

Towards a Multiscale Visions Active Particles Methods

N. BELLOMO⁽¹⁾ and F. BREZZI⁽²⁾

⁽¹⁾*Politecnico di Torino and Collegio Carlo Alberto, Torino, Italy*
nicola.bellomo@polito.it

⁽²⁾*IMATI - CNR,*
Via A. Ferrata 5/A, 27100 Pavia, Italy
brezzi@imati.cnr.it

Received (Day Month Year)
 Revised (Day Month Year)
 Communicated by (xxxxxxxxxx)

The aim of this paper is firstly to provide a presentation of the papers published in a special issues devoted to modeling, qualitative analysis, control and simulations of large systems of living entities, viewed as active particles, related to real life applications, and subsequently to present some perspective ideas on possible future research programs for the development of mathematical tools to model living systems.

Keywords: Complexity, active particles, nonlinear interactions, swarming, control, multiscale vision.

1. Introduction

This paper presents the contents and the aims of a special issue devoted to modeling, qualitative analysis, control, and simulations of the collective dynamics of large systems of interacting living entities. This topic has been the object of continuous attention from the journal *Mathematical Models and Methods in Applied Sciences*, where a sequence of special issues has been recently published on closely related topics, as for instance self-propelled particles and social systems^{8,9,10}.

We present the contents of this note in the next three sections with focus on the following topics:

- (1) Reasonings on the complexity features of collective dynamics;
- (2) Presentation of the articles, referring them to the specific topics and applications treated in the issue;
- (3) Research perspectives promoted by the articles in the issue.

In more details:

2 *Bellomo and Brezzi*

Section 2 presents some preliminary speculations on some complexity features of collective dynamics of living systems that need to be captured in the modeling approach.

Section 3 presents the contents of the issue, joining each paper to a well defined topic and research objective.

Section 4 is devoted to research perspectives focused on a multiscale vision which pervades all articles proposed here. Such a vision, in our opinion, is an unavoidable passage towards the design of a mathematics for living systems.

2. Complexity features of collective dynamics

A common feature of the overall collective dynamics of large systems of interacting social entities is that the collective dynamics is generated by nonlinearly additive interactions, where social behaviors can have an important influence over the said dynamics. These behaviors (as for instance emotional states, opinions, strategies, or the search of individual well-being) are communicated across individuals by a complex learning dynamics¹⁹, where each living entity acquires the information by individual-based interactions as well as by the overall system viewed as a whole¹.

The interest of mathematicians towards this specific research field is rapidly growing due to the joint action of the stimuli exerted by highly challenging mathematical problems and by the related interest in our society. This includes the search of safety conditions as well as the search of mathematical tools to support decision making in social sciences and economy, with the ultimate goal of improving the overall well-being of the society.

Possible examples are the interest in technological innovation which might contribute to the ability of nations to compete and share the international market^{25,26}, modeling social systems and behavioral economy^{1,24}, safety problems related to crowd dynamics^{11,51}, modeling of criminality^{21,50} and its propagation in the territory. Additional examples can be found in the modeling of biological systems, for instance multicellular systems related to genetic diseases^{22,27}.

In all these examples, a sort of common *research dream* pervades the whole activity focusing on the possible development of a mathematics for living systems. We will return to this aspect in the last section.

3. Contents of the issue

The presentation of the contents of the special issue, rather than referring individually to each article, will firstly focus on the overall subject of the issue, and subsequently deal with the articles that treat some particular aspects of the whole subject. This strategy allows to enlighten the emerging research topics and to point out the new contributions with respect to the preceding literature. The specific topics and related papers published in this issue, as well as a selection of related literature, are presented here as sub-items of the following subjects:

Swarming phenomena by multiagent and kinetic theory models: As it is known, the mathematical modeling of swarming dynamics has been promoted by the pioneering paper by Cucker and Smale ²³, while methods of the kinetic theory have been developed in parallel towards the modeling of large systems of interacting living entities as witnessed in the several applications treated in the books ^{7,43}. The key modeling problem consists in deriving swarm and kinetic theory models from the dynamics of interactions at the microscopic scale. The contents of all published in this specific issue ^{2,16,18,29,35,40} include some aspects of this key problem focusing on specific applications.

In details, a study of emergent dynamics of continuous and discrete Cucker-Smale models equipped with temperature as an extra observable is developed in ³⁵, while social dynamics in swarms are treated in ² and ⁴⁰ in line with the research activity witnessed in recent papers for instance ^{4,20}.

A possible explanation of log-normal distribution forming in human activities is proposed in ²⁹, by resorting to classical methods of statistical mechanics of multi-agent systems with a model derived the framework of a linear Fokker-Planck type equation with lognormal equilibrium density. The study specifically refers to the modeling of call-center service times introduced in ²⁸.

The dynamics of consensus in various frameworks is treated in ¹⁶, having in mind control problems and the continuous approximation of discrete problems which will be further discussed in what follows.

The applications treated in ¹⁸ refer to multi-cellular systems, and specifically to the derivation of nonlinear diffusion problems which are somehow related to the Keller-Segel model ³⁷ reviewed in ³³ and ³⁶. The main contribution of this paper is the search for a unified approach to derive models at the macroscopic scale from the underlying description at the microscopic scale.

Control problems: Once models have been derived, applications generally require the development of a control theory suitable to drive the collective dynamics towards a properly selected behavior. The importance of this topic is witnessed by a rapidly growing literature, where significant new applications of the mathematical control theory have been developed for swarms of interacting living entities. Control problems are treated in ^{2,40,16} as a conceptual development of previous papers of the same authors, see ^{3,4,20,49}.

The authors of ⁴ provide a mean-field description for a leader-follower dynamics with mass transfer among the two populations. This model allows the transition from followers to leaders and vice versa, with scalar-valued transition rates depending nonlinearly on the global state of the system at each time. Existence and uniqueness of solutions for the leader-follower dynamics is proved under appropriate assumptions. Subsequently the authors establish the equivalence of the system with a continuity equation over the state space and an ODE for the transition from leader to follower or vice versa. Additional qualitative analysis and simulations follow. An interesting contribution is the modeling of the endogenous switch between the two

populations. A qualitative analysis of the solutions is developed and is followed by simulations that complete the study of the overall properties of the model.

The dynamics of multi-agent collective behavior models and its control theoretical properties are studied in ¹⁶ which focuses on a key conceptual difficulty, namely the attainment of explicit estimates on the behavior of the controls (the time of control, the norm of the control, etc.) as the number of agents tends to infinity. The authors first derive a large population limit to parabolic diffusive equations and also show that the non-local transport equations which are commonly derived as the mean-field limit, are subordinated to the first one. Namely, they show that the solution of the non-local transport model can be obtained by a suitable averaging of the solution of the diffusive one. A detailed estimate supports the aforementioned concepts, while applications and simulations refer to opinion formation in networks.

Multiscale topics: In ¹⁸ the authors propose a review on the mathematical tools for the derivation of macroscopic models in biology from the underlying description at the cell scale as delivered by a kinetic model. The derivation is inspired by the Hilbert's method, well known in classic kinetic theory, which is here applied to a broad class of kinetic equations modeling multicellular dynamics. The paper provides a unified approach to the aforementioned micro-macro derivation. The contents refers mainly to ¹⁴, where this research line was initiated, and to ¹⁷, which is the second step of the same research line. The method is developed in the case of the parabolic scaling, namely for diffusion phenomena, focusing on a variety of nonlinear diffusion cases (see for instance ⁵). Arguably, this can be further developed for hyperbolic scaling ^{42,45}, or for diffusion in complex environments.

The approach of ¹⁸ is inspired by the sixth Hilbert problem ³² which is addressed to the search for a unified mathematical approach to physical theories. This also means that one should look for the links between different scales when a given system is described by mathematical equations. As an example the equations of hydrodynamics should be obtained from the underlying description at the microscopic scale.

4. Looking ahead to research perspectives

A common feature of the papers published in this issue consists in the design of mathematical tools to transfer the dynamics at the low scale of individuals to the description of the collective dynamics of the system observed as a whole. It can be rapidly observed that different techniques and selections of modeling scales are known in the literature (as it is witnessed also in the present issue). In particular, we observe that even the mathematical structure of swarms has not yet found a commonly shared modeling strategy, as, for instance, possible alternatives are the kinetic theory approach ¹² (which allows to account for a broad variety of interactions at the microscopic scale) or stochastic models ⁴⁴, where a noise induces randomness in the modeling of interactions.

Therefore, the following key question can be raised out of the contents of the

issue as well as out of the related bibliography.

Which is the rationale and the related mathematical tools that are needed to design models of large systems of living entities?

This question is definitely consistent with the contents of the present special issue. In fact, it pervades all articles published therein. However, although we keep this question well in mind, we do not naively attempt to provide an answer in this short note, but we simply provide three specific reasonings on the modeling approach which might hopefully contribute to future research activity. Their statement also takes into account the opinion of the Nobel Laureate Hartwell³¹ pointing out that the mathematics of the inert matter cannot be straightforwardly applied to living systems, and that we should look for alternative mathematical tools suitable to capture the complexity of living systems.

Bearing all the above in mind, let us propose three specific arguments that might contribute to the development of mathematical tools for the study of living systems.

- Living systems are always shaped by evolution and, due to the struggle with the external environment, they can undergo a Darwinist type of selection. Accordingly, the fundamental ideas by Mayr^{38,39} are a precious gift to mathematics. Darwinist dynamics has to be taken into account not only in biology, for instance in cancer dynamics³⁰ and modeling the immune competition¹⁵, but for all living systems. Possible examples can be found in economy from³⁴ to^{47,48}, or in the evolution of languages, from⁴¹ to⁴⁶, and many others.
- A multiscale vision suggests to develop a unified modeling approach, where interactions at each scale are modeled by the same principles. In more details, each living entity interacts, at the micro-scale (typically in a non-local and non-linearly additive way), with all entities in its sensitivity domain. Further, the conjecture proposed in⁶ suggests that interactions involve only a fixed number of individuals rather than all individuals in their sensitivity domain.
- An important multiscale problem is the micro-macro derivation which can generate models that can be different from that of the direct phenomenological approach consisting in the use of conservation equations closed by heuristic models of material behaviors. In some cases, these new models show interesting qualitative features consistent with physical reality, that are not shown by purely phenomenological models. However, the classical approaches might not be sufficient for a full understanding of the validity and of the inner mathematical structure of models. In fact, the bridge between the submicroscopic and the microscopic scales is even much less understood. This is an important topic, as living entities are not classical particles whose physical properties might be constant for large time intervals, but they are active particles whose properties depend also on the dynamics at the lower scale.

The presentation of these three issues identifies three challenging research top-

ics. Needless to say, the presentation does not claim to be exhaustive of all different features of the class of systems under consideration. At best we can reasonably hope to have indicated a general path toward the development of an effective mathematical approach to modeling the collective behavior of living systems, with help, work and ideas of many many other scientists.

References

1. G. Ajmone Marsan, N. Bellomo, and L. Gibelli, Stochastic evolutionary differential games toward a systems theory of behavioral social dynamics, *Math. Models Methods Appl. Sci.*, **26**, 1051–1093, (2016).
2. G. Albi, M. Bongini, and F. Rossi, Leader formation with mean-field birth and death models, *Math. Models Methods Appl. Sci.*, **29**, Insert pages, (2019).
3. G. Albi, Y.-P. Choi, and A.-S. Häck, Pressureless Euler alignment system with control, *Math. Models Methods Appl. Sci.*, **28**, 1635–1664, (2018).
4. G. Albi, M. Herty, and L. Pareschi, Kinetic description of optimal control problems and application to opinion consensus, *Comm. Math. Sciences*, **13(6)**, 1407–1429, (2015).
5. M. Arias, J. Campos, and J. Soler, Cross-diffusion and traveling waves in porous-media flux-saturated Keller-Segel models, *Math. Models Methods Appl. Sci.*, **28**, 2103–2129 (2018).
6. M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study, *PNAS: Proc. Nat. Acad. Sci.*, **105(4)**, 1232–1237, (2008).
7. N. Bellomo, A. Bellouquid, L. Gibelli and N. Outada **A Quest Towards a Mathematical Theory of Living Systems**, Springer International Publishing, (2017).
8. N. Bellomo and F. Brezzi, Mathematical models of self-propelled particles, *Math. Models Methods Appl. Sci.*, **27**, 997–1004, (2017).
9. N. Bellomo, F. Brezzi, and M. Pulvirenti, Modeling behavioral social systems, *Math. Models Methods Appl. Sci.*, **27**, 1–11, (2017).
10. N. Bellomo and F. Brezzi, Challenges in active particles methods: Theory and applications *Math. Models Methods Appl. Sci.*, **28**, 1627–1633, (2018).
11. N. Bellomo, L. Gibelli, and N. Outada, On the interplay between behavioral dynamics and social interactions in human crowds, *Kinetic Related Models*, **12**, 397–409, (2019).
12. N. Bellomo, and S.-Y. Ha, A quest toward a mathematical theory of the dynamics of swarms. *Math. Models Methods Appl. Sci.*, **27**, 745–770, (2017).
13. N. Bellomo and M. Winkler, A degenerate chemotaxis system with flux limitation: Maximally extended solutions and absence of gradient blow-up, *Commun. Part. Diff. Eq.*, **42(3)**, 436–473, (2017).
14. A. Bellouquid and N. Chouhad, Kinetic models of chemotaxis towards the diffusive limit: asymptotic analysis, *Math. Models Methods Appl. Sci.*, **39**, 3136–3151 (2016).
15. A. Bellouquid, E. De Angelis and D. Konpoff, From the modeling of the immune hallmarks of cancer to a black swan in biology, *Math. Models Methods Appl. Sci.*, **23(5)**, 949–978, (2013).
16. U. Biccari, D. Koi, and E. Zuazua, Dynamics and control for multiagent networked systems: A finite difference PDE approach, *Math. Models Methods Appl. Sci.*, **29**, Insert pages, (2019).
17. D. Burini and N. Chouhad, An Hilbert perturbation method towards a multiscale

- analysis from kinetic to macroscopic models, *Math. Models Methods Appl. Sci.*, **27**, 1327–1353, (2017).
18. D. Burini and N. Chouhad, A multiscale view of nonlinear diffusion in biology: From cells to tissues, *Math. Models Methods Appl. Sci.*, **29**, Insert pages, (2019).
 19. D. Burini, S. De Lillo, and L. Gibelli, Collective learning dynamics modeling based on the kinetic theory of active particles, *Phys. Life Rev.*, **16**, 123–139, (2016).
 20. M. Caponigro, P. Piccoli, and Trelat, Mean-field sparse Jurdjevic-Quinn control, *Math. Models Methods Appl. Sci.*, **27**, 1223–1253, (2017).
 21. Chaohao Pan, Bo Li, Chuntian Wang, Yuqi Zhang, N. Geldner, Li Wang, and A. Bertozzi, Crime modeling with truncated Lévy flights for residential burglarly flights, *Math. Models Methods Appl. Sci.*, **28**, 1857–1880, (2018).
 22. G. Corbin, A. Hunt, A. Klar, F. Schneider, and C. Surulescu, Higher order models for glioma invasion: From a two scale description to effective equations for mass density and momentum, *Math. Models Methods Appl. Sci.*, **28**, 1771–1800, (2018).
 23. F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Automat. Control*, **52**, 852–862, (2007).
 24. M. Dolfin, L. Leonida, and N. Outada, Modelling human behaviour in economics and social science, *Phys. Life Rev.*, **22-23**, 1–21, (2017).
 25. G. Dosi, M.-C. Pereira, and M.-E. Virgillito, The footprint of evolutionary processes of learning and selection upon the statistical properties of industrial dynamics, *Industrial and Corporate Change*, doi:10.1093/icc/dtw044, 1–24, (2016).
 26. G. Dosi, M.-C. Pereira, and M.-E. Virgillito, On the robustness of the fat-tailed distribution of firm growth rates: a global sensitivity analysis, *J. Econ. Interact. Coord.*, 18 March, DOI 10.1007/s11403-017-0193-4, (2017).
 27. C. Engwer, C. Stinner, and C. Surulescu, A structured multiscale model for acid mediated tumor invasion: the effect of adhesion and tumor proliferation, *Math. Models Methods Appl. Sci.*, **27**, 1355–1390, (2018).
 28. S. Gualandi and G. Toscani, Call center service times are lognormal. A Fokker-Planck description, *Math. Models Methods Appl. Sci.*, **28**, 1513–1527, (2018).
 29. S. Gualandi and G. Toscani, Human behavior and lognormal distribution. A kinetic description, *Math. Models Methods Appl. Sci.*, **29**, Insert pages, (2019).
 30. D. Hanahan, and R.A. Weinberg, **The Hallmarks of Cancer**, *Cell*, 100 (2000) 57–70.
 31. H.L. Hartwell, J.J. Hopfield, S. Leibner, and A.W. Murray, From molecular to modular cell biology, *Nature*, 402 (1999) c47-c52.
 32. D. Hilbert, Mathematical problems, *Bull. Amer. Math. Soc.*, **8(10)**, 437–479, (1902).
 33. T. Hillen and K.J. Painter, A users guide to PDE models for chemotaxis, *J. Math. Biol.*, **58**, 183–217, (2009).
 34. J.H. Holland, The global economy as an adaptive process, in P. W. Anderson, K. J. Arrow and D. Pines, eds., *The economy as an evolving complex system*, 117–124, (1988).
 35. Jiu-Gang Dong, Seung-Yeal Ha, and Doeon Kim, Emergent behaviors of continuous and discrete thermomechanical Cucker-Smale models on general digraphs, *Math. Models Methods Appl. Sci.*, **29**, Insert pages, (2019).
 36. A. Juengel, **Entropy Methods for Diffusive Partial Differential Equations**, Springer Briefs in Mathematics, (2016).
 37. E.F. Keller and L.A. Segel, Model for chemotaxis, *J. Theor. Biol.*, **30**, 225–234, (1971).
 38. E. Mayr, The philosophical foundation of Darwinism, *Proceedings of the American Philosophical Society*, **145**, 488–495, (2001).
 39. E. Mayr, **What evolution is**, Basic Books, New York, (2001).

40. S. McQuade, B. Piccoli, and N. Pouradier Duteil, Social dynamics models with time-varying influence, *Math. Models Methods Appl. Sci.*, **29**, Insert pages, (2019).
41. M.A. Nowak, N.L. Komarova, and P. Niyogy, Computational and evolutionary aspects of language, *Nature*, **417**, 611–617, (2002).
42. N. Outada, N. Vauchelet, T. Akrid and M. Khaladi, From kinetic theory of multicellular systems to hyperbolic tissue equations: asymptotic limits and computing, *Math. Models Methods Appl. Sci.*, **26**, 2709–2734, (2016).
43. L. Pareschi, and G. Toscani, **Interacting Multiagent Systems: Kinetic Equations and Monte Carlo Methods**, Oxford University Press, Oxford, (2013).
44. R. Pinnau, C. Totzeck, O. Tse, and S. Martin, A consensus-based model for global optimization and its mean field limit, *Math. Models Methods Appl. Sci.*, **27**, 183–204, (2017).
45. D. Poyato and J. Soler, Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models, *Math. Models Methods Appl. Sci.*, **27**, 1089–1152, (2017).
46. K. Prochazka and G. Vogla, Quantifying the driving factors for language shift in a bilingual region, *PNAS*, **114(17)**, 4365–4369, (2017).
47. R.H. Thaler, *Behavioral Economics: Past, Present, and Future*, The American Economic Review, **106(7)**, 1577–1600, (2016).
48. R.H. Thaler and C. Sunstein, **Nudge: Improving Decisions About Health, Wealth, and Happiness**, Penguin, New York, (2016).
49. E. Trélat, J. Zhu, and E. Zuazua, Alee optimal control of a system ecology, *Math. Models Methods Appl. Sci.*, **28**, 1665–1697, (2018).
50. L. Wang, M. Short, and A.L. Bertozzi, Efficient numerical methods for multiscale crowd dynamics with emotional contagion, *Math. Models Methods Appl. Sci.*, **27**, 205–230, (2017).
51. N. Wijermans, C. Conrado, M. van Steen, C. Martella, and J.-L. Li, A landscape of crowd management support: An integrative approach, *Safety Science*, **86**, 142–164, (2016).