ISSN 1722-8964

Consiglio Nazionale delle Ricerche

Istituto di Matematica Applicata e Tecnologie Informatiche "*Enrico Magenes*"

PUBBLICAZIONI

L. Beirão da Veiga, F. Brezzi, L.D. Marini

VIRTUAL ELEMENTS FOR LINEAR ELASTICITY PROBLEMS

11PV12/10/0

C.N.R. - Istituto di Matematica Applicata e Tecnologie Informatiche *“Enrico Magenes”*

Sede di Pavia
Via Ferrata, 1 - 27100 PAVIA (Italy)
Tel. +39 0382 548211
Fax +39 0382 548300

Sezione di Genova
Via De Marini, 6 (Torre di Francia) - 16149 GENOVA (Italy)
Tel. +39 010 6475671
Fax +39 010 6475660

Sezione di Milano
Via E. Bassini, 15 - 20133 MILANO (Italy)
Tel. +39 02 23699521
Fax +39 02 23699538

URL: http://www.imati.cnr.it

VIRTUAL ELEMENTS FOR LINEAR ELASTICITY PROBLEMS

L. BEIRÃO DA VEIGA[1,2], F. BREZZI[3,2,7], AND L. D. MARINI[4,2]

Abstract. We discuss the application of *Virtual Elements* to linear elasticity problems, both for the compressible and the nearly incompressible case. Virtual Elements are very close to Mimetic Finite differences (see, for linear elasticity, [2]), and in particular to Higher Order Mimetic Finite Differences. As such, they share the good features of being able to represent in an *exact* way certain physical properties (conservation, incompressibility, etc.) and of being applicable in very general geometries. The advantage of Virtual Elements is the ductility that allows to have easily high order accuracy and high order continuity.

1. Introduction

In recent times the Mimetic Finite Difference approach has been successfully applied to a great variety of problems, from diffusion problems to electromagnetism, on fairly irregular decompositions, including polygons with rather weird shape, polyhedra in 3D with curved faces, hanging nodes and so on (for a partial list of citations we refer to [25, 21, 14, 20, 22, 11, 13, 7, 23, 1, 12, 6, 15, 18, 28, 9, 2, 8, 16]). Their use was limited to low order approximations until very recently, when people started to extend the methodology to include higher order approximations, in order to gain better accuracy in the numerical results. See, e.g., [19, 5, 4, 3]. The first results in this direction were very encouraging, and people started to look more closely to these extensions, analyzing advantages and limitations, and mostly looking for the key properties that could make things easier. This gave rise to a new interpretation of Mimetic Finite Differences (see, to start with, [16]) and to a subsequent new approach, much closer to Finite Elements, that we call Virtual Element Method (VEM). Other methods that extend the Finite Element philosophy to polygonal meshes can be found in [26, 27, 24].

The basic idea of the new method can be described, roughly speaking, as follows. We start as we do for the classical Finite Elements, of Lagrange or Hermite type, with one difference: in each element K, together with the usual polynomials, say $\mathcal{S}$ (in general: all the polynomials up to a given degree k), some additional functions are also considered (typically, solutions of PDE's within the element K) in order to help getting unisolvence. If things are properly done (good choice of the functions and of the degrees of freedom), the local stiffness matrix AE can be computed exactly *whenever one of the two entries is a polynomial of* $\mathcal{S}$, without solving the local PDE (virtual solution). For the other coefficients of AE it is enough to have

[1]Dipartimento di Matematica, Università di Milano Statale, Via Saldini 50, I-20133 Milano (Italy); lourenco.beirao@unimi.it.
[2]IMATI-CNR, Via Ferrata 1, 27100 Pavia (Italy).
[3]Istituto Universitario di Studi Superiori (IUSS), 27100 Pavia (Italy); brezzi@imati.cnr.it.
[7]KAU, Jeddah, Saudi Arabia.
[4]Dipartimento di Matematica, Università di Pavia, Via Ferrata 1, 27100 Pavia (Italy); marini@imati.cnr.it.

numbers that are bounded from above and from below by the "right" ones, with constants independent of h. The label *Virtual* actually depends on the fact that some of the basis functions are not explicitly known.

In this paper we apply the Virtual Element approach to linear elasticity problems, compressible and nearly incompressible, in two dimensions. We shall prove "optimal" a priori estimates, meaning that, if $\mathcal{S}$ contains all the polynomial of degree $\le k$ (for some integer $k \ge 1$) then the error between the exact and the approximate solution, in the energy norm, behaves as $O(h^k)$ (times some $(k+1)$-norm of the exact solution) where h is the maximum diameter of the elements in the decomposition.

Finally, we shall also prove optimal L^2-estimates, that is, $O(h^{k+1})$.

As for the traditional Mimetic Finite Difference approach the decomposition of the computational domain Ω can be done in a *very general* way, as for instance in [9]. We point out that for a decomposition in triangles the VEM will reproduce, essentially, the classical Finite Element Methods. This will not be the case for more general decompositions, including quadrilaterals. In particular, we stress the fact that general quadrilaterals do not require to be treated as *iso-parametric elements* (and, besides, we can allow aligned vertices and non convex elements). We also point out that small intrusions of a "practically arbitrary" shape, made of a material with different elastic properties, can be treated with a single element.

Throughout the paper C will denote, as usual, a generic positive constant independent of the mesh size. For the definition of Sobolev spaces and their norms we refer to [17]. In particular we shall use the notation $(\cdot,\cdot)_{\mathcal{O},0}$ to denote the inner product in $L^2(\mathcal{O})$ or $(L^2(\mathcal{O}))^2$, and simply $(\cdot,\cdot)_0$ whenever $\mathcal{O} = \Omega$. Moreover, for k integer ≥ 0, $\mathbb{P}_k$ will denote the space of polynomials of degree $\le k$.

2. Compressible linear elasticity

2.1. **The problem.** We consider the deformation problem of a linearly elastic body subjected to a volume load and with given boundary conditions, under the hypothesis of small deformations. Let Ω be a polygonal domain, and let Γ be its boundary. Let λ and μ be positive coefficients (Lamé coefficients) and let $\boldsymbol{f}$ be a vector valued function belonging to $(L^2(\Omega))^2$. For the sake of simplicity we will use (homogeneous) Dirichlet boundary conditions, and hence consider the space

$$\mathbf{V} := (H^1_0(\Omega))^2 \qquad \text{with} \quad \|\boldsymbol{v}\|^2_{\mathbf{V}} := \|\nabla \boldsymbol{v}\|_{(L^2(\Omega))^4} \; \forall \boldsymbol{v} \in \mathbf{V}. \tag{1}$$

However we will always write our bilinear form as

$$a(\boldsymbol{u},\boldsymbol{v}) := 2\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\mathbf{x} + \lambda \int_\Omega \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}\,\mathrm{d}\mathbf{x} \equiv 2\mu a_\mu(\boldsymbol{u},\boldsymbol{v}) + \lambda a_\lambda(\boldsymbol{u},\boldsymbol{v}), \tag{2}$$

(where $\boldsymbol{\varepsilon}$ represents as usual the *symmetric gradient operator*) and we will keep an eye on the set of *rigid body motions*

$$RM(\Omega) := \{\boldsymbol{v} \in (H^1(\Omega))^2 \text{ such that } \boldsymbol{\varepsilon}(\boldsymbol{v}) = 0\} = (c_1,c_2) + c_3(x_2,-x_1), \tag{3}$$

so that the extensions to more general cases will be immediate. To the bilinear form a we can associate, in the usual distributional sense, the linear elliptic operator $A_{\lambda,\mu}$ given by

$$A_{\lambda,\mu}\boldsymbol{u} := -\begin{pmatrix} 2\mu(u_{1,xx} + \frac{1}{2}(u_{1,yy} + u_{2,xy})) + \lambda(u_{1,xx} + u_{2,yx}) \\ 2\mu(\frac{1}{2}(u_{1,yx} + u_{2,xx}) + u_{2,yy}) + \lambda(u_{1,xy} + u_{2,yy}) \end{pmatrix}. \tag{4}$$

It is easy to see (possibly using Korn inequality in the presence of more general boundary conditions) that there exist two constants, $M > 0$ and $\alpha > 0$, depending only on Ω, λ and μ, such that

$$\alpha \, \|\boldsymbol{v}\|^2_{\mathbf{V}} \leq \, a(\boldsymbol{v}, \boldsymbol{v}) \leq M \, \|\boldsymbol{v}\|^2_{\mathbf{V}} \qquad \forall \, \boldsymbol{v} \in \mathbf{V}. \tag{5}$$

We note that we obviously have $\boldsymbol{f} \in \mathbf{V}'$, and we denote by $< \boldsymbol{f}, \boldsymbol{v} >$ the corresponding duality pairing (that here coincides with the usual $(L^2)^2$ inner product). We consider the problem: *find* $\boldsymbol{u} \in \mathbf{V}$ *such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) =< \boldsymbol{f}, \boldsymbol{v} > \quad \forall \, \boldsymbol{v} \in \mathbf{V} \tag{6}$$

that clearly has a unique solution that belongs at least to $(H^s(\Omega))^2$ for some $s > 3/2$ depending on the maximum angle in Γ. More generally we denote by $\mathbf{W}$ a space of vector valued functions that contains our solution $\boldsymbol{u}$ and where we are allowed to take point values. For instance, we can assume that $\mathbf{W} \subseteq (C^0(\overline{\Omega}))^2$.

Remark 2.1. *It is clear that problem* (6) *is equivalent to*

$$A_{\lambda,\mu}\boldsymbol{u} = \boldsymbol{f} \;\text{ in } \Omega \qquad \text{and} \qquad \boldsymbol{u} = 0 \;\text{ on } \Gamma. \tag{7}$$

□

2.2. **The decompositions and the discrete problems.** In order to approximate the solution of (6) we consider a sequence $\{\mathcal{T}_h\}_h$ of decompositions of Ω into subpolygons, such that:

H0 - There exists an integer N and a positive real number r such that for every h and for every $K \in \mathcal{T}_h$:

- the number of edges of K is $\leq N$,
- the ratio between the shortest edge and the diameter h_K of K is bigger than r, and
- K is star-shaped with respect to every point of a ball of radius rh_K. □

With obvious notation, we split the norm

$$\|\boldsymbol{v}\|^2_{\mathbf{V}} = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{v}\|^2_{\mathbf{V},K} \quad \forall \, \boldsymbol{v} \, \in \mathbf{V} \tag{8}$$

and the bilinear form a as

$$a(\boldsymbol{u}, \boldsymbol{v}) = \sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \, \boldsymbol{u}, \, \boldsymbol{v} \, \in \mathbf{V}. \tag{9}$$

H1 - We fix an integer $k \geq 1$ (that will be our order of accuracy) and consider for each h

- a space $\mathbf{V}_h \subset \mathbf{V}$,
- a symmetric bilinear form a_h from $\mathbf{V}_h \times \mathbf{V}_h$ to $\mathbb{R}$, and
- an element $\boldsymbol{f}_h \in \mathbf{V}'_h$.
- We also suppose that the approximate bilinear form a_h can also be split as we did for a in (9), namely:

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_K a^K_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \qquad \forall \, \boldsymbol{u}_h, \, \boldsymbol{v}_h \, \in \mathbf{V}_h. \tag{10}$$

□

2.3. **An abstract convergence theorem.** Together with **H0** and **H1** we further assume the following properties.

H2 - For all h, and for all K in $\mathcal{T}_h$

- $\forall \boldsymbol{p} \in (\mathbb{P}_k)^2$, $\forall \boldsymbol{v}_h \in \mathbf{V}_h$

$$a_h^K(\boldsymbol{p}, \boldsymbol{v}_h) = a^K(\boldsymbol{p}, \boldsymbol{v}_h). \tag{11}$$

- $\exists$ two positive constants α_* and α^*, independent of h and of K, such that

$$\forall \boldsymbol{v}_h \in \mathbf{V}_h \qquad \alpha_*\, a^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \le a_h^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \le \alpha^*\, a^K(\boldsymbol{v}_h, \boldsymbol{v}_h). \tag{12}$$

□

Note that the symmetry of a_h, (12) and (5) easily imply that

$$\begin{aligned} a_h^K(\boldsymbol{u}, \boldsymbol{v}) \le & \Big(a_h^K(\boldsymbol{u}, \boldsymbol{u})\Big)^{1/2} \Big(a_h^K(\boldsymbol{v}, \boldsymbol{v})\Big)^{1/2} \\ & \le \alpha^* \Big(a^K(\boldsymbol{u}, \boldsymbol{u})\Big)^{1/2} \Big(a^K(\boldsymbol{v}, \boldsymbol{v})\Big)^{1/2} \le \alpha^*\, M\, \|\boldsymbol{u}\|_{\mathbf{V},K}\, \|\boldsymbol{v}\|_{\mathbf{V},K} \end{aligned} \tag{13}$$

for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbf{V}_h$.

Remark 2.2. *It is clear that, in the classical terminology, assumption* (11) *is meant to ensure consistency of the bilinear form a_h, and* (12) *is meant to ensure stability.* □

From now on, since we will deal also with functions that belong to $\prod_K (H^1(K))^2$, but are not globally in $(H^1(\Omega))^2$, we will use the broken H^1- norm:

$$\|\boldsymbol{v}\|_{h,\mathbf{V}} := \Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{v}\|^2_{\mathbf{V},K}\Big)^{1/2}. \tag{14}$$

Theorem 2.1. *Under the assumptions* **H0**, **H1**, *and* **H2** *the discrete problem: Find $\boldsymbol{u}_h$ in $\mathbf{V}_h$ such that*

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = < \boldsymbol{f}_h, \boldsymbol{v}_h >, \qquad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h \tag{15}$$

has a unique solution $\boldsymbol{u}_h$. Moreover, for every approximation $\boldsymbol{u}_I$ of $\boldsymbol{u}$ in $\mathbf{V}_h$ and for every approximation $\boldsymbol{u}_\pi$ of $\boldsymbol{u}$ that is piecewise in $(\mathbb{P}_k)^2$, we have

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathbf{V}} \le C\Big(\|\boldsymbol{u} - \boldsymbol{u}_I\|_{\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \mathfrak{F}\Big)$$

where C is a constant depending only on Ω, λ, μ, α_*, α^*, and $\mathfrak{F}$ is the smallest constant such that

$$|< \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h >| \le \mathfrak{F}\|\boldsymbol{v}_h\|_{\mathbf{V}} \quad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h. \tag{16}$$

Remark 2.3. *Note that we cannot write $\|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}}$ as $\|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{\mathbf{V}}$ since, in general, $\boldsymbol{u}_\pi$ will not be an element of $\mathbf{V}_h$ (lacking the necessary continuity).* □

Proof. Existence and uniqueness of the solution of (15) follows immediately from (12) and (5). Next, setting $\boldsymbol{\delta}_h := \boldsymbol{u}_h - \boldsymbol{u}_I$ we have

$$\alpha_*\, \alpha \|\boldsymbol{\delta}_h\|^2_{\mathbf{V}} \le \alpha_*\, a(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \le a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) = a_h(\boldsymbol{u}_h, \boldsymbol{\delta}_h) - a_h(\boldsymbol{u}_I, \boldsymbol{\delta}_h) \text{ (use (15) and (10))} \tag{17}$$

$$=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K a_h^K(\boldsymbol{u}_I, \boldsymbol{\delta}_h) \text{ (use } \pm \boldsymbol{u}_\pi)$$

$$=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \Big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a_h^K(\boldsymbol{u}_\pi, \boldsymbol{\delta}_h) \Big) \text{ (use (11))}$$

$$=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \Big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K(\boldsymbol{u}_\pi, \boldsymbol{\delta}_h) \Big) \text{ (use } \pm u \text{ and (9))}$$

$$=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \Big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{\delta}_h) \Big) - a(\boldsymbol{u}, \boldsymbol{\delta}_h) \text{ (use (6))}$$

$$=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \Big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{\delta}_h) \Big) - < \boldsymbol{f}, \boldsymbol{\delta}_h >$$

$$= < \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{\delta}_h > - \sum_K \Big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K(\boldsymbol{u}_\pi - u, \boldsymbol{\delta}_h) \Big).$$

Now we use (16), (13), and the continuity of each a^K in (17) in order to obtain

$$\|\boldsymbol{\delta}_h\|_{\mathbf{V}}^2 \le C \Big(\mathfrak{F} + \|\boldsymbol{u}_I - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} \Big) \|\boldsymbol{\delta}_h\|_{\mathbf{V}} \tag{18}$$

for some constant C depending only on Ω, λ, μ, α_*, α^*. Then the result follows easily by the triangle inequality. □

2.4. **Construction of $\mathbf{V}_h$.** We can now tackle the second part of our construction, showing how, given the sequence of decompositions $\{\mathcal{T}_h\}_h$ and the integer k we can construct (and use!) the corresponding spaces $\mathbf{V}_h$ and the bilinear forms a_h such that the assumptions (10)-(12) of our Theorem are satisfied.

Definition 2.1. *For every decomposition $\mathcal{T}_h$ and for every $k \ge 1$ we define $\mathbf{V}_h$ as the space of (vector valued) continuous functions $\boldsymbol{v}_h$ that vanish on the boundary of Ω, that are polynomials of degree $\le k$ on each edge e of $\mathcal{T}_h$, and such that in each element K we have $(A_{\lambda,\mu}\boldsymbol{v}_h)_{|K} \in (\mathbb{P}_{k-2})^2$ when $k \ge 2$, and $(A_{\lambda,\mu}\boldsymbol{v}_h)_{|K} = 0$ when $k = 1$.*

It is not difficult to see that for an element $K \in \mathcal{T}_h$ having n edges, the set of all (vector valued) continuous functions on ∂K that are polynomials of degree $\le k$ on each edge of ∂K is a linear space of dimension $2n + 2n(k-1)$. Indeed, a continuous *scalar function* which is a polynomial of degree $\le k$ on each edge is uniquely determined by its values at the vertices (n conditions) plus, for $k > 1$, by the moments up to the order $k-2$ ($k-1$ conditions) on each edge (hence $n + n(k-1)$ conditions in total). And for a *vector valued* function we have 2 scalar functions. Therefore, denoting by $\mathbf{V}_h^K$ the restriction of $\mathbf{V}_h$ to K we can check easily that, say, for an internal element K with n vertices the dimension of $\mathbf{V}_h^K$ is given by

$$\mathrm{N}^K \equiv dim \mathbf{V}_h^K = 2n + 2n(k-1) + k(k-1), \tag{19}$$

where the last term corresponds to the dimension of vector valued polynomials of degree $\le k-2$ in two dimensions (and hence to the number of *internal* degrees of freedom). Similarly, we can easily see that the dimension of the whole space $\mathbf{V}_h$ is given by

$$\mathrm{N}^{tot} \equiv dim \mathbf{V}_h = 2N_V + 2N_E(k-1) + N_P k(k-1), \tag{20}$$

where N_V, N_E and N_P are, respectively, the total number of *internal Vertices*, *internal Edges*, and *elements (Polygons)* in $\mathcal{T}_h$.

In $\mathbf{V}_h$ we choose the following *degrees of freedom* :

- $\mathcal{V}$ - The values of $\boldsymbol{v}_h$ at the *internal vertices*.
- $\mathcal{E}$ - For $k > 1$, the moments $\displaystyle\int_e \boldsymbol{q}(t)\boldsymbol{v}_h(t)\mathrm{d}t$ for $\boldsymbol{q} \in (\mathbb{P}_{k-2}(e))^2$ on each *internal edge* e.
- $\mathcal{P}$ - For $k > 1$, the moments $\displaystyle\int_K \boldsymbol{q}(\mathbf{x})\boldsymbol{v}_h(\mathbf{x})\mathrm{d}\mathbf{x}$ for $\boldsymbol{q} \in (\mathbb{P}_{k-2}(K))^2$ in each *element* K.

It is not difficult to check that the dimension N^{tot} of $\mathbf{V}_h$, computed in (20), equals the total number of degrees of freedom $\mathcal{V}$ plus $\mathcal{E}$ plus $\mathcal{P}$. For future use, it will also be convenient to define the **interpolation operator** χ as the operator that to each smooth enough vector valued function $\boldsymbol{\varphi}$ associates the N^{tot} values $\chi_1(\boldsymbol{\varphi}), .., \chi_{N^{tot}}(\boldsymbol{\varphi})$ of its degrees of freedom, and, for each element K, its restriction χ^K to the degrees of freedom related to $\overline{K}$.

Remark 2.4. *We notice that, in each element K, the degrees of freedom $\mathcal{V}$ plus $\mathcal{E}$ uniquely determine a polynomial of degree $\leq k$ on each edge of K, that is, $\mathcal{V}$ plus $\mathcal{E}$ are equivalent to prescribe $\boldsymbol{v}_h$ on ∂K. On the other hand, the degrees of freedom $\mathcal{P}$ are equivalent to prescribe $\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h$ in K (where $\mathbf{\Pi}^K_{k-2}$ is the projection operator, in the $(L^2(K))^2$ norm, onto the space $(\mathbb{P}_{k-2})^2$).*

Proposition 2.1. *The degrees of freedom given by $\mathcal{V}$ plus $\mathcal{E}$ plus $\mathcal{P}$ are unisolvent in $\mathbf{V}_h$.*

Proof. According to Remark 2.4, to prove the proposition it is enough to see that, for each $K \in \mathcal{T}_h$, a vector valued function $\boldsymbol{v}_h$ such that

$$\boldsymbol{v}_h = 0 \qquad \text{on } \partial K \quad \forall K \in \mathcal{T}_h \tag{21}$$

and

$$\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h = 0 \qquad \text{in } K \quad \forall K \in \mathcal{T}_h \tag{22}$$

is actually identically zero in K. In order to prove this, we will prove that $A_{\lambda,\mu}\boldsymbol{v}_h = 0$ in K (that joined with (21) gives $\boldsymbol{v}_h \equiv 0$). In order to see this, we first solve, for every $\boldsymbol{q} \in (\mathbb{P}_{k-2}(K))^2$, the following auxiliary problem:

Find $\boldsymbol{w} \in (H^1_0(K))^2$ such that

$$a^K(\boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{q}, \boldsymbol{v})_{0,K} \qquad \forall\, \boldsymbol{v} \in (H^1_0(K))^2, \tag{23}$$

that, in agreement with Remark 2.1, could also be written

$$A^K_{\lambda,\mu}\boldsymbol{w} = \boldsymbol{q} \ \text{ in } K \qquad \text{and} \qquad \boldsymbol{w} = 0 \ \text{ on } \partial K. \tag{24}$$

Next, we consider the map R, from $(\mathbb{P}_{k-2}(K))^2$ into itself, defined by

$$R\boldsymbol{q} := \mathbf{\Pi}^K_{k-2}(A^K_{\lambda,\mu})^{-1}(\boldsymbol{q}) \equiv \mathbf{\Pi}^K_{k-2}\boldsymbol{w}. \tag{25}$$

We claim that R, with this definition, is an isomorphism. Indeed, from (25) and the definition of $\mathbf{\Pi}^K_{k-2}$ we have, for every $\boldsymbol{q} \in (\mathbb{P}_{k-2})^2$:

$$(R(\boldsymbol{q}), \boldsymbol{q})_{0,K} = (\mathbf{\Pi}^K_{k-2}(A^K_{\lambda,\mu})^{-1}(\boldsymbol{q}), \boldsymbol{q})_{0,K} = (\mathbf{\Pi}^K_{k-2}\boldsymbol{w}, \boldsymbol{q})_{0,K} = (\boldsymbol{w}, \boldsymbol{q})_{0,K} = a^K(\boldsymbol{w}, \boldsymbol{w}). \tag{26}$$

Since $\boldsymbol{w}$ is in $(H^1_0(K))^2$ we have then that

$$\{R(\boldsymbol{q}) = 0\} \Leftrightarrow \{a^K(\boldsymbol{w}, \boldsymbol{w}) = 0\} \Leftrightarrow \{\boldsymbol{w} = 0\} \Leftrightarrow \{\boldsymbol{q} = 0\}. \tag{27}$$

We notice that, if $\boldsymbol{v}_h = 0$ on ∂K, then

$$\boldsymbol{\Pi}^K_{k-2}\boldsymbol{v}_h = R(A^K_{\lambda,\mu}\boldsymbol{v}_h).$$

Hence, $\boldsymbol{\Pi}^K_{k-2}\boldsymbol{v}_h = 0 \Longrightarrow R(A^K_{\lambda,\mu}\boldsymbol{v}_h) = 0 \Longrightarrow A^K_{\lambda,\mu}\boldsymbol{v}_h = 0$, and the proof is concluded. □

Remark 2.5. *It follows easily from the above construction that for every smooth enough* $\boldsymbol{w} \in \mathbf{W}$ *there exists a unique element* $\boldsymbol{w}_I$ *of* V_h *such that*

$$\chi(\boldsymbol{w} - \boldsymbol{w}_I) = 0. \tag{28}$$

Moreover, by the usual Bramble-Hilbert/Deny-Lions technique (see e.g. [17]*) and using a scaling argument to get around the variability in the geometry (see e.g.* [10]*) it is not difficult to see that*

$$\|\boldsymbol{w} - \boldsymbol{w}_I\|_{r,\Omega} \le C\, h^{s-r}\, |w|_{s,\Omega} \qquad r = 0, 1,\ r \le s \le k+1 \tag{29}$$

(with a constant C *independent of* h*) as in the usual Finite Element framework.*

We finally note that the operator $A_{\lambda,\mu}$ appearing in Definition 2.1 is the most natural choice, but it could be replaced by any second order elliptic operator such as, for instance, a component-wise laplacian.

2.5. Construction of a_h.

We are left to show how to construct a (computable!) a_h that satisfies (11) and (12).

First of all we observe that, for every $K \in \mathcal{T}_h$ and for every $\boldsymbol{v}_h \in \mathbf{V}^K_h$, knowing the degrees of freedom that identify $\boldsymbol{v}_h$ we can compute

- The value of $\boldsymbol{v}_h$ on ∂K (that can be computed explicitly on every edge e)
- The value of $\boldsymbol{\Pi}^K_{k-2}\boldsymbol{v}_h$ (that comes immediately out of the degrees of freedom).

Then we observe that, on any $K \in \mathcal{T}_h$, if $\boldsymbol{p} \in (\mathbb{P}_k)^2$ and $\boldsymbol{v}_h$ is in $\mathbf{V}^K_h$, then

$$a^K_\mu(\boldsymbol{p}, \boldsymbol{v}_h) = \int_K \boldsymbol{\varepsilon}(\boldsymbol{p}) : \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\,\mathrm{d}\mathbf{x} = -\int_K \mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{p})) \cdot \boldsymbol{v}_h\,\mathrm{d}\mathbf{x} + \int_{\partial K} (\boldsymbol{\varepsilon}(\boldsymbol{p}) \cdot \mathbf{n}) \cdot \boldsymbol{v}_h\,\mathrm{d}s. \tag{30}$$

We note then that $\mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{p}))$ belongs to $(\mathbb{P}_{k-2})^2$ so that

$$\int_K \mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{p})) \cdot \boldsymbol{v}_h\,\mathrm{d}\mathbf{x} = \int_K \mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{p})) \cdot \boldsymbol{\Pi}^K_{k-2}\boldsymbol{v}_h\,\mathrm{d}\mathbf{x} \tag{31}$$

is computable using only the values of $\boldsymbol{\Pi}^K_{k-2}\boldsymbol{v}_h$. On the other hand, the boundary integral in (30) is also easily computable knowing $\boldsymbol{v}_h$ on ∂K. Similarly,

$$a^K_\lambda(\boldsymbol{p}, \boldsymbol{v}_h) = \int_K \mathrm{div}(\boldsymbol{p})\mathrm{div}(\boldsymbol{v}_h)\mathrm{d}\mathbf{x} = -\int_K \nabla(\mathrm{div}(\boldsymbol{p})) \cdot \boldsymbol{v}_h\mathrm{d}\mathbf{x} + \int_{\partial K} (\mathrm{div}(\boldsymbol{p})\, \boldsymbol{v}_h \cdot \mathbf{n}\,\mathrm{d}s \tag{32}$$

is computable knowing only $\boldsymbol{v}_h$ on ∂K and $\boldsymbol{\Pi}^K_{k-2}\boldsymbol{v}_h$. Hence we can *define*

$$a^K_h(\boldsymbol{p}, \boldsymbol{v}_h) := a^K(\boldsymbol{p}, \boldsymbol{v}_h) \text{ and } a^K_h(\boldsymbol{v}_h, \boldsymbol{p}) := a^K(\boldsymbol{v}_h, \boldsymbol{p}) \tag{33}$$

that come out to be *computable* whenever $\boldsymbol{p}$ is an element of $(\mathbb{P}_k)^2$ and $\boldsymbol{v}_h$ is any element of $\mathbf{V}^K_h$. Note that by this choice we took care already of (11), and we have only (12) to take care of. At this point however we are also able to compute a new basis for $\mathbf{V}^K_h$, by taking first all the elements of $(\mathbb{P}_k)^2$ (whose dimension is $(k+1)(k+2)$), and then **completing it** with

$$N^K - (k+1)(k+2) \equiv 2n + 2n(k-1) + k(k-1) - (k+1)(k+2) \equiv 2nk - 4k - 2$$

independent elements $\widetilde{\boldsymbol{v}}_h$ such that

$$a_h^K(\boldsymbol{p}, \widetilde{\boldsymbol{v}}_h) = a_h^K(\widetilde{\boldsymbol{v}}_h, \boldsymbol{p}) \; (= a^K(\boldsymbol{p}, \widetilde{\boldsymbol{v}}_h) = a^K(\widetilde{\boldsymbol{v}}_h, \boldsymbol{p})\,) = 0 \quad \forall \boldsymbol{p} \in (\mathbb{P}_k)^2. \tag{34}$$

Denoting by $\widetilde{\mathbf{V}}_h^K$ the set of elements of $\mathbf{V}_h^K$ that satisfy (34), we have that, in the new basis, the local stiffness matrices corresponding to a^K and to a_h^K (respectively) are both 2×2 block diagonal, one block being made of

$$a^K(\boldsymbol{p}, \boldsymbol{q}) \equiv a_h^K(\boldsymbol{p}, \boldsymbol{q}) \qquad \text{for } \boldsymbol{p} \text{ and } \boldsymbol{q} \text{ both in } (\mathbb{P}_k)^2 \tag{35}$$

and the other block concerning, respectively,

$$a^K(\widetilde{\boldsymbol{v}}_h, \widetilde{\boldsymbol{w}}_h) \text{ and } a_h^K(\widetilde{\boldsymbol{v}}_h, \widetilde{\boldsymbol{w}}_h) \qquad \text{for } \widetilde{\boldsymbol{v}}_h \text{ and } \widetilde{\boldsymbol{w}}_h \text{ both in } \widetilde{\mathbf{V}}_h^K. \tag{36}$$

Note however that the choice of this second part for a_h cannot jeopardize the property (11), that has been already taken care of in (35) and (34). On the other hand, it is easy to check that the bilinear form corresponding to the block (36), for the form a^K, has a maximum and a minimum *positive* eigenvalue that depend (continuously) on the geometry of K but not on its size (note that, in particular, all the rigid body motions are already considered in the first block). Hence, by simply taking, for the form a_h^K, the block corresponding to (36) as the identity matrix (or, if you prefer, the identity matrix multiplied by the trace of the first block) we will have that the last property, (12), is also satisfied.

2.6. **Construction of the loading term.** In order to build the loading term $< \boldsymbol{f}_h, \boldsymbol{v}_h >$ for $\boldsymbol{v}_h \in \mathbf{V}_h$, we define $\boldsymbol{f}_h$ on each element K as the $(L^2(K)^2)$ projection of the load $\boldsymbol{f}$ on the space of piecewise polynomials of degree $\overline{k}$, where $\overline{k} := \max\{k-2, 0\}$, that is:

$$\boldsymbol{f}_h = \boldsymbol{\Pi}_{\overline{k}}^K \boldsymbol{f}\ , \qquad \overline{k} = \max\{k-2, 0\}, \qquad \text{on each } K \in \mathcal{T}_h.$$

Then if $k \geq 2$, the associated loading term

$$< \boldsymbol{f}_h, \boldsymbol{v}_h >= \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{f}_h \cdot \boldsymbol{v}_h \mathrm{d}\mathbf{x} \equiv \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\Pi}_{k-2}^K \boldsymbol{f} \cdot \boldsymbol{v}_h \mathrm{d}\mathbf{x} = \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{f} \cdot \boldsymbol{\Pi}_{k-2}^K \boldsymbol{v}_h \mathrm{d}\mathbf{x}$$

can be exactly computed using the degrees of freedom for $\mathbf{V}_h$ that represent the internal moments, see Section 2.4. In such a case, standard L^2 orthogonality and approximation estimates on star-shaped domains yield

$$\begin{aligned} < \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h > &= \sum_{K\in\mathcal{T}_h} \int_K (\boldsymbol{\Pi}_{k-2}^K \boldsymbol{f} - \boldsymbol{f}) \cdot (\boldsymbol{v}_h - \boldsymbol{\Pi}_0 \boldsymbol{v}_h) \mathrm{d}\mathbf{x} \\ &\leq C \sum_{K\in\mathcal{T}_h} h_K^{k-1} |\boldsymbol{f}|_{k-1,K}\ h_K \|\boldsymbol{v}_h\|_{1,K} \leq C\, h^k \Big(\sum_{K\in\mathcal{T}_h} |\boldsymbol{f}|_{k-1,K}^2\Big)^{1/2} \|\boldsymbol{v}_h\|_V \end{aligned} \tag{37}$$

and thus, recalling (16),

$$\mathfrak{F} \leq C h^k \Big(\sum_{K\in\mathcal{T}_h} |\boldsymbol{f}|_{k-1,K}^2 \Big)^{1/2}. \tag{38}$$

If $k = 1$, an integration rule based on the vertex values of $\boldsymbol{v}_h$ needs to be used in order to compute $\int_K \boldsymbol{f}_h \cdot \boldsymbol{v}_h \mathrm{d}\mathbf{x} = \int_K \boldsymbol{\Pi}_0^K \boldsymbol{f} \cdot \boldsymbol{v}_h \mathrm{d}\mathbf{x}$. In this case the same procedure as in (37)-(38) gives again

$$\mathfrak{F} \leq Ch|\boldsymbol{f}|_{0,\Omega}. \tag{39}$$

Remark 2.6. *Both* (38) *and* (39) *become meaningless when* $\boldsymbol{f}$ *does not have enough regularity, since the right-hand side becomes* $+\infty$*. However, with a quite similar procedure one could easily get*

$$\mathfrak{F} \leq Ch^s \Big(\sum_{K \in \mathcal{T}_h} |\boldsymbol{f}|^2_{s-1,K} \Big)^{1/2} \qquad \text{for } 1 \leq s \leq k. \tag{40}$$

□

2.7. **Estimates in the L^2 norm.** In the present section we derive error estimates in the L^2 norm. We have the following result.

Lemma 2.1. *Assume that the domain* Ω *is convex and* $k \geq 2$*. Under the same assumptions and notation of Theorem 2.1 the following holds. For every approximation* $\boldsymbol{u}_I$ *of* $\boldsymbol{u}$ *in* $\mathbf{V}_h$ *and for every approximation* $\boldsymbol{u}_\pi$ *of* $\boldsymbol{u}$ *that is piecewise in* $(\mathbb{P}_k)^2$*, we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq Ch\Big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + h^{\widehat{k}} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega} \Big)$$

where $\widehat{k} = 0$ *if the polynomial degree* $k = 2$ *and* $\widehat{k} = 1$ *otherwise. The constant* C *depends only on* Ω, λ, μ, α^*.

Proof. We consider the usual auxiliary problem: Find $\boldsymbol{\psi} \in \mathbf{V}$ such that

$$a(\boldsymbol{\psi}, \boldsymbol{v}) = (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{V}.$$

Since Ω is a convex polygonal domain, the regularity of the problem guarantees that $||\boldsymbol{\psi}||_{2,\Omega} \leq C||\boldsymbol{u} - \boldsymbol{u}_h||_{0,\Omega}$ with $C = C(\Omega, \mu, \lambda)$.

Let $\boldsymbol{\psi}_I$ and $\boldsymbol{\psi}_\pi$ denote approximations of ψ, with $\boldsymbol{\psi}_I \in \mathbf{V}_h$ and $\boldsymbol{\psi}_\pi$ piecewise in $(\mathbb{P}_k)^2$. Standard approximation estimates combined with the above regularity result yield immediately

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_I\|_{0,\Omega} + h\|\boldsymbol{\psi} - \boldsymbol{\psi}_I\|_{\mathbf{V}} + h\|\boldsymbol{\psi} - \boldsymbol{\psi}_\pi\|_{h,\mathbf{V}} \leq Ch^2 \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}. \tag{41}$$

Simple manipulations, first adding and subtracting $\boldsymbol{\psi}_I$, then using (41) give

$$\begin{aligned} ||\boldsymbol{u} - \boldsymbol{u}_h||^2_{0,\Omega} &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\psi} - \boldsymbol{\psi}_I) + a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\psi}_I) \\ &\leq Ch\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{V}} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} + a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\psi}_I). \end{aligned} \tag{42}$$

For the second term above, we note that $\boldsymbol{\psi}_I \in \mathbf{V}$ and use (6), add and subtract $< \boldsymbol{f}_h, \boldsymbol{\psi}_I >$, and finally apply (15). We get

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\psi}_I) =< \boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{\psi}_I > + \Big(a_h(\boldsymbol{u}_h, \boldsymbol{\psi}_I) - a(\boldsymbol{u}_h, \boldsymbol{\psi}_I) \Big) =: T_1 + T_2. \tag{43}$$

In order to bound the term T_1, we first add and subtract $\boldsymbol{\psi}$, then use the orthogonality property of $\boldsymbol{f} - \boldsymbol{f}_h$. Following the same notation as in Section 2.6 we obtain

$$T_1 = \sum_{K \in \mathcal{T}_h} \Big(\int_K (\boldsymbol{f} - \boldsymbol{f}_h)(\boldsymbol{\psi}_I - \boldsymbol{\psi}) \mathrm{d}\mathbf{x} + \int_K (\boldsymbol{f} - \boldsymbol{f}_h)(\boldsymbol{\psi} - \boldsymbol{\Pi}^K_{\widehat{k}} \boldsymbol{\psi}) \mathrm{d}\mathbf{x} \Big),$$

where we recall that $\overline{k} = \max\{k-2, 0\}$.

Cauchy-Schwarz inequalities, bound (41), standard approximation estimates for the operator $\mathbf{\Pi}^K_{\overline{k}}$ and the regularity result for $\boldsymbol{\psi}$ yield

$$
\begin{aligned}
T_1 &\leq \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}\Big(\|\boldsymbol{\psi} - \boldsymbol{\psi}_I\|_{0,\Omega} + \|\boldsymbol{\psi} - \mathbf{\Pi}^K_{\overline{k}}\boldsymbol{\psi}\|_{0,\Omega}\Big) \\
&\leq C(h^2 + h^{\min(2,\overline{k}+1)})\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}||\boldsymbol{u} - \boldsymbol{u}_h||_{0,\Omega} \\
&\leq C(h^{\widehat{k}+1})\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}||\boldsymbol{u} - \boldsymbol{u}_h||_{0,\Omega}.
\end{aligned}
\tag{44}
$$

For the second term in (43) we use the consistency property **H2** twice, thus obtaining

$$
\begin{aligned}
T_2 &= \sum_{K\in\mathcal{T}_h} \Big(a_h^K(\boldsymbol{u}_h, \boldsymbol{\psi}_I) - a^K(\boldsymbol{u}_h, \boldsymbol{\psi}_I)\Big) \\
&= \sum_{K\in\mathcal{T}_h} \Big(a_h^K(\boldsymbol{u}_h, \boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi) - a^K(\boldsymbol{u}_h, \boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi)\Big) \\
&= \sum_{K\in\mathcal{T}_h} \Big(a_h^K(\boldsymbol{u}_h - \boldsymbol{u}_\pi, \boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi) - a^K(\boldsymbol{u}_h - \boldsymbol{u}_\pi, \boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi)\Big).
\end{aligned}
$$

From the above bound, the continuity of the bilinear form a and (13) yield

$$
T_2 \leq C\Big(\sum_{K\in\mathcal{T}_h} |\boldsymbol{u}_h - \boldsymbol{u}_\pi|^2_{1,K}\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} |\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi|^2_{1,K}\Big)^{1/2}.
$$

By a triangle inequality it now follows

$$
\Big(\sum_{K\in\mathcal{T}_h} |\boldsymbol{u}_h - \boldsymbol{u}_\pi|^2_{1,K}\Big)^{1/2} \leq C\Big(\sum_{K\in\mathcal{T}_h} |\boldsymbol{u} - \boldsymbol{u}_h|^2_{1,K} + |\boldsymbol{u} - \boldsymbol{u}_\pi|^2_{1,K}\Big)^{1/2},
$$

and, by the same argument and using (41),

$$
\Big(\sum_{K\in\mathcal{T}_h} |\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi|^2_{1,K}\Big)^{1/2} \leq Ch\,|\boldsymbol{\psi}|_{2,\Omega} \leq C\,h\,||\boldsymbol{u} - \boldsymbol{u}_h||_{0,\Omega}.
$$

Combining the above bounds we obtain for T_2

$$
T_2 \leq Ch\,\Big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}}\Big)\,||\boldsymbol{u} - \boldsymbol{u}_h||_{0,\Omega}.
\tag{45}
$$

The final bound follows applying bounds (42), (43), (44) and (45).
□

By application of standard approximation estimates and Theorem 2.1, if $k \geq 3$ from Lemma 2.1 it immediately follows

$$
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq Ch^{k+1}|\boldsymbol{u}|_{k+1,\Omega},
\tag{46}
$$

provided the solution $\boldsymbol{u}$ is sufficiently regular.

In the case $k = 1$ an analogous result can be derived. The steps are essentially identical to the ones in the proof of Lemma 2.1, the only difference being in the treatment of the loading term T_1. Assuming that the integration rule used for the evaluation of the loading is at least of first order, see Section 2.6, one finally obtains

$$\begin{aligned}\|\boldsymbol{u}_h-\boldsymbol{u}\|_{0,\Omega} &\le Ch\Big(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{V}}+\|\boldsymbol{u}-\boldsymbol{u}_\pi\|_{h,\mathbf{V}}+\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}+h\|\boldsymbol{f}\|_{0,\Omega}\Big)\\ &\le Ch^2\Big(|\boldsymbol{u}|^2_{2,\Omega}+\sum_{K\in\mathcal{T}_h}|\boldsymbol{f}|^2_{1,K}\Big)^{1/2}\end{aligned}$$

where the last bound holds if $\boldsymbol{u}$ is sufficiently regular.

In the case $k=2$, the proposed method does not guarantee a $O(h^{k+1})$ convergence rate in the L^2 norm. Such result is sharp and is not related to some theoretical limitation in the proof of Lemma 2.1. Indeed, the reason for this non optimal behavior is related to the load approximation, as is clear from the proof above, and can be cured in the following way.

It is immediate to check that, for any $\boldsymbol{v}_h\in\mathbf{V}_h$ and $K\in\mathcal{T}_h$, the (component-wise) averages

$$\overline{\boldsymbol{v}}_h=(\int_K\boldsymbol{v}_h\mathrm{d}\mathbf{x})/|K|\ ,\quad \overline{\boldsymbol{\nabla}\boldsymbol{v}_h}=(\int_K\boldsymbol{\nabla}\boldsymbol{v}_h\mathrm{d}\mathbf{x})/|K|$$

can be exactly computed using the degree of freedom of $\mathbf{V}_h$, where in the case of the gradient an integration by parts needs to be used.

Therefore, denoting by $\mathbf{x}_B$ the coordinates of the barycenter of K, we can define

$$\begin{aligned}\widetilde{\boldsymbol{v}}_h&=\overline{\boldsymbol{v}}_h+\overline{\boldsymbol{\nabla}\boldsymbol{v}_h}\cdot(\mathbf{x}-\mathbf{x}_B)\\ <\boldsymbol{f}_h,\boldsymbol{v}_h>&=\sum_{K\in\mathcal{T}_h}\int_K\boldsymbol{\Pi}_1^K\boldsymbol{f}\cdot\widetilde{\boldsymbol{v}}_h\mathrm{d}\mathbf{x}\end{aligned}\tag{47}$$

Then we have:

$$\begin{aligned}<\boldsymbol{f}_h-\boldsymbol{f},\boldsymbol{v}_h> &=\sum_{K\in\mathcal{T}_h}\int_K\Big(\boldsymbol{\Pi}_1^K\boldsymbol{f}\cdot\widetilde{\boldsymbol{v}}_h-\boldsymbol{f}\cdot\boldsymbol{v}_h\Big)\mathrm{d}\mathbf{x}\\ &=\sum_{K\in\mathcal{T}_h}\int_K\Big(\boldsymbol{\Pi}_1^K\boldsymbol{f}\cdot(\widetilde{\boldsymbol{v}}_h-\boldsymbol{v}_h)+(\boldsymbol{\Pi}_1^K\boldsymbol{f}-\boldsymbol{f})\cdot\boldsymbol{v}_h\Big)\mathrm{d}\mathbf{x}\\ &=\sum_{K\in\mathcal{T}_h}\int_K\Big((\boldsymbol{\Pi}_1^K\boldsymbol{f}-\boldsymbol{\Pi}_0^K\boldsymbol{f})\cdot(\widetilde{\boldsymbol{v}}_h-\boldsymbol{v}_h)+(\boldsymbol{\Pi}_1^K\boldsymbol{f}-\boldsymbol{f})\cdot(\boldsymbol{v}_h-\widetilde{\boldsymbol{v}}_h)\Big)\mathrm{d}\mathbf{x}\\ &\le C\sum_{K\in\mathcal{T}_h}\Big(\|\boldsymbol{\Pi}_1^K\boldsymbol{f}-\boldsymbol{\Pi}_0^K\boldsymbol{f}\|_{0,K}+\|\boldsymbol{\Pi}_1^K\boldsymbol{f}-\boldsymbol{f}\|_{0,K}\Big)\|\widetilde{\boldsymbol{v}}_h-\boldsymbol{v}_h\|_{0,K}\\ &\le C\,h^3(\sum_{K\in\mathcal{T}_h}|\boldsymbol{f}|^2_{1,K})^{1/2}|\boldsymbol{v}_h|_{2,K}.\end{aligned}\tag{48}$$

Clearly, $O(h^3)$ is more than we need for the error estimate in $\mathbf{V}$, and this is why we did not use it in Section 2.6. Moreover, although easily computable, the integral in (47) is not immediate as the one proposed in Section 2.6. Now, proceeding as in (48) we have:

$$\begin{aligned}T_1&=\sum_{K\in\mathcal{T}_h}\int_K\Big(\boldsymbol{\Pi}_1^K\boldsymbol{f}\cdot\widetilde{\boldsymbol{\psi}}_I-\boldsymbol{f}\cdot\boldsymbol{\psi}_I\Big)\mathrm{d}\mathbf{x}\\ &\le C\,h^3(\sum_{K\in\mathcal{T}_h}|\boldsymbol{f}|^2_{1,K})^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}.\end{aligned}\tag{49}$$

3. Nearly incompressible materials

As is well known, we say that the material is *nearly incompressible* when $\lambda >> \mu$. It is equally well known that, in such a case, it is more convenient (for the analysis and the discretization of the problem) to relax the incompressibility constraint introducing a projection operator or, equivalently, to shift to the so-called $(\boldsymbol{u}, p)$-formulation.

In the present section we shall follow the first option, and we shall derive error bounds which do not explode when $\lambda/\mu \to \infty$. Let hereafter Π_{k-1} denote the L^2 projection operator on the auxiliary space

$$Q_h := \{q \in L^2(\Omega)/\mathbb{R} \text{ such that } q_{|K} \in \mathbb{P}_{k-1}(K) \quad \forall\, K \in \mathcal{T}_h\}. \tag{50}$$

We define, for all $\boldsymbol{u}_h, \boldsymbol{v}_h \in \mathbf{V}_h$,

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \sum_{K\in\mathcal{T}_h} a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h), \\ a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) &= 2\mu a_{\mu,h}^K(\boldsymbol{u}_h, \boldsymbol{v}_h) + \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}_h, \Pi_{k-1}\mathrm{div}\boldsymbol{v}_h)_K \quad \forall K \in \mathcal{T}_h, \end{aligned} \tag{51}$$

where the bilinear forms $a_{\mu,h}^K(\boldsymbol{u}_h, \boldsymbol{v}_h)$ are constructed with the strategy that we followed to construct the bilinear forms a_h^K in Section 2.5 (think, for instance, to what you would get in the previous case if λ was equal to zero).

Following the obvious notation

$$a_\mu(\boldsymbol{u}, \boldsymbol{v}) = \sum_{K\in\mathcal{T}_h} a_\mu^K(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}, \tag{52}$$

the local bilinear forms $a_{\mu,h}^K(\boldsymbol{u}_h, \boldsymbol{v}_h)$ satisfy an analogous version of (11)-(12):

H2 - For all h, and for all K in $\mathcal{T}_h$

- $\forall \boldsymbol{p} \in (\mathbb{P}_k)^2$, $\forall \boldsymbol{v}_h \in \mathbf{V}_h$

$$a_{\mu,h}^K(\boldsymbol{p}, \boldsymbol{v}_h) = a_\mu^K(\boldsymbol{p}, \boldsymbol{v}_h). \tag{53}$$

- $\exists$ two positive constants α_* and α^*, independent of h, μ and of K, such that

$$\forall \boldsymbol{v}_h \in \mathbf{V}_h \qquad \alpha_*\, a_\mu^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \le a_{\mu,h}^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \le \alpha^*\, a_\mu^K(\boldsymbol{v}_h, \boldsymbol{v}_h). \tag{54}$$

We note now that for every $\boldsymbol{v}_h \in \mathbf{V}_h$ and for every $q \in Q_h$ the integral

$$\int_K \mathrm{div}\, \boldsymbol{v}_h\, q \mathrm{d}\mathbf{x} = \int_{\partial K} \boldsymbol{v}_h \cdot \mathbf{n}\, q \mathrm{d}s - \int_K \boldsymbol{v}_h \cdot \nabla q \mathrm{d}\mathbf{x} \tag{55}$$

is computable once we know $\boldsymbol{v}_h$ at the boundary and the projection of $\boldsymbol{v}_h$ on $(\mathbb{P}_{k-2})^2$ (that is $\boldsymbol{\Pi}_{k-2}\boldsymbol{v}_h$). Therefore the projection operator Π_{k-1} appearing in (51) is computable for any function in $\mathbf{V}_h$ by using the available degrees of freedom.

The discrete problem reads as (15), with the bilinear form given now by (51). In order to study the convergence properties of the proposed method we consider $k \geq 2$ and prove a discrete *inf-sup* condition:

$$\exists\, \beta^* > 0 \text{ such that } \forall\, h \inf_{\boldsymbol{q}\in Q_h} \sup_{\boldsymbol{v}\in\mathbf{V}_h} \frac{(\mathrm{div}\boldsymbol{v}, q)}{\|q\|_Q\, \|\boldsymbol{v}\|_{\mathbf{V}}} \geq \beta^* > 0. \tag{56}$$

We have the following theorem.

Theorem 3.1. *Let $k \geq 2$. In the above assumptions,* (56) *holds true.*

Proof. We recall the continuous inf-sup condition, see for instance [10],

$$\exists\, \beta > 0 \text{ such that } \inf_{q\in Q} \sup_{\boldsymbol{v}\in \mathbf{V}} \frac{(\mathrm{div}\boldsymbol{v}, q)}{\|q\|_Q\ \|\boldsymbol{v}\|_{\mathbf{V}}} \geq \beta > 0\ , \quad Q = L^2(\Omega). \tag{57}$$

For every $q^* \in Q_h \subset Q$, using (57) we have that there exists a $\boldsymbol{w} \in \mathbf{V}$ such that

$$2\frac{(\mathrm{div}\boldsymbol{w}, q^*)}{\|q^*\|_Q\ \|\boldsymbol{w}\|_{\mathbf{V}}} \geq \beta > 0. \tag{58}$$

Now, in order to use the so-called Fortin's trick, we want to construct a $\boldsymbol{w}_h \in \mathbf{V}_h$ such that

$$(\mathrm{div}(\boldsymbol{w} - \boldsymbol{w}_h), q) = 0 \quad \forall\, q \in Q_h \tag{59}$$

with

$$\|\boldsymbol{w}_h\|_{\mathbf{V}} \leq\ C_F\, \|\boldsymbol{w}\|_{\mathbf{V}} \tag{60}$$

for some constant C_F independent of h. For this, we proceed as in the classical Finite Element theory (see e.g. [10]). We first construct a $\overline{\boldsymbol{w}}_h \in \mathbf{V}_h$ such that

$$(\mathrm{div}(\boldsymbol{w} - \overline{\boldsymbol{w}}_h), \overline{q}) = 0 \quad \forall\, \overline{q} \text{ piecewise constant in } \Omega \tag{61}$$

with

$$\|\boldsymbol{w} - \overline{\boldsymbol{w}}_h\|_{r,K} \leq\ C\, h_K^{1-r} \|\boldsymbol{w}\|_{1,K}\ \forall\, K \in \mathcal{T}_h \qquad r = 0, 1 \tag{62}$$

with a constant C that does not depend on h. Note that for doing this, as is well known, we must use in an essential way the degree of freedom "average of the normal component of $\overline{\boldsymbol{w}}$ on each edge of K", that allows to enforce

$$\int_e (\boldsymbol{w} - \overline{\boldsymbol{w}}_h) \cdot \mathbf{n}\, \mathrm{d}s = 0\ \forall \text{ edge } e \text{ in } \mathcal{T}_h, \tag{63}$$

and this is the reason why we require $k \geq 2$. Once $\overline{\boldsymbol{w}}_h$ is constructed, always following the Finite Element track, we choose a "bubble" $\widetilde{\boldsymbol{w}}_h$ having all the degrees of freedom on each ∂K equal to zero, and relying only, in each K, on the $k(k-1)$ internal degrees of freedom, such that, on each K

$$(\mathrm{div}\, \widetilde{\boldsymbol{w}}_h, q)_{0,K} \equiv\ -(\widetilde{\boldsymbol{w}}_h, \nabla q)_{0,K} = (\mathrm{div}(\boldsymbol{w} - \overline{\boldsymbol{w}}_h), q)_{0,K} \quad \forall\, q \in (Q_h)_{|K}. \tag{64}$$

We note that (64) amounts to $k(k+1)/2 - 1$ conditions (as the dimension of the space of gradients of polynomials of degree $\leq k-1$). We note that this is $\leq k(k-1)$ (the number of internal degrees of freedom for the space $\mathbf{V}_h^K$) for $k \geq 2$. An additional scaling argument, together with (62), shows that we also have

$$\|\boldsymbol{w} - \widetilde{\boldsymbol{w}}_h\|_{r,K} \leq\ C\, h_K^{1-r} \|\boldsymbol{w}\|_{1,K}\ \forall\, K \in \mathcal{T}_h \qquad r = 0, 1. \tag{65}$$

Finally we set

$$\boldsymbol{w}_h := \overline{\boldsymbol{w}}_h + \widetilde{\boldsymbol{w}}_h$$

and we note that (64) implies (59), while (62) and (65) imply (60). Finally from (58), (59) and (60) we have, as usual,

$$2\frac{(\mathrm{div}\boldsymbol{w}_h, q^*)}{\|q^*\|_Q\ \|\boldsymbol{w}_h\|_{\mathbf{V}}} \geq \frac{2}{C_F}\frac{(\mathrm{div}\boldsymbol{w}_h, q^*)}{\|q^*\|_Q\ \|\boldsymbol{w}\|_{\mathbf{V}}} = \frac{2}{C_F}\frac{(\mathrm{div}\boldsymbol{w}, q^*)}{\|q^*\|_Q\ \|\boldsymbol{w}\|_{\mathbf{V}}} \geq \beta^* \tag{66}$$

that gives (56) with $\beta^* = 2\beta/C_F$.

□

As a consequence of the inf-sup condition (56) we have, with classical arguments (see for instance Proposition 2.5 in [10]) the following property: *For all smooth enough vector valued function* $\boldsymbol{u}$ *it exists* $\boldsymbol{u}_I \in V_h$ *such that*

$$\Pi_{k-1}\mathrm{div}\,\boldsymbol{u}_I = \Pi_{k-1}\mathrm{div}\,\boldsymbol{u} \tag{67}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_I\|_{\mathbf{V}} \le C \inf_{\boldsymbol{v}_h \in \mathbf{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathbf{V}}. \tag{68}$$

We are now able to prove the following result.

Theorem 3.2. *Under the assumptions* **H0**, **H1**, *and* **H2** *the discrete problem* (15) *with bilinear form* (51) *has a unique solution* $\boldsymbol{u}_h$. *Moreover, let* $\boldsymbol{u}_I$ *be the interpolant of* $\boldsymbol{u}$ *defined in* (67)-(68). *Then for every approximation* $\boldsymbol{u}_\pi$ *of* $\boldsymbol{u}$ *that is piecewise in* $(\mathbb{P}_k)^2$, *we have*

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathbf{V}} \le C\Big(\|\boldsymbol{u} - \boldsymbol{u}_I\|_{\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|p - \Pi_{k-1}p\|_Q + \mathfrak{F}\Big) \tag{69}$$

where C *is a constant depending only on* Ω, γ, α_*, α^*, $\mathfrak{F}$ *is still defined by* (16), *and*

$$p := \lambda\,\mathrm{div}\,\boldsymbol{u} \tag{70}$$

is the "pressure".

Proof. The existence of a unique $\boldsymbol{u}_h$ follows immediately from the definite positive property of the discrete bilinear form, see (54), on the space with boundary conditions $\mathbf{V}_h$. Setting $\boldsymbol{\delta}_h := \boldsymbol{u}_h - \boldsymbol{u}_I$ we have

(71)

$$\begin{aligned}
&\alpha_*\,2\mu\|\boldsymbol{\delta}_h\|^2_{\mathbf{V}} \le \alpha_* 2\mu\, a_\mu(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) \le a_h(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) = a_h(\boldsymbol{u}_h,\boldsymbol{\delta}_h) - a_h(\boldsymbol{u}_I,\boldsymbol{\delta}_h) \text{ (use (15) and (51))}\\
&=< \boldsymbol{f}_h,\boldsymbol{\delta}_h > - \sum_K \Big(2\mu a^K_{\mu,h}(\boldsymbol{u}_I,\boldsymbol{\delta}_h) + \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}_I, \Pi_{k-1}\mathrm{div}\boldsymbol{\delta}_h)_K\Big) \text{ (use } \pm\boldsymbol{u}_\pi \text{ and (67))}\\
&=< \boldsymbol{f}_h,\boldsymbol{\delta}_h > -2\mu\sum_K \Big(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi,\boldsymbol{\delta}_h) + a^K_{\mu,h}(\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)\Big)\\
&\quad - \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{\delta}_h)_\Omega \text{ (use (53))}\\
&=< \boldsymbol{f}_h,\boldsymbol{\delta}_h > -2\mu\sum_K \Big(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi,\boldsymbol{\delta}_h) + a^K_{\mu}(\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)\Big)\\
&\quad - \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{\delta}_h)_\Omega \text{ (use } \pm u \text{ and (52))}\\
&=< \boldsymbol{f}_h,\boldsymbol{\delta}_h > -2\mu\sum_K \Big(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi,\boldsymbol{\delta}_h) + a^K_{\mu}(\boldsymbol{u}_\pi - \boldsymbol{u},\boldsymbol{\delta}_h)\Big)\\
&\quad - 2\mu a_\mu(\boldsymbol{u},\boldsymbol{\delta}_h) - \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{\delta}_h)_\Omega \text{ (use (2), (6) and (70))}\\
&=< \boldsymbol{f}_h,\boldsymbol{\delta}_h > -2\mu\sum_K \Big(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi,\boldsymbol{\delta}_h) + a^K_{\mu}(\boldsymbol{u}_\pi - \boldsymbol{u},\boldsymbol{\delta}_h)\Big) - < \boldsymbol{f},\boldsymbol{\delta}_h > - (\Pi_{k-1}p - p, \mathrm{div}\boldsymbol{\delta}_h)_{0,\Omega}\\
&=< \boldsymbol{f}_h - \boldsymbol{f},\boldsymbol{\delta}_h > -2\mu\sum_K \Big(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi,\boldsymbol{\delta}_h) + a^K_{\mu}(\boldsymbol{u}_\pi - \boldsymbol{u},\boldsymbol{\delta}_h)\Big) - (\Pi_{k-1}p - p, \mathrm{div}\boldsymbol{\delta}_h)_{0,\Omega}.
\end{aligned}$$

Recalling (16), (54) and the continuity of each a^K_μ in (71) easily yields

$$\|\boldsymbol{\delta}_h\|^2_{\mathbf{V}} \le C\Big(\mathfrak{F} + \|\boldsymbol{u}_I - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|\Pi_{k-1}p - p\|_{0,\Omega}\Big)\,\|\boldsymbol{\delta}_h\|_{\mathbf{V}} \tag{72}$$

for some constant C depending only on Ω, α_*, α^*. Then the result follows easily by the triangle inequality.
□

Combining the above error bound with (68), (29), (38) and standard approximation results on star shaped domains gives the following uniform convergence estimate.

Remark 3.1. *Under the same assumptions of Theorem 3.2 it holds*

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathbf{V}} \leq C h^{s-1}\Big(|\boldsymbol{u}|_{s,\Omega} + |p|_{s-1,\Omega}\Big)\ , \quad 1 \leq s \leq k+1,$$

with C independent of the material constants and on the mesh $\mathcal{T}_h$.

References

[1] J.E. Aarnes, S. Krogstad, and K.-A. Lie. Multiscale mixed/mimetic methods on corner-point grids. *Comput. Geosci.*, 12:297–315, 2008. Special Issue on Multiscale Methods.

[2] L. Beirão da Veiga. A mimetic finite difference method for linear elasticity. *M2AN: Math. Model. Numer. Anal.*, 44(2):231–250, 2010.

[3] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. Arbitrary-Order Nodal Mimetic Discretizations of Elliptic Problems on Polygonal Meshes. *SIAM J. Numer. Anal.*, 49(5):1737–1760, 2011.

[4] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. Convergence analysis of the high-order mimetic finite difference method. *Numer. Math.*, 113(3):325–356, 2009.

[5] L. Beirão da Veiga and G. Manzini. A higher-order formulation of the mimetic finite difference method. *SIAM, J. Sci. Comput.*, 31(1):732–760, 2008.

[6] L. Beirao da Veiga and G. Manzini. An a-posteriori error estimator for the mimetic finite difference approximation of elliptic problems. *Int. J. Numer. Meth. Engng.*, 76(11):1696–1723, 2008.

[7] P. Bochev and J. M. Hyman. Principle of mimetic discretizations of differential operators. In D. Arnold, P. Bochev, R. Lehoucq, R. Nicolaides, and M. Shashkov, editors, *Compatible discretizations. Proceedings of IMA hot topics workshop on compatible discretizations*, IMA Volume 142. Springer-Verlag, 2006.

[8] F. Brezzi and A. Buffa. Innovative mimetic discretizations for electromagnetic problems. *J. Comput. Appl. Math.*, 234(6):1980–1987, 2010.

[9] F. Brezzi, A. Buffa, and K. Lipnikov. Mimetic finite differences for elliptic problems. *M2AN: Math. Model. Numer. Anal.*, 43:277–295, 2009.

[10] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, 1991.

[11] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Num. Anal.*, 43:1872–1896, 2005.

[12] F. Brezzi, K. Lipnikov, M. Shashkov, and V. Simoncini. A new discretization methodology for diffusion problems on generalized polyhedral meshes. *Comp. Meth. Appl. Mech. Engrg.*, 196:3682–3692, 2007.

[13] F. Brezzi, K. Lipnikov, and V. Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *Math. Models Methods Appl. Sci.*, 15:1533–1553, 2005.

[14] J. C. Campbell and M. J. Shashkov. A tensor artificial viscosity using a mimetic finite difference algorithm. *J. Comput. Phys.*, 172(2):739–765, 2001.

[15] A. Cangiani and G. Manzini. Flux reconstruction and pressure post-processing in mimetic finite difference methods. *Comput. Methods Appl. Mech. Engrg.*, 197/9-12:933–945, 2008.

[16] A. Cangiani, G. Manzini, and A. Russo. A Finite Element re-formulation of the nodal Mimetic Finite Difference Method for elliptic problems on polygonal and polyhedral meshes. Work in progress.

[17] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, 1978.

[18] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. *Math. Models Methods Appl. Sci. (M3AS)*, 20(2):265–295, 2010.

[19] V. Gyrya and K. Lipnikov. High-order mimetic finite difference method for diffusion problems on polygonal meshes. *J. Comput. Phys.*, 227:8841–8854, 2008.

[20] J. Hyman and M. Shashkov. Mimetic discretisations of Maxwell's equations and the equations of magnetic diffusion. *Progress in Electromagnetic Research*, 32:89–121, 2001.

[21] J. M. Hyman and M. Shashkov. The orthogonal decomposition theorems for mimetic finite difference methods. *SIAM J. Numer. Anal.*, 36(3):788–818, 1999.

[22] Yu. Kuznetsov and S. Repin. New mixed finite element method on polygonal and polyhedral meshes. *Russ. J. Numer. Anal. Math. Modelling*, 18(3):261–278, 2003.

[23] J.B. Perot and V. Subramanian. Higher-order mimetic methods for unstructured meshes. *J. Comput. Phys.*, 219(1):68–85, 2006.

[24] S. Rjasanow and S. Weißer. Higher order BEM-based FEM on polygonal meshes. Preprint Nr. 297, Fachrichtung 6.1 - Mathematik, Saarbrücken 2011.

[25] M. Shashkov and S. Steinberg. Solving diffusion equations with rough coefficients in rough grids. *J. Comput. Phys.*, 129(2):383–405, 1996.

[26] A. Tabarraei and N. Sukumar. Application of polygonal finite elements in linear elasticity. *Int. J. Comput. Methods*, 3(4):503–520, 2006.

[27] A. Tabarraei and N. Sukumar. Extended finite element method on polygonal and quadtree meshes. *Comput. Methods Appl. Mech. Engrg.*, 197(5):425–438, 2007.

[28] K.A. Trapp. Inner products in covolume and mimetic methods. *M2AN: Math. Model. Numer. Anal.*, 42:941–959, 2008.