that*

$$\int_e \boldsymbol{v}\cdot(\boldsymbol{\tau}\cdot\mathbf{n})\,ds \le C_{p,\theta,kmax}\,h_T^{-1/2}\|\boldsymbol{v}\|_{0,e}\,\big(h_T\|\mathbf{div}\boldsymbol{\tau}\|_{0,T} + h_T^{\frac{p-2}{p}}\|\boldsymbol{\tau}\|_{0,p,T}\big) \tag{4.21}$$

for every $\boldsymbol{\tau}\in(L^p(\Omega))^{2\times2}_{sym}$ with divergence in $\boldsymbol{L}^2(T)$ and for every $\boldsymbol{v}\in\boldsymbol{P}^{k_{max}}(e)$.

Then we have

$$\begin{aligned}\mathcal{I}(\boldsymbol{v},\boldsymbol{\eta}) &= \int_\Omega \boldsymbol{v}\cdot\boldsymbol{\eta}\,\mathrm{d}x = -\int_\Omega \boldsymbol{v}\cdot(\boldsymbol{bf}\,\boldsymbol{div}\,\boldsymbol{\tau})\,\mathrm{d}x\\ &= (\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\tau})_{\mathcal{T}_h} - \langle[\![\boldsymbol{v}_t]\!]:\{\boldsymbol{\tau}\}\rangle_{\mathcal{E}_h^o}\end{aligned} \tag{4.22}$$

having taken into account that at the interelement boundaries the normal component of $\boldsymbol{v}$ is continuous and on Γ both the normal component of $\boldsymbol{v}$ and $(\boldsymbol{\tau})_{nt}$ are zero.

At this point, we can apply (4.21) to each e of the last term in (4.22). We apply the usual Cauchy-Schwarz inequality on the first term and we use instead the generalized Hölder inequality (with $q = 1/2$ and $r = 2p/(p-2)$, so that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$) on the second one.

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Then we obtain

$$\sum_{e\in\mathcal{E}_h^o}\int_e [\![\boldsymbol{v}_t]\!] : \{\boldsymbol{\tau}\}\,\mathrm{d}s \le \sum_{T\in\mathcal{T}_h}\sum_{e\in\partial T} C_{p,\theta,kmax}\Big(h_T^{-1/2}\|\boldsymbol{v}\|_{0,e}\, h_T\|bf\,div\boldsymbol{\tau}\|_{0,T}$$

$$+h_T^{-1/2}\|\boldsymbol{v}\|_{0,e}\|\, h_T^{\frac{p-2}{p}}\ \|\boldsymbol{\tau}\|_{0,p,T}\Big)$$

$$\le C|[\![\boldsymbol{v}_t]\!]|_*\, h\,\|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega}$$

$$+C\Big(\sum_{e\in\mathcal{E}_h^o} h_e^{-1}|[\![\boldsymbol{v}_t]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^o}\|\boldsymbol{\tau}\|_{0,p,T(e)}^p\Big)^{1/p}\Big(\sum_{e\in\mathcal{E}_h^o} h_e^{\frac{p-2}{p}r}\Big)^{1/r}$$

$$\le Ch\,|[\![\boldsymbol{v}_t]\!]|_*\,\|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega} + C\,|[\![\boldsymbol{v}_t]\!]|_*\,\|\boldsymbol{\tau}\|_{0,p,\Omega}\,\mu(\Omega)^{1/r} \tag{4.23}$$

where for each $e\in\mathcal{E}_h^o$ with $e=\partial T^+\cap\partial T^-$, the set $T(e)$ refers to $T(e):=T^+\cup T^-$. In the second line, $\mu(\Omega)$ denotes the measure of the domain Ω, whereas the constant C still depends on p, k_{max} and on the maximum angle in the decomposition $\mathcal{T}_h$.

From (4.22), (4.23), and the bound (4.20) we then obtain

$$|\mathcal{I}(\boldsymbol{v},\boldsymbol{\eta})| \le C\left(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h} + |[\![\boldsymbol{v_t}]\!]|_*\right)\|\boldsymbol{\eta}\|_{0,\Omega} \tag{4.24}$$

which gives (4.16). Thus the proof of the lemma is complete. □

Remark 4.4 The fact that in inequality (4.13) only the jumps over the interior edges $e\in\mathcal{E}_h^o$ (but not on the boundary edges) are included, prevents a direct and straightforward application of the results from [12]. The proof presented here is surely too elaborate, and we believe that a simpler proof is possible. However some of the machinery used here is likely to be of use elsewhere. Therefore, we decided that it would be worthwhile to present the proof we have obtained to date.

The stability of $a_h(\cdot,\cdot)$ in the $\|\cdot\|_{DG}$-norm can now be easily checked with the usual DG machinery. We have

$$\left|\int_e \{\boldsymbol{\varepsilon}(\boldsymbol{v})\} : [\![\boldsymbol{v}_t]\!]\,\mathrm{d}s\right| \le h^{1/2}\|\{\boldsymbol{\varepsilon}(\boldsymbol{v})\}\|_{0,e}\|h^{-1/2}[\![\boldsymbol{v}_t]\!]\|_{0,e},$$

which when we proceed as in [5] (or as in (4.23) with $p=2$) yields

$$\left|\sum_{e\in\mathcal{E}_h^o}\int_e \{\boldsymbol{\varepsilon}(\boldsymbol{v})\} : [\![\boldsymbol{v}_t]\!]\,\mathrm{d}s\right| \le C|\boldsymbol{v}|_{1,h}\,|[\![\boldsymbol{v}_t]\!]|_*. \tag{4.25}$$

Using (4.25) in (4.7), we then have

$$a_h(\boldsymbol{v},\boldsymbol{v}) \ge 2\nu\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 + 2\nu\,\alpha\,|[\![\boldsymbol{v}_t]\!]|_*^2 - 4\nu C|\boldsymbol{v}|_{1,\mathcal{T}_h}\,|[\![\boldsymbol{v}_t]\!]|_*.$$

Now using the Korn inequality (4.13) and the usual arithmetic-geometric mean inequality, we easily have a big enough α :

$$a_h(\boldsymbol{v},\boldsymbol{v}) \ge \gamma\|\boldsymbol{v}\|_{DG}^2 \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}_h.$$

We close this section with the following theorem.

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Theorem 4.5 *Let $(\boldsymbol{V}_h, \mathcal{Q}_h)$ be as in one of our three examples. Then problem (4.6) has a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times \mathcal{Q}_h$ that verifies*

$$\operatorname{div} \boldsymbol{u}_h = 0 \quad in\ \Omega. \tag{4.26}$$

Moreover, there exists a positive constant C, independent of h, such that for every $\boldsymbol{v}_h \in \boldsymbol{V}_h$ with $\operatorname{div} \boldsymbol{v}_h = 0$ and for every $q_h \in \mathcal{Q}_h$ the following estimate holds:

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{DG} &\le C\, \|\boldsymbol{u} - \boldsymbol{v}_h\|_{DG}, \\ \|p - p_h\|_{0,\Omega} &\le C\,(\|p - q_h\|_{0,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{DG}), \end{aligned} \tag{4.27}$$

with $(\boldsymbol{u}, p)$ solution of (2.6).

Proof The existence and uniqueness of the solution of (4.6) follow from (4.10), (4.11). The divergence-free property (4.26) is implied by (3.16), which holds for all our choices of spaces. Let $\boldsymbol{v}_h \in \boldsymbol{V}_h$ also be divergence-free; then we obviously have that $b(\boldsymbol{v}_h - \boldsymbol{u}_h, q) = 0$ for every $q \in L^2(\Omega)/\mathbb{R}$. In particular, $b(\boldsymbol{v}_h - \boldsymbol{u}_h, p - p_h) = 0$. Hence, from the coercivity (4.10), consistency (4.8), and continuity of $a_h(\cdot,\cdot)$ we deduce immediately

$$\begin{aligned} \gamma\|\boldsymbol{v}_h - \boldsymbol{u}_h\|^2_{DG} &\le a_h(\boldsymbol{v}_h - \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \\ &= a_h(\boldsymbol{v}_h - \boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}_h) \le \|\boldsymbol{v}_h - \boldsymbol{u}\|_{DG}\|\boldsymbol{v}_h - \boldsymbol{u}_h\|_{DG}. \end{aligned}$$

On the same basis we deduce that the first estimate in (4.27) follows by triangle inequality. For every $\boldsymbol{w}_h \in \boldsymbol{V}_h$, using the consistency and continuity of $a_h(\cdot,\cdot)$, we have

$$\begin{aligned} b(\boldsymbol{w}_h, q_h - p_h) &= b(\boldsymbol{w}_h, q_h - p) + b(\boldsymbol{w}_h, p - p_h) = b(\boldsymbol{w}_h, q_h - p) - a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) \\ &\le (\|q_h - p\|_{0,\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{DG})\|\boldsymbol{w}_h\|_{DG}. \end{aligned} \tag{4.28}$$

By dividing (4.28) by $\|\boldsymbol{w}_h\|_{DG}$ and then using the *inf-sup* condition (4.11), we immediately deduce that

$$\beta\|q_h - p_h\|_{0,\Omega} \le \|q_h - p\|_{0,\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{DG},$$

and that the second estimate in (4.27) follows again by triangle inequality. □

Remark 4.6 In the assumptions of Theorem4.5, we could obviously consider any trio of finite element spaces satisfying **H0**. However, for choices like $\mathbf{RT_0}$, not considered in our three examples, the estimate (4.27) could be meaningless, as the term $\|\boldsymbol{u} - \boldsymbol{v}_h\|_{DG}$ does not, in general, go to zero with h. Still, this choice could be profitably used, in some cases, as a preconditioner, as it does satisfy **H0**, **H1**, and **H2**.

5 Discrete Helmholtz Decompositions

In this section we provide results related to the discrete Helmholtz decomposition, introduced in Sect. 3 that plays a key role in the design of the preconditioner. We wish to note that Discrete Helmholtz or Hodge decompositions have been shown and used in several contexts for similar spaces but with other boundary conditions (typically, homogeneous Dirichlet) in [6,7,16,20]. A nice and short proof in the language of Finite Element Exterior Calculus can be also found in ([9], p. 72). Here, together with the proof of the decomposition with our boundary conditions, we provide an estimate in the DG-norm for the components in the splitting, that will be essential in the analysis of the solver, and that, to the best of our knowledge, has not been obtained or used in any previous work.

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

So far, we have assumed that the computational domain Ω is a polygon (or polyhedron). From now on, for the sake of simplicity, we are going to work under the stronger assumption that Ω is *a convex* polygon or polyhedron. As is well known, this allows the use of better regularity results, and in particular the H^2-regularity for elliptic second-order operators.

Following [19] we define the discrete gradient operator $\mathcal{G}_h : \mathcal{Q}_h \longrightarrow \boldsymbol{V}_h$ as

$$(\mathcal{G}_h q_h, \boldsymbol{v}_h)_{0,\Omega} := -(q_h, \operatorname{div} \boldsymbol{v}_h)_{0,\Omega} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.1}$$

Lemma 5.1 *Assume that together the three spaces $(\boldsymbol{V}_h, \mathcal{Q}_h, \mathcal{N}_h)$ (resp. $(\boldsymbol{V}_h, \mathcal{Q}_h, \boldsymbol{\mathcal{N}}_h)$) satisfy assumption* **H0** *(given in Definition 4.1). Then, in $d = 2$, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ a unique $q_h \in \mathcal{Q}_h$ and a unique $\varphi_h \in \mathcal{N}_h$ exist such that*

$$\boldsymbol{v}_h = \mathcal{G}_h q_h + \mathbf{curl}\, \varphi_h, \tag{5.2}$$

that is,

$$\boldsymbol{V}_h = \mathcal{G}_h(\mathcal{Q}_h) \oplus \mathbf{curl}\, \mathcal{N}_h.$$

If $d = 3$, there exists a $\boldsymbol{\psi} \in \boldsymbol{\mathcal{N}}_h$ such that

$$\boldsymbol{v}_h = \mathcal{G}_h q_h + \mathbf{curl}\, \boldsymbol{\psi}_h, \tag{5.3}$$

and therefore

$$\boldsymbol{V}_h = \mathcal{G}_h(\mathcal{Q}_h) \oplus \mathbf{curl}\, \boldsymbol{\mathcal{N}}_h.$$

Moreover, in both cases there exists a constant C independent of h such that the following estimate holds:

$$\|\mathcal{G}_h q_h\|_{DG} \leq C \|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega}. \tag{5.4}$$

We present the proof in two dimensions; see however Remark 5.2 after this proof, where the differences for the case $d = 3$ are discussed.

Proof For $\boldsymbol{v}_h \in \boldsymbol{V}_h$, consider the auxiliary problem:

$$-\Delta q = \operatorname{div} \boldsymbol{v}_h \quad \text{in } \Omega, \qquad \frac{\partial q}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad \text{and} \quad \int_\Omega q \, \mathrm{d}x = 0. \tag{5.5}$$

Owing to the boundary conditions in $\boldsymbol{V}_h$, we have that $\operatorname{div} \boldsymbol{v}_h$ has zero mean value in Ω. Hence, problem (5.5) has a unique solution, that satisfies

$$\|q\|_{2,\Omega} \leq C_{reg} \|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega}. \tag{5.6}$$

We write (5.5) in mixed form:

$$\boldsymbol{\sigma} = -\nabla q \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{\sigma} = \operatorname{div} \boldsymbol{v}_h \text{ in } \Omega, \quad \boldsymbol{\sigma} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

and we consider directly the approximation of the mixed formulation: *Find $(\boldsymbol{\sigma}_h, q_h) \in \boldsymbol{V}_h \times \mathcal{Q}_h$ such that :*

$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau})_{0,\Omega} - (q_h, \operatorname{div} \boldsymbol{\tau})_{0,\Omega} = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{V}_h, \\ (\operatorname{div} \boldsymbol{\sigma}_h, s_h)_{0,\Omega} = (\operatorname{div} \boldsymbol{v}_h, s_h)_{0,\Omega} & \forall s_h \in \mathcal{Q}_h. \end{cases} \tag{5.7}$$

Problem (5.7) obviously has a unique solution, which moreover satisfies

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \leq C\, h\, |\boldsymbol{\sigma}|_{1,\Omega} \leq C C_{reg}\, h\, \|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega}, \tag{5.8}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

given that (5.6) was used in the last step. As both $\boldsymbol{v}_h$ and $\boldsymbol{\sigma}_h$ are in $\boldsymbol{V}_h$ (and as (3.16) holds), the second equation in (5.7) directly implies that

$$\operatorname{div}(\boldsymbol{\sigma}_h - \boldsymbol{v}_h) = 0.$$

Hence, the exact sequence (3.8) implies that

$$\text{a unique } \varphi_h \in \mathcal{N}_h \text{ exists such that } \boldsymbol{\sigma}_h - \boldsymbol{v}_h = \mathbf{curl}\,\varphi_h. \tag{5.9}$$

Next, by using the first equation in (5.7) and then applying definition (5.1), we deduce that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau})_{0,\Omega} = (q_h, \operatorname{div}\boldsymbol{\tau})_{0,\Omega} = -(\mathcal{G}_h q_h, \boldsymbol{\tau})_{0,\Omega} \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{V}_h,$$

which implies $\boldsymbol{\sigma}_h = -\mathcal{G}_h q_h$, that joined to (5.9) gives (5.2).

In order to prove (5.4), we recall that

$$\|\mathcal{G}_h q_h\|_{DG}^2 = \|\boldsymbol{\sigma}_h\|_{DG}^2 = \|\nabla\boldsymbol{\sigma}_h\|_{0,\mathcal{T}_h}^2 + |[\![(\boldsymbol{\sigma}_h)_t]\!]|_*^2. \tag{5.10}$$

For the first term, by adding and subtracting the interpolant $\boldsymbol{\sigma}^I$ of $\boldsymbol{\sigma}$ and then using inverse inequality and (3.21), we have:

$$\begin{aligned} \|\nabla\boldsymbol{\sigma}_h\|_{0,\mathcal{T}_h} &\le \|\nabla(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^I)\|_{0,\mathcal{T}_h} + \|\nabla\boldsymbol{\sigma}^I\|_{0,\mathcal{T}_h} \\ &\le C_{inv} h^{-1} \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^I\|_{0,\mathcal{T}_h} + C\|\nabla\boldsymbol{\sigma}\|_{0,\mathcal{T}_h}. \end{aligned} \tag{5.11}$$

From triangle inequality, (5.8), and standard approximation properties (see (3.21)), we have

$$\|\nabla\boldsymbol{\sigma}_h\|_{0,\mathcal{T}_h} \le C\, \|\operatorname{div}\boldsymbol{v}_h\|_{0,\Omega}. \tag{5.12}$$

The jump term in (5.10) is estimated similarly. First, we remark that $\boldsymbol{\sigma} = -\nabla q$ with $q \in H^2(\Omega)$ so that $[\![\boldsymbol{\sigma}]\!] = 0$ on each $e \in \mathcal{E}_h^o$, and therefore

$$|[\![(\boldsymbol{\sigma}_h)_t]\!]|_*^2 = |[\![(\boldsymbol{\sigma}_h)_t - \boldsymbol{\sigma}_t]\!]|_*^2.$$

Then, using Agmon trace inequalities (5.8) and the boundedness of $\boldsymbol{\sigma}_h$ and $\boldsymbol{\sigma}$, we have

$$\begin{aligned} |[\![(\boldsymbol{\sigma}_h)_t - \boldsymbol{\sigma}_t]\!]|_*^2 &= \sum_{e\in\mathcal{E}_h^o} h_e^{-1} \|[\![(\boldsymbol{\sigma}_h)_t - \boldsymbol{\sigma}_t]\!]\|_{0,e}^2 \\ &\le C_t h^{-2} \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{0,\mathcal{T}_h}^2 + C_t \|\nabla(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|_{0,\mathcal{T}_h}^2 \\ &\le C C_{reg} \|\operatorname{div}\boldsymbol{v}_h\|_{0,\Omega}^2. \end{aligned}$$

Thus the proof is complete. □

Remark 5.2 For $d = 3$, instead of (5.9), the exact sequence (3.9) property implies

$$\exists\, \boldsymbol{\psi}_h \in \boldsymbol{\mathcal{N}}_h \text{ such that } \boldsymbol{\sigma}_h - \boldsymbol{v}_h = \mathbf{curl}\,\boldsymbol{\psi}_h.$$

The vector potential $\boldsymbol{\psi}_h$ would be uniquely determined by adding the condition $\operatorname{div}\boldsymbol{\psi} = 0$. In fact, on a simply connected domain, $\operatorname{div}\boldsymbol{\psi} = 0$ and $\mathbf{curl}\,\boldsymbol{\psi} = 0$ together with $\boldsymbol{\psi} \in \boldsymbol{H}_{0,t}(\mathbf{curl}, \Omega)$ imply $\boldsymbol{\psi} = 0$. However, in general, the solution of $\operatorname{div}\boldsymbol{\psi} = 0$ and $\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{v}_h$ together with $\boldsymbol{\psi} \in \boldsymbol{H}_{0,t}(\mathbf{curl}, \Omega)$ (which is uniquely determined) does not belong to $\boldsymbol{\mathcal{N}}_h$. A possibility to select a vector potential $\boldsymbol{\psi}_h$ in a unique way could be to compute it as the approximation to the following continuous problem: Find $(\boldsymbol{\psi}, \theta)$ in $\boldsymbol{H}_{0,t}(\mathbf{curl}; \Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} (\mathbf{curl}\,\boldsymbol{\psi}, \mathbf{curl}\,\boldsymbol{\phi})_{\mathcal{T}_h} + (\nabla\theta, \boldsymbol{\phi})_{\mathcal{T}_h} &= (\boldsymbol{v}_h, \boldsymbol{\phi})_{\mathcal{T}_h} \qquad && \forall\, \boldsymbol{\phi} \in \boldsymbol{H}_{0,t}(\mathbf{curl}; \Omega), \\ (\boldsymbol{\psi}, \nabla s)_{\mathcal{T}_h} &= 0 && \forall\, s \in H_0^1(\Omega). \end{aligned}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Setting

$$\overset{o}{\mathcal{W}}_h := \left\{ w \in H_0^1(\Omega) \;:\; w|_T \in \mathbb{P}^{k+1}(T) \; \forall T \in \mathcal{T}_h \right\},$$

the discrete problem reads: Find $(\boldsymbol{\psi}_h, \theta_h) \in \boldsymbol{\mathcal{N}}_h \times \overset{o}{\mathcal{W}}_h$ such that

$$\begin{aligned} (\mathbf{curl}\,\boldsymbol{\psi}_h, \mathbf{curl}\,\boldsymbol{\phi}_h)_{\mathcal{T}_h} + (\nabla\theta_h, \boldsymbol{\phi}_h)_{\mathcal{T}_h} &= (\boldsymbol{v}_h, \boldsymbol{\phi}_h)_{\mathcal{T}_h} \qquad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{\mathcal{N}}_h, \\ (\boldsymbol{\psi}_h, \nabla w_h)_{\mathcal{T}_h} &= 0 \qquad \forall\, w_h \in \overset{o}{\mathcal{W}}_h \,. \end{aligned} \tag{5.13}$$

Problem (5.13) has a unique solution satisfying $\mathbf{curl}\,\boldsymbol{\psi}_h = \boldsymbol{v}_h$ (from the first equation), and $\mathrm{div}\boldsymbol{\psi}_h = 0$ (from the second equation).

6 Preconditioner: Fictitious Space Lemma and Auxiliary Space Framework

6.1 Preconditioner for the Semi-Definite System

Assume V is a Hilbert space equipped with the norm $\|\cdot\|_V$ and that $A : V \mapsto V'$ is a bounded linear operator. We define the bilinear form

$$(u, v)_A = \langle Au, v\rangle.$$

We say A is symmetric if the bilinear form $(u, v)_A$ is symmetric. We say that A is semi-positive definite if

$$(v, v)_A \geq 0, \quad \forall v \in V$$

and $\alpha > 0$ exists such that

$$(v, v)_A \geq \alpha \|v\|^2_{V/N(A)}, \quad \forall v \in V/N(A).$$

And we say that A is SPD (Symmetric Positve Definite) if it is symmetric and $\alpha > 0$ exists such that

$$(v, v)_A \geq \alpha \|v\|^2_V, \quad \forall v \in V.$$

One useful property of symmetric semi-positive definite operators is that

$$Av = 0 \text{ iff } \langle Av, v\rangle = 0. \tag{6.1}$$

A preconditioner for A is another symmetric semi-positive definite operator $B : V' \mapsto V$. Again, we consider the bilinear form

$$(f, g)_B = \langle f, Bg\rangle.$$

The operator $BA : V \mapsto V$ satisfies

$$(BAu, v)_A = \langle Av, BAu\rangle = (Au, Av)_B.$$

Lemma 6.1 *If $A : V \mapsto V'$ and $B : V' \mapsto V$ are both symmetric semi-positive definite such that B is positive definite on $R(A)$, then*

(1) *$B : R(A) \mapsto R(BA)$ is an isomorphism (with the inverse satisfying trivially that $B^{-1}(BAv) = Av$).*
(2) *The bilinear form $(\cdot, \cdot)_{B^{-1}}$ defines an inner product on $R(BA)$.*

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

(3) *The bilinear form* $(\cdot,\cdot)_A$ *defines an inner product on* $R(BA)$.
(4) BA *is symmetric positive definite on* $R(BA)$ *with either of the above two inner products.*

Proof All these results are pretty obvious, and their proofs are similar. Let us give the proof for 3 as an example.

We only need to verify that $(\cdot,\cdot)_A$ is positive definite on $R(BA)$. If $v \in R(BA)$ is such that $(v,v)_A = 0$, then, by (6.1), we have $Av = 0$. We write $v = BAw$ for some $w \in V$, then $ABAw = 0$ and hence $(Aw, Aw)_B = 0$. As B is positive definite on $R(A)$, we have $Aw = 0$. Thus, $v = ABAw = 0$, as desired. □

For the system $Au = f$, we can apply the preconditioner B and the preconditioned conjugate gradient (PCG) method with respect to the inner product $(\cdot,\cdot)_{B^{-1}}$ with the following convergence estimate:

$$\|u - u^k\|_A \le 2\left(\frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1}\right)^k \|u - u^0\|_A.$$

The condition number can then be estimated by $\kappa(BA) \le c_1/c_0$, either where

$$c_0(v,v)_{B^{-1}} \le (BAv, v)_{B^{-1}} \le c_1(v,v)_{B^{-1}}, \quad \forall v \in R(BA),$$

or equivalently where

$$c_0(w,w)_B \le (Bw, Bw)_A \le c_1(w,w)_B, \quad \forall w \in R(A),$$

or where

$$c_1^{-1}(v,v)_A \le (B^{-1}v, v) \le c_0^{-1}(v,v)_A \quad \forall v \in R(BA).$$

6.2 Fictitious Space Lemma and Generalizations

Let us present and prove a refined version of the Fictitious Space Lemma originally proposed by Nepomnyaschikh [34] (see also [41]).

Lemma 6.2 *Let* $\tilde{V}$ *and* V *be two Hilbert spaces, and let* $\Pi : \tilde{V} \mapsto V$ *be a surjective map. Let* $\tilde{B} : \tilde{V}' \mapsto \tilde{V}$ *be a symmetric and positive definite operator. Then* $B := \Pi\tilde{B}\Pi'$ *is also symmetric and positive definite (here* $\Pi' : V' \mapsto \tilde{V}'$ *is such that* $\langle \Pi' g, \tilde{v}\rangle = \langle g, \Pi\tilde{v}\rangle$*, for all* $g \in V'$ *and* $\tilde{v} \in \tilde{V}$*). Furthermore,*

$$\langle B^{-1}v, v\rangle = \inf_{\Pi\tilde{v}=v} \langle \tilde{B}^{-1}\tilde{v}, \tilde{v}\rangle.$$

Proof It is obvious that B is symmetric and positive semi-definite. Note that if $v \in V'$ is such that $\langle Bv, v\rangle = 0$, then $\langle \tilde{B}\Pi' v, \Pi' v\rangle = \langle Bv, v\rangle = 0$. This means that $\Pi' v = 0$ as $\tilde{B}$ is SPD. Hence, $v = 0$ as Π' is injective. This proves that B is positive definite.

For any $\tilde{v} \in \tilde{V}$, let $v = \Pi\tilde{v}$ and $\tilde{v}^* = \tilde{B}\Pi' B^{-1} v$. As we obviously have $\Pi\tilde{v}^* = v$, we can write $\tilde{v} = \tilde{v}^* + \tilde{w}$ with $\Pi\tilde{w} = 0$. Thus,

$$\begin{aligned}
\inf_{\Pi\tilde{v}=v} \langle \tilde{B}^{-1}\tilde{v}, \tilde{v}\rangle &= \inf_{\Pi\tilde{w}=0} \langle \tilde{B}^{-1}(\tilde{v}^* + \tilde{w}), \tilde{v}^* + \tilde{w}\rangle \\
&= \langle \tilde{B}^{-1}\tilde{v}^*, \tilde{v}^*\rangle + \inf_{\Pi\tilde{w}=0} \left(\langle \tilde{B}^{-1}\tilde{w}, \tilde{w}\rangle + 2\langle \tilde{B}^{-1}\tilde{v}^*, \tilde{w}\rangle\right)
\end{aligned}$$

From the definition of $\tilde{v}^*$ we have

$$\langle \tilde{B}^{-1}\tilde{v}^*, \tilde{v}^*\rangle = \langle B^{-1}v, \Pi\tilde{v}^*\rangle = \langle B^{-1}v, v\rangle,$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

and also

$$\langle \tilde{B}^{-1}\tilde{v}^*, \tilde{w}\rangle = \langle \tilde{B}^{-1}\tilde{B}\Pi' B^{-1}v, \tilde{w}\rangle = \langle \Pi' B^{-1}v, \tilde{w}\rangle = \langle B^{-1}v, \Pi\tilde{w}\rangle = 0.$$

The last two identities lead to the desired result. □

Theorem 6.3 *Assume that $\tilde{A} : \tilde{V} \mapsto \tilde{V}'$ and $A : V \mapsto V'$ are symmetric semi-definite operators. We assume that $\Pi : \tilde{V} \mapsto V$ is surjective and that $\Pi(N(\tilde{A})) = N(A)$. Then for any SPD operator $\tilde{B} : \tilde{V}' \mapsto \tilde{V}$, we have, for $B = \Pi\tilde{B}\Pi'$,*

$$\kappa(BA) \le \kappa(\Pi)\kappa(\tilde{B}\tilde{A}).$$

Here $\kappa(\Pi)$ is the smallest ratio c_1/c_0 that satisfies

$$c_1^{-1}\langle Av, v\rangle \le \inf_{\Pi\tilde{v}=v}\langle \tilde{A}\tilde{v}, \tilde{v}\rangle \le c_0^{-1}\langle Av, v\rangle, \quad \forall v \in R(BA). \tag{6.2}$$

Proof Denote $\kappa(\tilde{B}\tilde{A}) = b_1/b_0$ with b_1 and b_0 satisfying

$$b_1^{-1}(\tilde{v}, \tilde{v})_{\tilde{A}} \le (\tilde{B}^{-1}\tilde{v}, \tilde{v}) \le b_0^{-1}(\tilde{v}, \tilde{v})_{\tilde{A}}, \quad \forall \tilde{v} \in R(\tilde{B}\tilde{A}).$$

By (6.2), we obtain

$$b_1^{-1}c_1^{-1}\|v\|_A^2 \le \inf_{\Pi\tilde{v}=v, \tilde{v}\in R(\tilde{B}\tilde{A})}(\tilde{B}^{-1}\tilde{v}, \tilde{v}) \le b_0^{-1}c_0^{-1}\|v\|_A^2, \quad \forall v \in R(BA).$$

By the assumption that $\Pi(N(\tilde{A})) = N(A)$, we can prove that $\Pi'(R(A)) \subset R(\tilde{A})$ and

$$\{\tilde{v}|\Pi\tilde{v} = v \in R(BA)\} = \{\tilde{v}|\Pi\tilde{v} = v \in R(BA), \tilde{v} \in R(\tilde{B}\tilde{A})\}.$$

By Lemma 6.2,

$$\inf_{\Pi\tilde{v}=v, \tilde{v}\in R(\tilde{B}\tilde{A})}(\tilde{B}^{-1}\tilde{v}, \tilde{v}) = \inf_{\Pi\tilde{v}=v}(\tilde{B}^{-1}\tilde{v}, \tilde{v}) = (B^{-1}v, v), \quad \forall v \in R(BA).$$

Therefore,

$$b_1^{-1}c_1^{-1}\|v\|_A^2 \le (B^{-1}v, v) \le b_0^{-1}c_0^{-1}\|v\|_A^2 \quad \forall v \in R(BA).$$

□

Theorem 6.4 *Assume that the following two conditions are satisfied for Π. First,*

$$\|\Pi\tilde{v}\|_A \le c_1\|\tilde{v}\|_{\tilde{A}}, \quad \forall \tilde{v} \in \tilde{V}.$$

Second, for any $v \in V$ there exists $\tilde{v} \in \tilde{V}$ such that $\Pi\tilde{v} = v$ and

$$\|\tilde{v}\|_{\tilde{A}} \le c_0\|v\|_A.$$

Then $\kappa(\Pi) \le c_1/c_0$ and, under the assumptions of Theorem 6.3,

$$\kappa(BA) \le \left(\frac{c_1}{c_0}\right)^2 \kappa(\tilde{B}\tilde{A}).$$

Remark 6.5 In view of the application of the above results to our two dimensional case (as we shall see in the next subsection), it would have been enough to restrict ourselves to the symmetric positive definite case (instead of the semi-definite case treated in the last two subsections). However we preferred to have them in the present more general setting, as in this form they are likely to be useful in many other circumstances (starting, as natural, from the extension of the present theory to the three-dimensional case).

6.3 Application to Our Problem

In this section we design a simple preconditioner for the linear system resulting from the approximation of the Stokes problem (2.6) defined in (4.6), (4.7). Note that the bilinear form $a_h(\cdot,\cdot)$ defined in (4.7) provides a discretization of the vector Laplacian problem

$$-\mathbf{div}(2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u}\cdot\mathbf{n} = 0, \quad (\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\mathbf{n})\cdot\boldsymbol{t} = 0 \text{ on } \Gamma.$$

We denote by A_h the operator associated with $a_h(\cdot,\cdot)$. As the solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$ of (4.6) is divergence-free, the discrete Helmholtz decomposition (5.2) implies that

$$\text{a unique } \psi_h \in \mathcal{N}_h \quad \text{exists such that } \boldsymbol{u}_h = \mathbf{curl}\,\psi_h.$$

At this point, it is convenient to introduce the space $\mathring{\boldsymbol{V}}_h$ as

$$\mathring{\boldsymbol{V}}_h := \boldsymbol{V}_h \cap \boldsymbol{H}_0(\mathrm{div}^0;\Omega). \tag{6.3}$$

We note that as the sequence (3.8) is exact, we have

$$\mathring{\boldsymbol{V}}_h \equiv \mathbf{curl}\,\mathcal{N}_h, \tag{6.4}$$

and that the mapping is one-to-one. Therefore, restricting the bilinear form $a_h(\cdot,\cdot)$ to $\mathring{\boldsymbol{V}}_h$, in the spirit of Remark 3.1, corresponds here to restricting the trial and test space to $\mathring{\boldsymbol{V}}_h \equiv \mathbf{curl}(\mathcal{N}_h)$. The discrete problem (4.6) then reduces to the following problem: *Find* $\psi_h \in \mathring{\boldsymbol{V}}_h$ *such that*

$$a_h(\psi_h, \varphi_h) = (\boldsymbol{f}, \varphi_h) \qquad \forall \varphi_h \in \mathring{\boldsymbol{V}}_h \tag{6.5}$$

Defining the operator $A_h : \mathring{\boldsymbol{V}}_h \mapsto \mathring{\boldsymbol{V}}_h'$ by $\langle A_h\psi_h, \varphi_h\rangle = a_h(\psi_h,\varphi_h)$, $\psi_h, \varphi_h \in \mathring{\boldsymbol{V}}_h$, we can write (6.5) as

$$A_h\psi_h = f_h.$$

We now use the original space $\boldsymbol{V}_h$ as the auxiliary space for $\mathring{\boldsymbol{V}}_h$. Define $\tilde{A}_h : \boldsymbol{V}_h \mapsto \boldsymbol{V}_h'$ by $\langle \tilde{A}_h u_h, v_h\rangle = a_h(u_h, v_h)$, $u_h, v_h \in \boldsymbol{V}_h$. We note that $\tilde{A}_h$ is a discrete Laplacian. We assume that $\tilde{B}_h$ is an optimal preconditioner for $\tilde{A}_h$.

We now define the operator

$$\Pi_h : \boldsymbol{V}_h \longrightarrow \mathring{\boldsymbol{V}}_h \equiv \mathbf{curl}(\mathcal{N}_h) \tag{6.6}$$

according to (5.2), namely

$$\Pi_h \boldsymbol{v}_h = \mathbf{curl}\,\varphi_h.$$

Note that Π_h is a surjective operator and that Π_h acts as the identity on the subspace $\mathring{\boldsymbol{V}}_h$. The auxiliary space preconditioner for A_h is then defined by

$$B_h = \Pi_h \tilde{B}_h \Pi_h^*. \tag{6.7}$$

Lemma 6.6 *Assume that the spaces* $(\boldsymbol{V}_h, \mathcal{Q}_h, \mathcal{N}_h)$ *satisfy assumption* **H0**. *Then* B_h *given by* (6.7) *is an optimal preconditioner for* A_h *as long as* $\tilde{B}_h$ *is an optimal preconditioner for* $\tilde{A}_h$.

Proof Following the auxiliary space techniques (Theorem 6.4), we need to check that the following two properties are satisfied:

(A1): Local Stability: there exists a positive constant C_1 independent of h such that

$$\|\Pi_h \boldsymbol{v}_h\|_{DG} \le C_1 \|\boldsymbol{v}_h\|_{DG} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \tag{6.8}$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

(A2): Stable decomposition: there exists a positive constant C_2 independent of h such that for any $\boldsymbol{w}_h \in \mathring{\boldsymbol{V}}_h$ there exists $\boldsymbol{v}_h \in \boldsymbol{V}_h$ such that $\Pi_h \boldsymbol{v}_h = \boldsymbol{w}_h$ and

$$\|\boldsymbol{v}_h\|_{DG} \leq C_2 \|\boldsymbol{w}_h\|_{DG}. \tag{6.9}$$

To prove (6.8) from the Helmholtz decomposition (5.2) and the definition (6.6) of Π_h, we have

$$\boldsymbol{v}_h = \mathcal{G}_h q_h + \mathbf{curl}\,\varphi_h = \mathcal{G}_h q_h + \Pi_h \boldsymbol{v}_h. \tag{6.10}$$

Using estimate (5.4) from Lemma 5.1 and the clear fact that div $\boldsymbol{v}_h$ is the trace of $\boldsymbol{\varepsilon v}_h$, we have

$$\|\mathcal{G}_h q_h\|_{DG} \leq C \|\mathrm{div}\, \boldsymbol{v}_h\|_{0,\Omega} \leq C \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h} \leq C \|\boldsymbol{v}_h\|_{DG}. \tag{6.11}$$

Hence, (6.8) follows from (6.10) and (6.11):

$$\|\Pi_h \boldsymbol{v}_h\|_{DG} = \|\boldsymbol{v}_h - \mathcal{G}_h q_h\|_{DG} \leq \|\boldsymbol{v}_h\|_{DG} + \|\mathcal{G}_h q_h\|_{DG} \leq C \|\boldsymbol{v}_h\|_{DG}.$$

Finally, the inequality (6.9) holds with $C_2 = 1$ by taking $\boldsymbol{v}_h = \boldsymbol{w}_h$. □

7 Numerical Experiments

7.1 Setup

The tests presented in this section use discretization by the lowest order, namely, $\mathbf{BDM}_1$ elements paired with piece-wise constant space for the pressure. They verify the *a priori* estimates given in Theorem 4.5 and confirm the uniform bound on the condition number of the preconditioned system for the velocity.

As previously set up, the discrete problem under consideration is given by Eq. (4.6) with bilinear forms $a_h(\cdot, \cdot)$ and $b(\cdot, \cdot)$ defined in (4.7). In the numerical tests presented here, we take $\nu = 1/2$ and the penalty parameter $\alpha = 6$ in (4.7). We present two sets of tests with A corresponding to the Stokes equation discretized on a sequence of successively refined unstructured meshes as shown in Figs. 1 and 2. On the square the coarsest mesh (level of refinement $J = 0$) has 160 elements and 97 vertices with 448 BDM degrees of freedom. The finer triangulations of the square domain are obtained via $1, \ldots, 5$ regular refinements (every element divided in 4) and the finest one is with 163,840 elements, 82,433 vertices and 490,496 BDM degrees of freedom. Similarly for the L-shaped domain we start with a coarsest grid ($J = 0$) with 64 vertices and 97 elements. For the L-shaped domain the finest grid (for $J = 5$) has 99,328 elements, 50,129 vertices and 297,056 $\mathbf{BDM}_1$ degrees of freedom. In the computations, we approximate the velocity component $\boldsymbol{u}_h$ of the solution of the Stokes equation by solving several simpler equations (such as scalar Laplace equations). After we obtain the velocity, the pressure then is found via a postprocessing step at low computational cost. Further, for this sequence of grids the $\mathbf{BDM}_1$ interpolant of a function $\boldsymbol{v}$ on the $k-th$ grid is denoted by $\boldsymbol{v}^{I_k}$. Accordingly the piece-wise constant, L_2-orthogonal projection of p is denoted by p^{I_k}. We also use the notation $(\boldsymbol{u}_k, p_k)$ for the solution of (4.6) on the $k-th$ grid, $k = 0, \ldots, 5$.

7.2 Discretization Error

We now present several tests related to the error estimates given in the previous sections. We computed and tabulated approximations of the order of convergence of the discrete solution

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

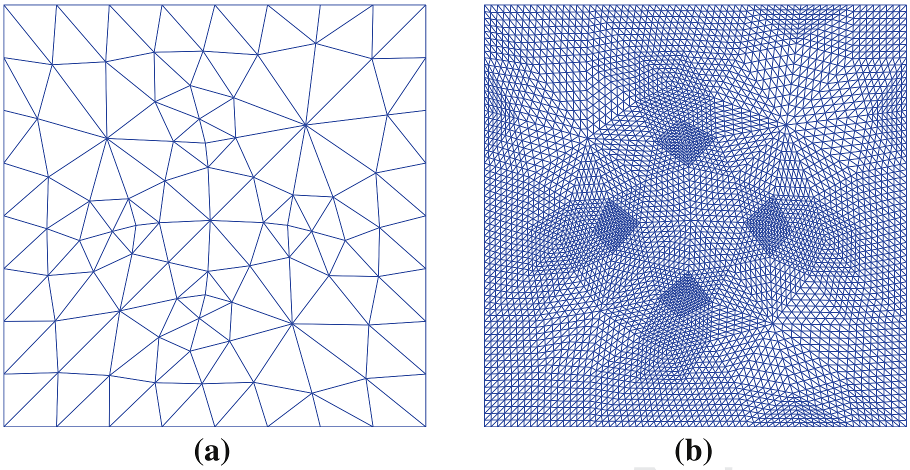

Fig. 1 Meshes used in the tests for the unit square domain $\Omega = (0, 1) \times (0, 1)$. **a** Coarsest mesh. **b** Mesh for level of refinement $J = 3$

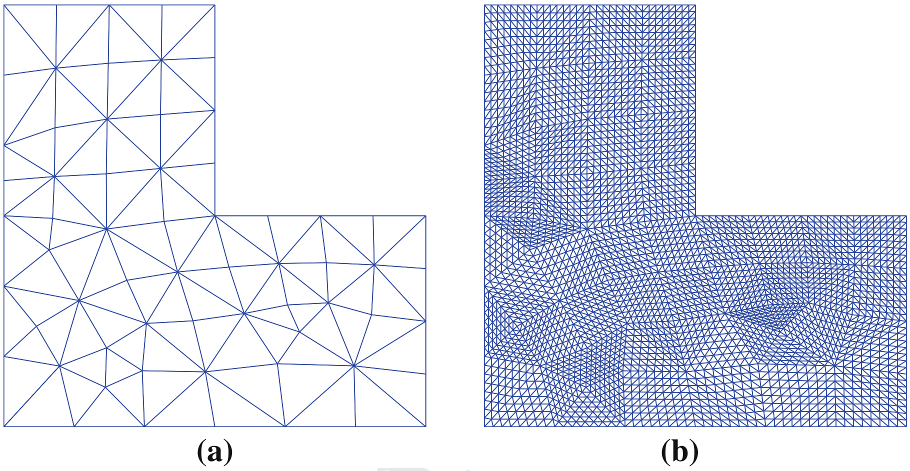

Fig. 2 Meshes used in the tests for the L-shaped domain $\Omega = ((0, 1) \times (0, 1)) \setminus ([\frac{1}{2}, 1) \times [\frac{1}{2}, 1))$. **a** Coarsest mesh. **b** Mesh for level of refinement $J = 3$

in different norms. These approximations are denoted by $\gamma_0 \approx \beta_0, \gamma_{DG} \approx \beta_{DG}, \gamma_p \approx \beta_p$, and $\gamma_* \approx \beta_*$. The actual orders of convergence $\beta_0,\ \beta_{DG},\ \beta_p$, and β_* are

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \approx C(\boldsymbol{u})h^{\beta_0}, \quad \|\boldsymbol{u} - \boldsymbol{u}_h\|_{DG} \approx C(\boldsymbol{u})h^{\beta_{DG}},$$
$$\|p - p_h\|_{0,\Omega} \approx C(\boldsymbol{u}, p)h^{\beta_p}, \quad |[\![\boldsymbol{u}_h]\!]|_* \approx C(\boldsymbol{u})h^{\beta_*}.$$

Here, as in (4.12), we denote

$$|[\![\boldsymbol{v}]\!]|_*^2 = \sum_{e \in \mathcal{E}_h^o} h_e^{-1} \int_e [\![\boldsymbol{u}_t]\!]^2\, ds.$$

Note that β_* is the order with which the jumps in the approximate solution (not in the error) go to zero.

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Table 1 Approximate order of convergence for the difference $(\boldsymbol{u}^I - \boldsymbol{u}_h)$ and $(p^I - p_h)$ and the jumps $|[\![\boldsymbol{u}_h]\!]|_*$ for the square and L-shaped domains

Square domain						L-shaped domain					
k	1	2	3	4	5	k	1	2	3	4	5
γ_0	1.75	1.87	1.94	1.98	1.99	γ_0	1.69	1.79	1.90	1.96	1.98
γ_{DG}	0.98	1.0	1.00	1.00	1.00	γ_{DG}	0.97	1.01	1.01	1.00	1.00
γ_p	0.94	0.95	0.97	0.99	0.99	γ_p	0.93	0.92	0.95	0.97	0.99
γ_*	0.77	0.89	0.95	0.98	0.99	γ_*	0.73	0.85	0.93	0.97	0.99

Here, $\boldsymbol{u}$ and p are given in (7.1) and (7.2)

We present two sets of experiments to illustrate the results given in Theorem 4.5. First, we consider the exact given solution and calculate the right–hand side and the boundary conditions from this solution. We set

$$\phi = xy(1-x)(2x-1)(y-1)(2y-1), \qquad \boldsymbol{u} = \mathbf{curl}\phi. \tag{7.1}$$

Clearly, the function ϕ vanishes on the boundary of both the domains under consideration and we take $\boldsymbol{u}$ defined in (7.1) as exact solution for the velocity for both the square and the L-shaped domains. For the pressure we choose as exact solutions functions with zero mean value and select p different for the square and the L-shaped domain, namely

$$\begin{aligned} p &= x^2 - 3y^2 + \tfrac{8}{3}xy, \ \ \text{(square domain)},\\ p &= x^2 - 3y^2 + \tfrac{24}{7}xy, \ \ \text{(L-shaped domain)}. \end{aligned} \tag{7.2}$$

The right hand side $\boldsymbol{f}$ is calculated by plugging $(\boldsymbol{u}, p)$ defined in (7.1), (7.2) in (2.1). Table 1 shows tabulation of the order of convergence of $(\boldsymbol{u}_h, p_h)$ to $(\boldsymbol{u}^I, p^I)$ for both the square domain and the L-shaped domain. The values approximating the order of convergence displayed in Table 1 are

$$\gamma = \log_2 \frac{\|\boldsymbol{u}^I_{k-1} - \boldsymbol{u}_{k-1}\|}{\|\boldsymbol{u}^I_k - \boldsymbol{u}_k\|}, \quad \gamma_* = \log_2 \frac{|[\![\boldsymbol{u}_k]\!]|_*}{|[\![\boldsymbol{u}_{k-1}]\!]|_*},$$

$$\gamma_p = \log_2 \frac{\|p^I_{k-1} - p_{k-1}\|_{0,\Omega}}{\|p^I_k - p_k\|_{0,\Omega}}, \quad k = 1, \ldots, 5.$$

Here $\|\cdot\|$ stands for any of the DG or L_2 norms. The quantity γ is the corresponding γ_0 or γ_{DG}. From the results in this table, we can conclude that in the $\|\cdot\|_{DG}$ norm the dominating error is the interpolation error, and as the next example shows, in general, the order of convergence in $\|\cdot\|_{DG}$ is 1.

The second test is for a fixed right hand side $\boldsymbol{f} = 2(1, x)$. We calculate approximations to the order of convergence of the numerical solutions on successively refined grids as follows:

$$\gamma = \log_2 \frac{\|\boldsymbol{u}_k - \boldsymbol{u}_{k-1}\|}{\|\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\|}, \quad \gamma_* = \log_2 \frac{|[\![\boldsymbol{u}_k]\!]|_* - |[\![\boldsymbol{u}_{k-1}]\!]|_*}{|[\![\boldsymbol{u}_{k+1}]\!]|_* - |[\![\boldsymbol{u}_k]\!]|_*},$$

$$\gamma_p = \log_2 \frac{\|p_k - p_{k-1}\|_{0,\Omega}}{\|p_{k+1} - p_k\|_{0,\Omega}}, \quad k = 1, \ldots, 4.$$

Again, $\|\cdot\|$ denotes any of the (semi)-norms of interest and γ approximates the corresponding order of convergence. Table 2 shows the tabulated values of γ_0, γ_{DG}, γ_p, and γ_*. It is

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Table 2 Approximate order of convergence of the error for square and L-shaped domains and right–hand side $\boldsymbol{f} = 2(1, x)$

Square domain						L-shaped domain					
k	1	2	3	4	5	k	1	2	3	4	5
γ_0	1.70	1.85	1.93	1.97	1.98	γ_0	1.65	1.79	1.86	1.74	1.24
γ_{DG}	0.86	0.95	0.98	0.99	1.00	γ_{DG}	0.84	0.92	0.92	0.86	0.74
γ_p	0.94	0.94	0.97	0.98	0.99	γ_p	0.91	0.89	0.88	0.82	0.70
γ_*	0.70	0.86	0.94	0.97	0.99	γ_*	0.63	0.81	0.89	0.89	0.83

clear from these values that the order of approximation for the velocity and the pressure is optimal for the square domain, whereas for the L-shaped domain the convergence is not of optimal order, due to the singularity of the solution near the reentrant corner. The numerical experiments and also the approximations for the orders of convergence presented in Tables 1 and 2 are computed using the FEniCS package http://fenicsproject.org.

7.3 Uniform Preconditioning

The tests presented in this subsection illustrate the efficient solution of the system (7.3) below by Preconditioned Conjugate Gradient (PCG) with the preconditioner given in (7.4). We introduce the matrices representing the bilinear forms defined in (4.6), (4.7), and also the mass matrix for the $\mathbf{BDM}_1$ space. We denote by $\mathbf{M}$ the mass matrix on $\boldsymbol{V}_h$ and by $\widetilde{\mathbf{A}}$ the stiffness matrix associated with $a_h(\cdot,\cdot)$ on $\boldsymbol{V}_h$ in (4.6), (4.7). We note that $\mathbf{A}$, without the divergence–free constraint, is spectrally equivalent to two scalar Laplacians.

It is known that the null space of $b(\cdot,\cdot)$ in (4.6) is made of vector fields that are curls of continuous, piecewise quadratic functions vanishing on the boundary. We denote by $\mathbf{P}_{\text{curl}}$ the matrix representation of these curls in the BDM space. Namely,

$$\mathbf{curl}(\text{basis functions in } \mathcal{N}_h) = (\text{basis functions in } \boldsymbol{V}_h)\mathbf{P}_{\text{curl}}.$$

It is easy to see that

$$\mathbf{A}_q = \mathbf{P}_{\text{curl}}^T \mathbf{M} \mathbf{P}_{\text{curl}}.$$

where $\mathbf{A}_q$ is the discretization of the Laplacian on N_h with homogeneous Dirichlet boundary conditions.

The problem of finding the solution of (6.5) then amounts to solving the following algebraic system of equations

$$\mathbf{P}_{\text{curl}}^T \widetilde{\mathbf{A}} \mathbf{P}_{\text{curl}} \mathbf{U} = \mathbf{P}_{\text{curl}}^T \mathbf{F}. \tag{7.3}$$

Here the superscript T means that the adjoint is taken with respect to the ℓ_2-inner product, $\mathbf{U}$ is the vector containing the velocity degrees of freedom, and $\mathbf{F}$ is the vector representing the right–hand side $(\boldsymbol{f}, \boldsymbol{v})$ of the problem (4.6).

The matrix representation $\mathbf{B}$ of the preconditioner B described in the previous section has the following form:

$$\mathbf{B} = \mathbf{A}_q^{-1} \mathbf{P}_{\text{curl}}^T \mathbf{M} \widetilde{\mathbf{A}}^{-1} \mathbf{M} \mathbf{P}_{\text{curl}} \mathbf{A}_q^{-1} \tag{7.4}$$

In the numerical experiments below we have used the preconditioned conjugate gradient provided by MATLAB with the above preconditioner. We note that one may further make

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Table 3 Preconditioning results for square domain (top) and L-shaped domain (bottom)

J	0	1	2	3	4	5
Square domain						
n_{it}	4	4	4	5	5	4
ρ	0.016	0.023	0.031	0.034	0.033	0.031
L-shaped domain						
n_{it}	5	5	5	5	5	5
ρ	0.044	0.061	0.061	0.058	0.055	0.053

The PCG iterations are terminated when the relative residual is smaller than 10^{-6}

the algorithm more efficient by incorporating approximations $\widetilde{\mathbf{B}}$ (for $\widetilde{\mathbf{A}}^{-1}$) and B_q (for A_q^{-1}) in (7.4). In our tests the inverses needed to compute the action of the preconditoner, namely A_q^{-1} and $\widetilde{\mathbf{A}}^{-1}$, are calculated by the MATLAB's backslash "\" operator (which in turn calls the direct solver from UMFPACK http://www.cise.ufl.edu/research/sparse/umfpack/). The tests presented here exactly match the theory for the auxiliary space preconditioner given in Sect. 6.3.

In summary, the action of the preconditioner requires the solution of systems corresponding to 4 scalar Laplacians. It is also worth noting that suitable multigrid packages for performing these tasks are available today.

The convergence rate results are summarized in Table 3. The legend for the symbols used in the table is as follows: n_{it} is the number of PCG iterations; ρ is the average reduction per one such iteration defined as $\rho = \left[\frac{\|r_{n_{it}}\|_{\ell 2}}{\|r_0\|_{\ell 2}}\right]^{1/n_{it}}$; J is the refinement level, for which $h \approx 2^{-J} h_0$, where h_0 is the characteristic mesh size on the coarsest grid. From the results in Table 3, we can conclude that the preconditioner is uniform with respect to the mesh size. It is also evident that this method is in fact quite efficient in terms of the number of iterations and the reduction factor.

Let us point out that when the preconditioner is implemented in 3D the action of Π_h requires an implementation of the action of L^2-orthogonal (or orthogonal in equivalent inner product) projection on the divergence free subspace $\mathring{V}_h$. This is done by solving an auxiliary mixed FE discretization of the Laplacian, as discussed in Sect. 5 and in practice it can be accomplished by considering a projection orthogonal in the inner product provided by the lumped mass matrix for BDM. In such case the solution to the auxiliary mixed FE problem corresponds to a solution of a system with an M-matrix and classical AMG methods [11] AMG yield optimal solvers for such problems. The application of the preconditioner in the 3D case requires the (approximate) solution of 5 scalar Laplacians.

Such extensions to 3D and also efficient approximations to $\widetilde{\mathbf{A}}^{-1}$ and A_q^{-1} in (7.4) are subject of current research and implementation and are to be included in a future release of the Fast Auxiliary Preconditioning Package http://fasp.sf.net.

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8 Appendix: Proof of Proposition 4.3

We now state and prove a result, Proposition 8.1 given below, used in Sect. 4 to show Korn inequality (cf. Lemma 4.2). After giving its proof, we comment briefly on how the result can be applied to show the corresponding Korn inequality (4.13) (cf. Lemma 4.2) for $d = 3$.

Proposition 8.1 *Let T be a triangle (or a tetrahedron for $d = 3$) with minimum angle $\theta > 0$, and let e be an edge (resp. face) of T. Then for every $p > 2$ and for every integer k_{max} there exists a constant $C_{p,\theta,kmax}$ such that*

$$\int_e \boldsymbol{v} \cdot (\boldsymbol{\tau} \cdot \mathbf{n})\, ds \le C_{p,\theta,k_{max}}\, h_T^{-1/2} \|\boldsymbol{v}\|_{0,e} \left(h_T \|\boldsymbol{div}\,\boldsymbol{\tau}\|_{0,T} + h_T^{\frac{d(p-2)}{2p}} \|\boldsymbol{\tau}\|_{0,p,T} \right) \tag{8.1}$$

for every $\boldsymbol{\tau} \in (L^p(\Omega))^{d\times d}_{sym}$ having divergence in $\boldsymbol{L}^2$ and for every $\boldsymbol{v} \in \boldsymbol{P}^{k_{max}}(T)$.

Proof First we go to the reference element $\hat{T}$:

$$\Big| \int_e \boldsymbol{v} \cdot (\boldsymbol{\tau} \cdot \mathbf{n})\ \mathrm{d}s \le C_\theta |e| \Big| \int_{\hat{e}} \hat{\boldsymbol{v}} \cdot (\hat{\boldsymbol{\tau}} \cdot \hat{\mathbf{n}})\, d\hat{s} \Big| \le C_\theta h_e^{d-1} \Big| \int_{\hat{e}} \hat{\boldsymbol{v}} \cdot (\hat{\boldsymbol{\tau}} \cdot \hat{\mathbf{n}})\, d\hat{s} \Big| \tag{8.2}$$

where $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{\tau}}$ are the usual covariant and contra-variant images of $\boldsymbol{v}$ and $\boldsymbol{\tau}$, respectively. And, here and throughout his proof, the constants C_θ and $C_{\theta,k_{max}}$ may assume different values at different occurrences. Note that $\hat{\boldsymbol{v}}$ will still be a vector-valued polynomial of degree $\le k_{max}$ and the space $H(\mathrm{div}, T)$ is effectively mapped into $H(\mathrm{div}, \hat{T})$ by means of the contra-variant mapping. Then for every component $\hat{v}$ of $\hat{\boldsymbol{v}}$, we construct the auxiliary function φ_v as follows. First we define φ_v on $\partial\hat{T}$ by setting it as equal to $\hat{v}$ on $\hat{e}$ and zero on the rest of $\partial\hat{T}$. Then we define φ_v in the interior using the harmonic extension. It is clear that φ_v will belong to $W^{1,p'}(\hat{T})$ (remember that $p > 2$ so that its conjugate index p' will be smaller than 2). Using the fact that $\hat{\boldsymbol{v}}$ is a polynomial of degree $\le k_{max}$, it is not difficult to see that

$$\|\boldsymbol{\varphi}_v\|_{W^{1,p'}(\hat{T})} \le \hat{C}_{\theta,k_{max}} \|\hat{\boldsymbol{v}}\|_{0,\hat{e}}. \tag{8.3}$$

Integration by parts then gives

$$\begin{aligned} \int_{\hat{e}} \hat{\boldsymbol{v}} \cdot (\hat{\boldsymbol{\tau}} \cdot \hat{\mathbf{n}})\, d\hat{s} &= \int_{\partial\hat{T}} \boldsymbol{\varphi}_v \cdot (\hat{\boldsymbol{\tau}} \cdot \hat{\mathbf{n}})\, d\hat{s} \\ &= \int_{\hat{T}} \nabla\boldsymbol{\varphi}_v : \hat{\boldsymbol{\tau}}\, d\hat{x} - \int_{\hat{T}} \boldsymbol{\varphi}_v \cdot \boldsymbol{div}\,\hat{\boldsymbol{\tau}}\, d\hat{x} \\ &\le |\boldsymbol{\varphi}_v|_{W^{1,p'}(\hat{T})} \|\hat{\boldsymbol{\tau}}\|_{(L^p(\hat{T}))^{d\times d}_{sym}} + \|\boldsymbol{\varphi}_v\|_{0,\hat{T}} \|\boldsymbol{div}\,\hat{\boldsymbol{\tau}}\|_{0,\hat{T}} \\ &\le \hat{C} \Big(\|\hat{\boldsymbol{v}}\|_{0,\hat{e}} \|\hat{\boldsymbol{\tau}}\|_{(L^p(\hat{T}))^{d\times d}_{sym}} + \|\boldsymbol{\varphi}_v\|_{0,\hat{e}} \|\boldsymbol{div}\,\hat{\boldsymbol{\tau}}\|_{0,\hat{T}} \Big) \\ &\le \hat{C}\, \|\hat{\boldsymbol{v}}\|_{0,\hat{e}} \Big(\|\hat{\boldsymbol{\tau}}\|_{(L^p(\hat{T}))^{d\times d}_{sym}} + \|\boldsymbol{div}\,\hat{\boldsymbol{\tau}}\|_{0,\hat{T}} \Big). \end{aligned} \tag{8.4}$$

Then we recall the inverse transformations (from $\hat{T}$ to T):

$$\|\hat{\boldsymbol{v}}\|_{0,\hat{e}} \le C_\theta h_e^{-\frac{d-1}{2}} \|\boldsymbol{v}\|_{0,e}, \quad \|\hat{\boldsymbol{\tau}}\|_{(L^p(\hat{T}))^{d\times d}_{sym}} \le C_\theta h_T^{-\frac{d}{p}} \|\boldsymbol{\tau}\|_{(L^p(T))^{d\times d}_{sym}},$$

$$\|\boldsymbol{div}\,\hat{\boldsymbol{\tau}}\|_{0,\hat{T}} \le C_\theta h_T^{\frac{2-d}{2}} \|\boldsymbol{div}\,\boldsymbol{\tau}\|_{0,T}.$$

Journal: 10915 Article No.: 9758 TYPESET DISK LE CP Disp.:2013/7/25 Pages: 31 Layout: Small

Inserting this into (8.4) and then in (8.2) we have then

$$\int_e \boldsymbol{v}\cdot(\boldsymbol{\tau}\cdot\mathbf{n})\,\mathrm{d}s \le C_{p,\theta,k_{max}}\, h_e^{d-1}\, h_e^{-\frac{d-1}{2}}\, \|\boldsymbol{v}\|_{0,e}\Big(h_T^{-\frac{d}{p}}\, \|\boldsymbol{\tau}\|_{(L^p(T))^{d\times d}_{sym}} + h_T^{\frac{2-d}{2}}\, \|\boldsymbol{div}\,\boldsymbol{\tau}\|_{0,T}\Big).$$

Now we note that

$$-\frac{1}{2}+\frac{d(p-2)}{2p} = d-1-\frac{d-1}{2}-\frac{d}{p},$$

and that

$$-\frac{1}{2}+1 = d-1-\frac{d-1}{2}+\frac{2-d}{2},$$

and the proof then follows immediately. □

With this result in hand, we can show the Korn inequality (4.13) given in Lemma 4.2 for $d=3$. It is necessary to modify the proof in only two places: the definition of the space of rigid motions on Ω, $\boldsymbol{RM}(\Omega)$, and the application of Proposition 4.21. The space $\boldsymbol{RM}(\Omega)$ is now defined by:

$$\boldsymbol{RM}(\Omega) = \Big\{\boldsymbol{a}+\boldsymbol{b}\boldsymbol{x} \;:\; \boldsymbol{a}\in\mathbb{R}^d \quad \boldsymbol{b}\in so(d)\Big\}$$

with $so(d)$ denoting the space of the skew-symmetric $d\times d$ matrices.

To prove (4.16) (and so conclude the proof of (4.13)), estimate (4.23) is replaced by estimate (8.5) below, which is obtained as follows: first, by applying (8.1) (instead of (4.21)) from Proposition 8.1 to each e in the last term in (4.22) and then by using the generalized Hölder inequality with the same exponents as for $d=2$ (with $q=1/2$ and $r=2p/(p-2)$, so that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$)

$$\begin{aligned}
\sum_{e\in\mathcal{E}_h^o}\int_e [\![\, \boldsymbol{v}_t \,]\!] : \{\boldsymbol{\tau}\} &\le C_{p,\theta,kmax}\sum_{T\in\mathcal{T}_h}\sum_{e\in\partial T} h_T^{-1/2}\|[\![\, \boldsymbol{v}_t \,]\!]\|_{0,e}\, h_T\,\|\mathbf{div}\boldsymbol{\tau}\|_{0,T}\\
&\quad + C_{p,\theta,kmax}\sum_{T\in\mathcal{T}_h}\sum_{e\in\partial T} h_T^{-1/2}\|[\![\, \boldsymbol{v}_t \,]\!]\|_{0,e}\|\, h_T^{\frac{d(p-2)}{2p}}\, \|\boldsymbol{\tau}\|_{0,p,T}\\
&\le Ch\,|[\![\, \boldsymbol{v}_t \,]\!]|_*\,\|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega} \qquad\qquad (8.5)\\
&\quad + C\Big(\sum_{e\in\mathcal{E}_h^o} h_e^{-1}|[\![\, \boldsymbol{v}_t \,]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^o}\|\boldsymbol{\tau}\|_{0,p,T(e)}^p\Big)^{1/p}\Big(\sum_{e\in\mathcal{E}_h^o} h_e^{\frac{d(p-2)}{2p}r}\Big)^{1/r}\\
&\le C|[\![\, \boldsymbol{v}_t \,]\!]|_*\, h\,\|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega} + C\,|[\![\, \boldsymbol{v}_t \,]\!]|_*\,\|\boldsymbol{\tau}\|_{0,p,\Omega}\,\mu(\Omega)^{1/r}
\end{aligned}$$

Here, as in estimate (4.23), $\mu(\Omega)$ denotes the measure of the domain Ω, and the constant C still depends on p, k_{max}, and on the maximum angle in the decomposition $\mathcal{T}_h$. The rest of the proof of Lemma 4.2 proceeds as for $d=2$.

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Journal: 10915 Article No.: 9758 ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:2013/7/25 Pages: 31 Layout: Small

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