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# Efficient Solutions of Elliptic Systems

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ON THE STABILIZATION OF FINITE ELEMENT  
APPROXIMATIONS OF THE STOKES EQUATIONS

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ABSTRACT

Consider finite element approximation of the Stokes equations. We present a systematic way of stabilizing it by adding bubble functions to the discrete velocity field. Another way of stabilization is also presented where the finite element spaces are kept unchanged but the discrete incompressibility condition is modified instead.

1. INTRODUCTION

Assume for the sake of simplicity that  $\Omega$  is a given polyhedron in  $\mathbb{R}^n$  ( $n \geq 2$ ) and set  $V = (H_0^1(\Omega))^n$ ,  $P = L^2(\Omega)$ ; consider now for a given  $\underline{f}$ , say, in  $(L^2(\Omega))^n$  and  $g \in P/\mathbb{R}$  the (generalized) Stokes problem

$$\left\{ \begin{array}{l} \text{Find } \underline{u} \in V, p \in P \text{ such that:} \\ \text{(i) } a(\underline{u}, \underline{v}) - \int_{\Omega} p \operatorname{div} \underline{v} \, dx = \int_{\Omega} \underline{f} \cdot \underline{v} \, dx \quad \forall \underline{v} \in V, \\ \text{(ii) } \int_{\Omega} q \operatorname{div} \underline{u} \, dx = \int_{\Omega} g q \, dx \quad \forall q \in P \end{array} \right. \quad (1.1)$$

where, as usual

$$a(\underline{u}, \underline{v}) := \int_{\Omega} \sum_{r=1}^n \sum_{i=1}^n \frac{\partial u_r}{\partial x_i} \frac{\partial v_r}{\partial x_i} \, dx. \quad (1.2)$$

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It is well known that problem (1.1) has a unique solution. Consider now a finite element discretization of (1.1) consisting (as usual) of two families  $\{V_h\}, \{P_h\}$  of finite dimensional subspaces of  $V$  and  $P$  respectively, and the corresponding discretized problems

$$\left\{ \begin{array}{l} \text{Find } \underline{u}_h \in V_h, p_h \in P_h \text{ such that:} \\ a(\underline{u}_h, \underline{v}_h) - \int_{\Omega} p_h \operatorname{div} \underline{v}_h dx = \int_{\Omega} \underline{f} \cdot \underline{v}_h dx \quad \forall \underline{v}_h \in V_h, \\ \int_{\Omega} q_h \operatorname{div} \underline{u}_h dx = \int_{\Omega} g q_h dx \quad \forall q_h \in P_h. \end{array} \right. \quad (1.3)$$

It is also well known that one is not allowed to take independent choices for  $V_h$  and  $P_h$  in order to have stability and convergence results (c.f. [2], [3]). In the present paper we consider the possibility of modifying an unstable scheme with a minimal cost so that stability can be recovered.

In the first case we show that by adding suitable bubble functions to the spaces  $V_h$ , one can stabilize "any" given pair of families  $\{V_h\}, \{P_h\}$ . More precisely, taking for instance triangular elements, this can be done for any choice of  $\{P_h\}$  provided that the spaces  $\{V_h\}$  contain at least all piecewise quadratic continuous functions (vanishing on  $\partial\Omega$ ); this can also be done under the weaker assumption that the  $\{V_h\}$ 's contain at least all piecewise linear continuous functions (vanishing on  $\partial\Omega$ ) provided that  $P_h \subseteq H^1(\Omega)$  for all  $h$ . The choice of bubble functions (that is: functions whose support is contained in a single element) as stabilizing correction relies on the fact that one can eliminate (i.e., condense) such degrees of freedom without affecting the structure of the stiffness matrix. Thus if direct solution techniques are used, the bubble functions have only a minor contribution to the total computational cost.

We consider also another way of achieving stability where we do not modify the discrete velocity or pressure fields but instead modify the discrete equations by adding a stabilizing

term into the discrete equations. Unlike in the bubble function approach, this type of stabilization affects also the consistency of the scheme. However, if the extra consistency error is not larger than that of the original scheme, this approach may be useful when iterative (such as multigrid) techniques are used in the solution of the linear system.

Stabilization by adding bubble functions is discussed in Section 2 below. The proof in the case  $P_h \subset H^1(\Omega)$  is essentially contained in [1], but we report it below for the sake of completeness. In Section 3 we give a simple example of stabilization by modifying the discrete equations. Another example is given in [7], where the same type of stabilization is applied to the "bilinear velocities - constant pressure" approximation of the Stokes problem.

## 2. STABILIZATION WITH BUBBLE FUNCTIONS

For the sake of simplicity we consider only the case of "triangular" elements: however it will be clear from the proofs that the results hold for "quadrilateral" elements as well, and also for more general discretizations.

Let  $\{C_h\}$  be a family of decompositions of  $\Omega$  into  $n$ -simplexes and let  $\{V_h\}, \{P_h\}$  be the corresponding given families of f.e.m. Let moreover, for any  $n$ -simplex  $K$ ,  $b_K(x)$  be the corresponding bubble function of degree  $n+1$ :

$$b_K(x) = c \lambda_1(x) \lambda_2(x) \dots \lambda_{n+1}(x) \quad (2.1)$$

where the  $\lambda_i$ 's are the barycentric coordinates (= equations of the faces) and  $c$  is a normalizing factor so that, say

$$\sup_K b_K(x) = 1; \quad (2.2)$$

note that  $b_K(x) > 0$  in the interior of  $K$ . We are looking for corrections of  $V_h$  of type



$$\left\{ \begin{array}{l} \tilde{V}_h = V_h \oplus B_h ; \quad B_h = \bigoplus_{K \in C_h} B_h^K ; \\ B_h^K = \begin{cases} b_K \nabla p_h|_K & \text{in } K \\ 0 & \text{elsewhere;} \end{cases} \end{array} \right. \quad (2.3)$$

clearly  $B_h \subseteq (H_0^1(\Omega))^n$ ; we want to show that the corresponding modified f.e.m:

$$\left\{ \begin{array}{l} \text{Find } \tilde{u}_h \in \tilde{V}_h, p_h \in P_h \text{ such that:} \\ a(\tilde{u}_h, \tilde{v}_h) - \int_{\Omega} p_h \operatorname{div} \tilde{v}_h dx = \int_{\Omega} f \cdot \tilde{v}_h dx \quad \forall \tilde{v}_h \in \tilde{V}_h \\ \int_{\Omega} q_h \operatorname{div} \tilde{u}_h dx = \int_{\Omega} g q_h dx \quad \forall q_h \in P_h \end{array} \right. \quad (2.4)$$

is stable and gives "optimal" error bounds:

$$\begin{aligned} \|\underline{u} - \tilde{u}_h\|_1 + \|p - p_h\|_{0/R} &\leq c \left( \inf_{\tilde{v}_h \in \tilde{V}_h} \|u - \tilde{v}_h\|_1 + \right. \\ &\left. + \inf_{q_h \in P_h} \|p - q_h\|_{0/R} \right), \end{aligned} \quad (2.5)$$

with  $c$  independent of  $h$ . As is well known (c.f. [2], [3]), the stability condition required for (2.5) to hold is: there is a constant  $C$  independent of  $h, f$  and  $g$  such that  $(\tilde{u}_h, p_h)$  satisfies

$$\|\tilde{u}_h\|_1 + \|p_h\|_{0/R} \leq C(\|f\|_{-1} + \|g\|_{0/R}) \quad (2.6)$$

where  $\|\cdot\|_{-1}$  denotes the dual norm of  $(H_0^1(\Omega))^2$ . It is also known that a sufficient condition in order to have (2.6) is (c.f. [3], [6])

$$\left\{ \begin{array}{l} \exists \tilde{\pi}_h \in L(V, \tilde{V}_h) \text{ such that} \\ \text{i) } \|\tilde{\pi}_h\| \leq c \text{ (indep. of } h) \text{ in } L(V, V) \\ \text{ii) } \int_{\Omega} q_h \operatorname{div}(\underline{v} - \tilde{\pi}_h \underline{v}) dx = 0 \quad \forall \underline{v} \in V, \forall q_h \in P_h. \end{array} \right. \quad (2.7)$$

The following two theorems give sufficient conditions on  $V_h$  so that  $\tilde{V}_h$ , as defined in (2.3), satisfy (2.7) (and hence (2.5)).

THEOREM 1. Assume that:

$$\left\{ \begin{array}{l} \exists \pi_h \in L(V, V_h) \text{ s.t. } \sum_K h_K^{2r-2} \|\underline{v} - \pi_h \underline{v}\|_r^2 \leq c \|\underline{v}\|_1^2 \\ \text{for } r = 0, 1, \text{ with } c \text{ independent of } h \\ \text{where } h_K = \text{diameter of } K, \end{array} \right. \quad (2.8)$$

$$P_h \subseteq H^1(\Omega); \quad (2.9)$$

then the pair  $\tilde{V}_h, P_h$  obtained from (2.3) satisfies (2.5).

PROOF. We look for  $\tilde{\pi}_h$  of the form

$$\tilde{\pi}_h \underline{v} = \pi_h \underline{v} + \underline{\beta}_h(\underline{v}), \quad \underline{\beta}_h \in B_h. \quad (2.10)$$

Then (2.7 ii) is satisfied if, for all  $K$  in  $C_h$ ,

$$\int_K (\underline{v} - \pi_h \underline{v} - \underline{\beta}_h) \cdot \nabla q_h = 0 \quad \forall q_h \in P_h. \quad (2.11)$$

This uniquely determines  $\underline{\beta}_h \in B_h^K$ . Moreover one has

$$\|\underline{\beta}_h\|_{1,K} \leq ch_K^{-1} \|\underline{v} - \pi_h \underline{v}\|_{0,K} \quad (2.12)$$

so that (2.8), (2.10) and (2.12) give (2.7 i).

COROLLARY. If  $P_h \subseteq H^1(\Omega)$  and  $V_h$  contains at least all piecewise linear continuous functions vanishing on  $\partial\Omega$ , then  $\tilde{V}_h, P_h$  satisfies (2.5).

PROOF. It is clear (see e.g. [1], [4]) that the present assumption on  $V_h$  implies (2.8).

THEOREM 2. Assume that  $\exists \pi_h \in L(V, V_h)$  such that:

$$\int_K \operatorname{div}(\underline{v} - \pi_h \underline{v}) dx = 0 \quad \forall K, \forall \underline{v} \in V \quad (2.13)$$

$$\|\pi_h\| \leq c \quad \text{independent of } h \quad (2.14)$$

then the pair  $\tilde{V}_h, P_h$  obtained from (2.3), satisfies (2.5).

PROOF. We look again for  $\tilde{\pi}_h$  of the form (2.10). Now for any given  $q_h \in P_h$  we consider the decomposition  $q_h = q_h^{(1)} + q_h^{(2)}$  with  $q_h^{(1)}$  piecewise constant and  $\int_K q_h^{(2)} dx = 0$  for all  $K$ . Then

$$\int_K \operatorname{div}(\underline{v} - \tilde{\pi}_h \underline{v}) q_h dx = \int_K \underline{\beta}_h \cdot \nabla q_h dx + \int_K \operatorname{div}(\underline{v} - \pi_h \underline{v}) q_h^{(2)} dx \quad (2.15)$$

so that (2.7 ii) holds if

$$\int_K \underline{\beta}_h \cdot \nabla q_h dx = - \int_K q_h^{(2)} \operatorname{div}(\underline{v} - \pi_h \underline{v}) dx \quad (2.16)$$

which again determines uniquely  $\underline{\beta}_h \in B_h$ . Since  $q_h^{(2)}$  has locally zero mean value, a simple scaling argument shows that

$$\|\underline{\beta}_h\|_1 \leq c \|\operatorname{div}(\underline{v} - \pi_h \underline{v})\|_0 \leq c \|\underline{v} - \pi_h \underline{v}\|_1 \quad (2.17)$$

with  $c$  independent of  $h$ . Now (2.14), (2.10) and (2.17) give (2.7 i).

COROLLARY. If  $V_h$  contains at least all piecewise quadratic continuous functions vanishing on  $\partial\Omega$ , then  $\tilde{V}_h, P_h$  satisfies (2.5).

PROOF. The result follows immediately from the fact that the "quadratic velocities - constant pressure" approximation of the Stokes problem satisfies (2.6) (c.f. [5]): hence for any given  $\underline{v} \in V$  we may choose  $\pi_h \underline{v}$  as the discretized solution of  $-\Delta \underline{u} + \Delta p = -\Delta \underline{v}$  and  $\operatorname{div} \underline{u} = \operatorname{div} \underline{v}$  (that is  $\underline{u} = \underline{v}$ ,  $p = 0$ ), by means of the quadratic-constant scheme. In our assumptions we have  $\pi_h \underline{v} \in V_h$  and hence (2.13), (2.14) are satisfied.

REMARK. With a slightly different proof we see that we could also deal with nonconforming  $V_h$ ; in that case we could accept, for any choice of  $P_h$ , to start with spaces  $V_h$  that contain at least  $P_1$ -nonconforming spaces like in [5].

### 3. STABILIZATION BY MODIFYING THE DISCRETE EQUATIONS

To present the idea, let us consider a simple example where  $P_h$  is the space of piecewise linear continuous functions associated to a triangulation  $C_h$  of a two-dimensional polygonal domain, and  $V_h = P_h^2 \cap H_0^1(\Omega)$ . The pair  $(V_h, P_h)$  is not stable in the sense of (2.7), but we can modify the discrete equations so that the scheme becomes stable in the sense of (2.6). To this end, let us define the approximate solution to (1.1) as the pair  $(\underline{u}_h, p_h)$  which satisfies (1.3 i) and

$$\begin{aligned} \sum_{K \in C_h} h_K^2 \int_K \nabla p_h \cdot \nabla q_h \, dx + \int_{\Omega} q_h \operatorname{div} \underline{u}_h \, dx \\ = \int_{\Omega} g q_h \, dx \quad \forall q_h \in Q_h. \end{aligned} \quad (3.1)$$

THEOREM 3. If  $(\underline{u}_h, p_h)$  satisfies (1.3i), (3.1) then (2.6) holds (with  $\tilde{u}_h := \underline{u}_h$ ) and one has the error estimate

$$\|\underline{u} - \underline{u}_h\|_1 + \|p - p_h\|_{0/R} \leq Ch(|\underline{u}|_2 + |p|_1). \quad (3.2)$$

REMARK. The error estimate does not follow directly from (2.6) because we have modified (1.3 ii) thus affecting the consistency of the scheme. In other words, we need a separate estimate for the extra consistency error caused by the added stabilizing term in (3.1).

PROOF. Let the triangles of  $C_h$  be grouped together to form disjoint polygonal "macroelements" each containing at most a fixed number, say  $N$ , triangles of  $C_h$ . Denote the coarse partitioning of  $\Omega$  so obtained by  $\tilde{C}_h$  and let  $\tilde{P}_h$  be the space of functions which are piecewise constant on the macro-



elements. Now it is well known (c.f. [5], [8]) that by a suitable choice of  $\tilde{C}_h$  the pair  $(v_h, \tilde{p}_h)$  is stable in the sense of (2.7), i.e., for any  $\tilde{p}_h \in \tilde{P}_h$  there exists  $w_h \in V_h$  such that

$$\begin{cases} (\tilde{p}_h, \operatorname{div} w_h) \geq \|\tilde{p}_h\|_{0/R}^2 \\ \|w_h\| \leq C \|\tilde{p}_h\|_{0/R} \end{cases} \quad (3.3)$$

For example, if  $C_h$  is obtained from a coarser triangulation  $C_h^0$  by subdividing each triangle into four equal subtriangles, then it suffices to take  $\tilde{C}_h = C_h^0$ .

Now with  $u_h, p_h$  satisfying (1.3 i) and (3.1) define  $\tilde{p}_h \in \tilde{P}_h$  as the local average of  $p_h$  on each macroelement. Then setting  $v_h = u_h - \delta w_h$  and  $q_h = p_h$  in (1.3 i) and (3.1), with  $\delta$  a constant and  $w_h$  satisfying (3.3), we obtain

$$\begin{aligned} a(u_h, u_h) + \delta \|\tilde{p}_h\|_{0/R}^2 + \sum_{T \in C_h} h_K^2 \int_K |\nabla p_h|^2 dx \\ - \delta a(u_h, w_h) + \delta (p_h - \tilde{p}_h, \operatorname{div} w_h) = (f, v_h) + (g, p_h). \end{aligned}$$

Now for any macroelement  $M \in \tilde{C}_h$  we have the inequalities

$$C^{-1} \int_M |p_h - \tilde{p}_h|^2 dx \leq \sum_{\substack{K \in C_h \\ K \subseteq M}} h_K^2 \int_K |\nabla p_h|^2 dx \leq C \int_M |p_h - \tilde{p}_h|^2 dx \quad (3.4)$$

where  $C$  depends only on  $N$  and on the minimal angle of the triangles contained in  $M$ , so we may assume  $C$  to be an absolute constant. Choosing then a sufficiently small positive value for the parameter  $\delta$  we have combining (3.3) and (3.4) that

$$\|u_h\|_1^2 + \|p_h\|_{0/R}^2 \leq C \{ (f, v_h) + (g, p_h) \}. \quad (3.5)$$

Applying on the right side the obvious estimate  $\|v_h\|_1 \leq C(\|u_h\|_1 + \|p_h\|_{0/R})$ , the asserted stability estimate (2.6) follows.

Using the stability estimate we obtain by usual manipulations an error bound of the form

$$\begin{aligned} & \| \underline{u} - \underline{u}_h \|_1 + \| p - p_h \|_{0/\mathbb{R}} \\ & \leq C \inf_{(\underline{v}_h, q_h) \in V_h \times P_h} \{ \| \underline{u} - \underline{v}_h \|_1 + \| p - q_h \|_{0/\mathbb{R}} \\ & + \sup_{\substack{r_h \in P_h \\ \| r_h \|_{0/\mathbb{R}} = 1}} \sum_{K \in \mathcal{C}_h} h_K^2 \int_K \nabla q_h \cdot \nabla r_h dx \} \end{aligned}$$

where the last term on the right side arises from the added stabilizing term in (3.1). Since  $\sum_K h_K^2 \int_K |\nabla r_h|^2 dx \leq C \| r_h \|_{0/\mathbb{R}}^2$  for  $r_h \in P_h$ , (3.2) follows by choosing  $(\underline{v}_h, q_h)$  to be an appropriate interpolant of  $(\underline{u}, p)$ .

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