Innovative mimetic discretizations for electromagnetic problems

Franco Brezzi

IUSS - University of Pavia and IMATI-CNR, Pavia, Italy

Annalisa Buffa

IMATI-CNR, Pavia, Italy

Abstract

In this paper we introduce a discretization methodology for Maxwell equations based on Mimetic Finite Differences (MFD). Following the lines of the recent advances in MFD techniques (see [10] and the references therein) and using some of the results of [7], we propose mimetic discretizations for several formulations of electromagnetic problems both at low and high frequency in the time-harmonic regime. The numerical analysis for some of the proposed discretizations have already been developed, whereas for others the convergence study is object of ongoing research.

Key words: Mimetic finite differences, electromagnetics

1 Introduction

Electromagnetic phenomena are governed by Maxwell equations which involve four fields: the electric and magnetic fields $\mathbf{E}$, $\mathbf{H}$, and the electric and magnetic inductions $\mathbf{D}$, $\mathbf{B}$, respectively. These fields obey the Maxwell equations:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{1}$$

Email addresses: franco.brezzi@imati.cnr.it (Franco Brezzi), annalisa.buffa@imati.cnr.it (Annalisa Buffa).

URLs: www.imati.cnr.it/brezzi (Franco Brezzi), www.imati.cnr.it/annalisa (Annalisa Buffa).

$$\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = -\mathbf{J}, \tag{2}$$

$$\operatorname{div} \mathbf{D} = \rho, \tag{3}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{4}$$

where ρ denotes the charge density and $\mathbf{J}$ the current density. It is easy to see that taking the divergence of (2) and inserting the time derivative of (3) one gets the charge conservation equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{J} = 0$$

as a *necessary condition* for the existence of a solution. Fields and inductions are related to each other by constitutive laws, which in the linear case are:

$$\mathbf{E} = \boldsymbol{\varepsilon} \mathbf{D} \tag{5}$$

$$\mathbf{B} = \boldsymbol{\mu}\, \mathbf{H} \tag{6}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are the electric permittivity and the magnetic permeability, respectively. In general, they are represented as 3×3 Hermitian, uniformly bounded and uniformly positive-definite tensors which account for the material behavior and which may depend on the space variables. Here we suppose that electric and magnetic fields are confined in a bounded region Ω of $\mathbb{R}^3$ which is supposed to have a Lipschitz continuous polyhedral boundary. This is the case for perfect conducting boundary conditions:

$$\mathbf{E} \times \mathbf{n} = 0 \qquad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{ on } \partial\Omega \tag{7}$$

where $\mathbf{n}$ denotes the outward unit vector on $\partial\Omega$.

In this paper, we discuss some innovative discretization techniques for the numerical simulations in electromagnetics. The techniques we propose belong to the family of *Mimetic Finite Differences* (MFD) and are intimately related to the cochain approximations of differential forms. In particular, they are strongly connected with the geometric structure of Maxwell equations. There are several papers exploring the use of differential geometry for the discretization of field equations: for a finite difference approach we refer to e.g., [16] while in the finite element context, the pioneering paper is surely [6]. The first attempt of a unified view and analysis is probably [14]. On the other hand, the extension of a, possibly unified, analysis to high order schemes is an open problem.

Since we are mainly concerned with spatial discretizations, we shall focus on two simplifications of Maxwell equations: *(i)* the "static" cases, i.e., the fields are assumed to be steady in time and the terms with time derivatives are dropped; *(ii)* the time-harmonic case, i.e., the fields are assumed to have a harmonic behavior in time:

$$\mathbf{H}(t, \mathbf{x}) = \exp(-i\omega t)\mathbf{H}(\mathbf{x}) \qquad \mathbf{E}(t, \mathbf{x}) = \exp(-i\omega t)\mathbf{E}(\mathbf{x}). \tag{8}$$

In the time harmonic regime, on bounded domains, another problem of interest is the computation of eigenfrequencies (see, e.g., [3]).

The paper is organized as follows: in Section 2 we introduce the continuous differential problems we want to discretize. In Section 3 we set the basic notation and we introduce the main concepts for mimetic finite differences, and, finally, in Section 4 we propose mimetic discretizations for the problems introduced in Section 2 and provide several comments.

2 Continuous problems

2.1 Strong formulations

When neglecting the dependence on time, the electric and magnetic part of Maxwell equations naturally decouple. In the sequel we introduce first the problems and formulations we are interested in. Then we will discuss their discretization.

Electrostatics: Neglecting the dependence on time, we have that $\nabla \times \mathbf{E} = 0$, i.e., the electric field can be represented in terms of a scalar potential $p : \mathbf{E} = \nabla p$. We then have the following equations governing the electric field:

$$\operatorname{div} \mathbf{D} = \rho \qquad \mathbf{D} = \boldsymbol{\varepsilon} \nabla p, \tag{9}$$

and, after elimination of the the electric induction $\mathbf{D}$, we obtain

$$-\operatorname{div} (\boldsymbol{\varepsilon} \nabla p) = \rho \tag{10}$$

where the electric charge density ρ is supposed to be given. Note that the scalar electric potential (that in the present notation would correspond to $-p$) is often denoted by V .

Magnetostatics: We assume that we are given a *divergence free* current density $\mathbf{J}$. Then the equations to be solved are

$$\mathbf{curl}\, \mathbf{H} = \mathbf{J} \qquad \mathbf{B} = \boldsymbol{\mu}\, \mathbf{H} \qquad \operatorname{div} \mathbf{B} = 0. \tag{11}$$

On the other hand the absence of magnetic charges ($\operatorname{div} \mathbf{B} = 0$) implies that the magnetic induction $\mathbf{B}$ can be represented in terms of a magnetic *vector potential* $\mathbf{u}$:

$$\mathbf{B} = \nabla \times \mathbf{u}.$$

We then have the following equations governing the magnetic field:

$$\mathbf{curl}\, \mathbf{u} = \boldsymbol{\mu}\, \mathbf{H} \qquad \mathbf{curl}\, \mathbf{H} = \mathbf{J} \tag{12}$$

and, after elimination of the magnetic field $\mathbf{H}$, we find:

$$\mathbf{curl}\,\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u} = \mathbf{J}. \tag{13}$$

Often the problem (13) is made wellposed by adding the gauge:

$$\operatorname{div}\mathbf{u} = 0.$$

Note that usually the magnetic vector potential is denoted by $\mathbf{A}$.

Time-harmonic regime: It is well known that, also in this case, the magnetic and electric part decouple. When solving for the electric field only, for example, we have:

$$\mathbf{curl}\,\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E} - \omega^2\boldsymbol{\varepsilon}\mathbf{E} = i\omega\,\mathbf{J} \tag{14}$$

endowed with the boundary conditions $\mathbf{E}\times\mathbf{n} = 0$ on $\partial\Omega$. At a fixed frequency ω away from resonances, this problem is well-posed.

The corresponding eigenvalue problem is: Find solutions $\mathbf{u}\neq 0$, $\omega\in\mathbb{C}\,,\ \omega\neq 0$ of the equation:

$$\mathbf{curl}\,\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u} = \omega^2\boldsymbol{\varepsilon}\mathbf{u}. \tag{15}$$

The equations governing the magnetic field have exactly the same structure and we will not detail it here.

2.2 Variational formulations

In this subsection we briefly recall the variational formulation for the differential problems introduced above. We refer the reader to the book [15] for an exhaustive description of the functional framework.

Let us introduce a few well known functional spaces. Let $L^2(\Omega)$ denote the space of complex valued, square integrable functions, and let $\|\cdot\|_0$ be the corresponding norm. We set:

$$\begin{aligned} H^1(\Omega) &= \{u\in L^2(\Omega)\ :\ \nabla u\in (L^2(\Omega))^3\}\\ H(\mathbf{curl}\,,\Omega) &= \{\mathbf{u}\in (L^2(\Omega))^3\ :\ \mathbf{curl}\,\mathbf{u}\in (L^2(\Omega))^3\}\\ H(\operatorname{div},\Omega) &= \{\mathbf{u}\in (L^2(\Omega))^3\ :\ \operatorname{div}\mathbf{u}\in L^2(\Omega)\} \end{aligned} \tag{16}$$

endowed with their graph norms. All the derivatives are obviously intended in the distributional sense. We also denote by $H^1_0(\Omega)$, $H_0(\mathbf{curl}\,,\Omega)$, and $H_0(\operatorname{div},\Omega)$ the closures of regular compactly supported functions (or vector-valued functions) in the spaces $H^1(\Omega)$, $H(\mathbf{curl}\,,\Omega)$, and $H(\operatorname{div},\Omega)$, respectively.

We list now the variational formulations associated to the problems introduced in the previous section. We denote by $(\cdot\,,\cdot)$ the $L^2(\Omega)$ scalar product (for both scalar

and vector fields):

$$(u, v) = \int_\Omega u\, \overline{v}\; d\Omega \quad \text{for } u\,,\ v \in L^2(\Omega),$$

and

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\; d\Omega, \quad \text{for } \mathbf{u}\,,\ \mathbf{v} \in (L^2(\Omega))^3.$$

Problem (9) can now be written as: *Find* $\mathbf{D} \in H(\mathrm{div}\,, \Omega)$ *and* $p \in L^2(\Omega)$ *such that*

$$\begin{aligned} (\boldsymbol{\varepsilon}^{-1}\mathbf{D}, \delta\mathbf{D}) + (p, \mathrm{div}\ \delta\mathbf{D}) &= 0 \qquad && \forall\, \delta\mathbf{D} \in H(\mathrm{div}\,, \Omega) \\ (\delta p, \mathrm{div}\, \mathbf{D}) &= (\delta p, \rho) && \forall\, \delta p \in L^2(\Omega) \end{aligned} \tag{17}$$

whereas the variational formulation for problem (10) reads: *Find* $p \in H_0^1(\Omega)$ *such that*

$$(\boldsymbol{\varepsilon}\nabla p, \nabla\delta p) = (\rho, \delta p) \qquad \forall\, \delta p \in H_0^1(\Omega). \tag{18}$$

For magnetostatics problems, we define first

$$H(\mathrm{div}^{\,0}, \Omega) := H(\mathrm{div}\,, \Omega) \cap \mathrm{Ker}\{\mathrm{div}\,\}.$$

The variational formulation for (12) is then: *Find* $\mathbf{H} \in H(\mathbf{curl}\,, \Omega)$, $\mathbf{u} \in H(\mathrm{div}^{\,0}, \Omega)$ *such that*

$$\begin{aligned} (\boldsymbol{\mu}\mathbf{H}, \delta\mathbf{H}) - (\mathbf{u}, \mathbf{curl}\, \delta\mathbf{H}) &= 0 \qquad && \forall\, \delta\mathbf{H} \in H(\mathbf{curl}\,, \Omega) \\ (\delta\mathbf{u}, \mathbf{curl}\, \mathbf{H}) &= (\delta\mathbf{u}, \mathbf{J}) && \forall\, \delta\mathbf{u} \in H(\mathrm{div}^{\,0}, \Omega). \end{aligned} \tag{19}$$

This is a mixed formulation for magnetostatics which has been object of several studies, in relation with eigenvalue problems (see e.g., [5]). On the other hand, the variational formulation of (13) together with the gauge $\mathrm{div}\ \mathbf{u} = 0$ is: *Find* $\mathbf{u} \in H_0(\mathbf{curl}\,, \Omega)$ *and* $p \in H_0^1(\Omega)$ *such that*:

$$\begin{aligned} (\boldsymbol{\mu}^{-1}\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \delta\mathbf{u}) - (\nabla p, \delta\mathbf{u}) &= (\mathbf{J}, \delta\mathbf{u}) \qquad && \forall \delta\mathbf{u} \in H_0(\mathbf{curl}\,, \Omega) \\ (\mathbf{u}, \nabla\delta p) &= 0 && \forall\, \delta p \in H_0^1(\Omega). \end{aligned} \tag{20}$$

For time-harmonic problems, the variational formulation for (14) is the following: *Find* $\mathbf{E} \in H_0(\mathbf{curl}\,, \Omega)$ *such that*

$$(\boldsymbol{\mu}^{-1}\mathbf{curl}\, \mathbf{E}, \mathbf{curl}\, \delta\mathbf{E}) - \omega^2(\boldsymbol{\varepsilon}\mathbf{E}, \delta\mathbf{E}) = i\omega(\mathbf{J}, \delta\mathbf{E}) \qquad \forall \delta\mathbf{E} \in H_0(\mathbf{curl}\,, \Omega). \tag{21}$$

Note that this problem is well posed only when ω is away from resonances [15]. Indeed, the corresponding eigenvalue problem is: *Find* $\mathbf{u} \neq 0$ *in* $H_0(\mathbf{curl}\,, \Omega)$ *and* $\omega \neq 0$ *such that*

$$(\boldsymbol{\mu}^{-1}\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \delta\mathbf{u}) = \omega^2(\boldsymbol{\varepsilon}\mathbf{u}, \delta\mathbf{u}) \qquad \forall \delta\mathbf{u} \in H_0(\mathbf{curl}\,, \Omega). \tag{22}$$

3 Mimetic finite differences

Given the domain Ω, we consider a polyhedral partition $\mathcal{T}_h$ of Ω having N vertices $V_1, V_2, ...V_{\mathsf{N}}$, E edges $e_1, e_2, ..., e_{\mathsf{E}}$, F faces $f_1, f_2, ..., f_{\mathsf{F}}$, and P elements $P_1, \ldots, P_{\mathsf{P}}$.

In order to avoid pathological situations, a few minimal assumptions on the partition $\mathcal{T}_h$ need to be made. Since the convergence analysis is beyond the scope of this paper, we don't need to make these assumptions explicit here and we refer the reader to [9] and [8].

In a natural way we can consider four types of unknowns attached to a partition $\mathcal{T}_h$:

• node unknowns, whose values are attached to vertices and are to be interpreted as the *value of a scalar function* at each node;

• edge unknowns, whose values are attached to edges and are to be interpreted as the *work of a vector field* along each edge;

• face unknowns, whose values are attached to faces and are to be interpreted as the *flux of a vector field* across each face;

• element unknowns, whose values are attached to elements and are to be interpreted as the *integral of a scalar function* over each element.

We denote the corresponding space of all node unknowns by $\mathcal{N}$, that of all edge unknowns by $\mathcal{E}$, that of all face unknowns by $\mathcal{F}$, and that of all element unknowns by $\mathcal{P}$.

The sign of the elements in $\mathcal{E}$ and $\mathcal{F}$ will depend on the orientation of edges and faces, respectively. We will consider that such an orientation is fixed once and for all.

In the Mimetic Finite Difference (MFD) framework these spaces can be used as discretization spaces for the problems mentioned above. $\mathcal{N}$ is the natural discretization space for functions which can be interpreted as $0-$ forms as the scalar potential p in (18), $\mathcal{E}$ is the natural discretization space for $1-$forms, as the ungauged vector potential $\mathbf{u}$ in (20), the magnetic field $\mathbf{H}$ in (19), or the electric field $\mathbf{E}$ in (21). In its turn, $\mathcal{F}$ is the natural discretization space for $2-$ forms as the electric displacement $\mathbf{D}$ in (17), the magnetic induction $\mathbf{B}$, or the current density $\mathbf{J}$. Finally, the right candidate to discretize $3-$ forms (as the charge density ρ), is clearly $\mathcal{P}$. Note that the same physical variable (as for instance the electric potential p) can be discretized in different ways according to the different circumstances (using $\mathcal{N}$ in (18) or using $\mathcal{P}$ in (17)). In equation (17), in order to guarantee the local charge conservation, the scalar potential p should be discretize by using $\mathcal{P}$, i.e., as a $3-$ form. Indeed, in the setting of differential forms, this choice can be interpreted as follows: we identify

p with its (Hodge-) dual $\star p$: $\star p$ is a $3-$ form and we discretize it.

From the point of view of algebraic topology, $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$ are cochain spaces and form a complex (a cochain complex) together with the co-boundary operator. We refer the reader to [2] for an application of these concepts to the formalization of MFD.

The co-boundary operator is a collection of operators linking our spaces one to the other. When cochains are interpreted as discrete differential forms, then the coboundary operator can be seen as a discretization of the standard differential operator d, that is, in our simplified setting, as $\mathbf{grad}$, $\mathbf{curl}$, or div depending on the space on which it acts. Here we adopt a self evident notation (as it is standard in MFD):

• The $\mathcal{GRAD}$ operator, from $\mathcal{N}$ to $\mathcal{E}$, defined as follows:
for each edge e with vertices (V_1, V_2) and oriented from V_1 to V_2

$$\Big(\mathcal{GRAD}\, u\Big)\Big|_e = u|_{V_2} - u|_{V_1}. \tag{23}$$

• The $\mathcal{CURL}$ operator, from $\mathcal{E}$ to $\mathcal{F}$, defined as follows:
for each element $\varphi \in \mathcal{E}$ and for each face f we denote by $e_1, e_2, ..., e_{\mathsf{E}_f}$ the edges sharing the face f and we suppose they are endowed with the orientation induced by the orientation of f. We consider the corresponding values $\varphi_1, \varphi_2, ..., \varphi_{\mathsf{E}_f}$ of φ with the sign corresponding to the orientation just chosen. Then $\mathcal{CURL}\,\varphi$ *on the face* f *is defined as*

$$\Big(\mathcal{CURL}\,\varphi\Big)\Big|_f = \sum_{i=1}^{\mathsf{E}_f} \varphi_i. \tag{24}$$

• The $\mathcal{DIV}$ operator, from $\mathcal{F}$ to $\mathcal{P}$, defined as follows:
let $f_1, ..., f_{\mathsf{F}_P}$ be all the faces of an element P, and for each $\sigma \in \mathcal{F}$ let $\sigma_1, ..., \sigma_{\mathsf{F}_P}$ be its values on each face that we assume to be oriented outward with respect to P. Then $\mathcal{DIV}\,\sigma$ *is defined as*

$$\Big(\mathcal{DIV}\,\sigma\Big)|_P = \sum_{k=1}^{\mathsf{F}_P} \sigma_k. \tag{25}$$

It is interesting to note that, taking in the spaces $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{P}$ the obvious canonical basis (after choosing an orientation of the edges, faces and elements in an arbitrary way, but once and for all), then the matrices associated with the operators $\mathcal{GRAD}$, $\mathcal{CURL}$, and $\mathcal{DIV}$ are the incidence matrices.

3.1 Interpolation operators

We shall now define interpolation operators $\Pi_{\mathcal{N}}$, $\Pi_{\mathcal{E}}$, $\Pi_{\mathcal{F}}$, and $\Pi_{\mathcal{P}}$ from spaces of smooth enough scalar or vector valued functions to the discrete spaces $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$, respectively. In particular for each smooth scalar function u and for each smooth vector valued function $\boldsymbol{\theta}$ we can set

- $\Pi_{\mathcal{N}} u \in \mathcal{N}$ defined by

$$(\Pi_{\mathcal{N}} u)|_V = u(V) \qquad \text{for all vertex } V; \tag{26}$$

- $\Pi_{\mathcal{E}} \boldsymbol{\theta} \in \mathcal{E}$ defined by

$$(\Pi_{\mathcal{E}} \boldsymbol{\theta})|_e = \int_e \boldsymbol{\theta} \cdot \mathbf{t} \, \mathrm{d}s \qquad \text{for all edge } e, \tag{27}$$

  where the unit tangent vector $\mathbf{t}$ indicates the orientation of e;
- $\Pi_{\mathcal{F}} \boldsymbol{\theta} \in \mathcal{F}$ defined by

$$(\Pi_{\mathcal{F}} \boldsymbol{\theta})|_f = \int_f \boldsymbol{\theta} \cdot \mathbf{n} \, \mathrm{d}S \qquad \text{for all face } f \tag{28}$$

  where the unit normal vector $\mathbf{n}$ indicates the orientation of f;
- $\Pi_{\mathcal{P}} u \in \mathcal{P}$ defined by

$$(\Pi_{\mathcal{P}} u)|_P = \int_P u \, \mathrm{d}P. \tag{29}$$

The above definitions are rather formal, and will be made precise in the sequel.

Note that the interpolation operators and the differential operators introduced above have interesting commutation properties. Namely

$$\mathcal{GRAD}\, \Pi_{\mathcal{N}} = \Pi_{\mathcal{E}}\, \mathbf{grad}\ , \quad \mathcal{CURL}\, \Pi_{\mathcal{E}} = \Pi_{\mathcal{F}}\, \mathbf{curl}\ , \quad \mathcal{DIV}\, \Pi_{\mathcal{F}} = \Pi_{\mathcal{P}}\, \mathrm{div}\, . \tag{30}$$

This property reproduces, on general polyhedral meshes, the commuting properties that link the corresponding finite elements spaces and which are fundamental for a correct discretization of mixed formulations (see e.g. [4], [12]).

3.2 Scalar products

If one wants to use the above cochain spaces in order to approximate boundary value problems as the ones discussed in the previous section, then, apparently, little can be done unless we introduce suitable *scalar products* of cochains. From the point of view of differential geometry, scalar products are an implicit discretization of the discretized Hodge-* operators that, in turn, are substantial in describing the

properties of the material we are dealing with (even when the problem is set in the vacuum).

If we want to obtain robust and reliable numerical methods, we need to mimic in a way or another the variational principles introduced in Section 2.2. This means that, at least, we have to define scalar products able to *mimic* the L^2-inner product, possibly weighted with material parameters. *To mimic* means here that the scalar products need to have "some" exactness properties that will guarantee "some" consistency.

Here we use the scalar products introduced in [7]. As it is natural, scalar products are constructed element by element, and the global L^2-like inner product is then obtained by summing over the elements. For this reason, we consider a single element P and we denote by $\mathcal{N}_P$, $\mathcal{E}_P$, $\mathcal{F}_P$, and $\mathcal{P}_P$ the corresponding cochain spaces.

We introduce the notation $[\cdot,\cdot]_{\mathcal{K},P}$ for scalar products in $\mathcal{K}$ (with $\mathcal{K} = \mathcal{N},\ \mathcal{E},\ \mathcal{F}$, or $\mathcal{P}$, respectively) *without material parameters* (or, equivalently, with material parameters set to the identity). The basic consistency requirements are imposed by following the general strategy based on reconstruction operators. The case of $\mathcal{P}_P$ actually requires a very little amount of work. Indeed we can consider the obvious reconstruction operator $R_{\mathcal{P}}$ mapping each element of $\mathcal{P}$ to the element-wise constant function having the prescribed integral on each element. Then we set

$$[p, q]_{\mathcal{P},P} := \int_P R_{\mathcal{P}} p\, \overline{R_{\mathcal{P}} q} \qquad \forall\, p,\ q \in \mathcal{P}_P. \tag{31}$$

For the other cases, we need to be a little more subtle. To start with, we ask that for all $c \in \mathbb{C}$ and $\mathbf{c} \in \mathbb{C}^3$ we have:

$$\begin{aligned}
[u, \Pi_{\mathcal{N}} c]_{\mathcal{N},P} &= \int_P R_{\mathcal{N}} u\, \bar{c} \qquad && \forall\, u \in \mathcal{N}_P \\
[\varphi, \Pi_{\mathcal{E}} \mathbf{c}]_{\mathcal{E},P} &= \int_P R_{\mathcal{E}} \varphi\, \overline{\mathbf{c}} \qquad && \forall\, \varphi \in \mathcal{E}_P \\
[\sigma, \Pi_{\mathcal{F}}\, \mathbf{c}]_{\mathcal{F},P} &= \int_P R_{\mathcal{F}} \sigma\, \overline{\mathbf{c}} \qquad && \forall\, \sigma \in \mathcal{F}_P
\end{aligned} \tag{32}$$

where $R_{\mathcal{N}}$, $R_{\mathcal{E}}$, $R_{\mathcal{F}}$ are suitable reconstruction operators which map the cochain spaces to the space of (vector) fields defined inside Ω. The reconstruction operators are supposed to verify two main properties:

(i) they are, indeed, reconstructions:

$$\Pi_{\mathcal{N}} R_{\mathcal{N}} = I_{\mathcal{N}}\,, \quad \Pi_{\mathcal{E}} R_{\mathcal{E}} = I_{\mathcal{E}}\,, \quad \Pi_{\mathcal{F}} R_{\mathcal{F}} = I_{\mathcal{F}}$$

where $I_{\mathcal{K}}$ denotes the identity on the space $\mathcal{K}$, for $\mathcal{K} = \mathcal{N},\ \mathcal{E},\ \mathcal{F}$;

(ii) they are *constant preserving*, i.e.:

$$R_{\mathcal{N}} \Pi_{\mathcal{N}} c = c \qquad R_{\mathcal{E}} \Pi_{\mathcal{E}} \mathbf{c} = \mathbf{c} \qquad R_{\mathcal{F}} \Pi_{\mathcal{F}} \mathbf{c} = \mathbf{c}\,, \qquad \forall\, c \in \mathbb{C}\,,\ \mathbf{c} \in \mathbb{C}^3.$$

Note that the operator $R_{\mathcal{P}}$, constructed above, trivially satisfies the analogue properties.

In [9], [11] and [7], it is proved that, for suitable (and rather general) families of reconstructions, the L^2 inner products with constants (or constant vectors) appearing in the right-hand sides of (32) are actually *independent of the specific reconstruction operator* (within its family) and can be computed just using a few geometric information about the polyhedron P (and thus at a very low computational cost). To introduce these results, we need some additional notation.

We denote by $\mathbf{x}_P$ the barycenter of P, and by $\boldsymbol{\xi}_f$ the barycenter of the face f. Then we denote by $\mathbf{n}_f$ the outward normal unit vector on f and finally, by $T_f(\mathbf{c})$ the tangential part of a vector $\mathbf{c}$, namely $T_f(\mathbf{c}) := \mathbf{c} - (\mathbf{c}\cdot\mathbf{n}_f)\mathbf{n}_f$. In [7] it is proved that there exists a class of constant preserving reconstructions such that for all $c \in \mathbb{C}$ and $\mathbf{c} \in \mathbb{C}^3$, for any $P \in \mathcal{T}_h$, and for any $u \in \mathcal{N}_P$, $\varphi \in \mathcal{E}_P$, and $\sigma \in \mathcal{F}_P$:

$$
\begin{aligned}
[u, \Pi_{\mathcal{N}} c]_{\mathcal{N},P} &= \int_P R_{\mathcal{N}} u\, \overline{c} = \sum_{f\in\partial P} [u|_f, \Pi^f_{\mathcal{N}}(c(\mathbf{x}-\mathbf{x}_P)\cdot\mathbf{n}_f)]_{\mathcal{N},f} \\
[\varphi, \Pi_{\mathcal{E}} \mathbf{c}]_{\mathcal{E},P} &= \int_P R_{\mathcal{E}} \varphi\, \overline{\mathbf{c}} = \sum_{f\in\partial P} [\varphi|_f, \Pi^f_{\mathcal{E}} \boldsymbol{\chi}_f]_{\mathcal{E},f} \\
&\qquad \text{with } \boldsymbol{\chi}_f = (\mathbf{n}_f\cdot(\mathbf{x}-\mathbf{x}_P))\, T_f(\mathbf{c}) + (\mathbf{n}_f\cdot\mathbf{c})\,(T_f(\mathbf{x}_P) - \boldsymbol{\xi}_f) \\
[\sigma, \Pi_{\mathcal{F}}\, \mathbf{c}]_{\mathcal{F},P} &= \int_P R_{\mathcal{F}} \sigma\, \overline{\mathbf{c}} = \sum_{f\in\partial P} \frac{1}{|f|} \sigma|_f \int_f \overline{\mathbf{c}}\cdot(\mathbf{x}-\mathbf{x}_P)\, \mathrm{d}P
\end{aligned}
\tag{33}
$$

where $\Pi^f_{\mathcal{N}}$ and $\Pi^f_{\mathcal{E}}$ are the interpolation operators restricted to the face f, and $[\cdot,\cdot]_{\mathcal{N},f}$ and $[\cdot,\cdot]_{\mathcal{E},f}$ denote the 2 dimensional scalar products. In particular $[\cdot,\cdot]_{\mathcal{N},f}$ and $[\cdot,\cdot]_{\mathcal{E},f}$ are constructed according to the same rationale described above in the 3 dimensional case, and the 2 dimensional counterpart of (32)-(33) provide the following explicit formulae: $\forall\, c \in \mathbb{C}\,,\ \mathbf{c} \in \mathbb{C}^2$:

$$
\begin{aligned}
[u, \Pi^f_{\mathcal{N}} c]_{\mathcal{N},f} &= \sum_{e\in\partial f} \overline{c}\, \left((\boldsymbol{\xi}-\boldsymbol{\xi}_f)\cdot\mathbf{n}_f\right)\Big|_e\, |e|\, \frac{u|_{V_e^1} + u|_{V_e^2}}{2} \\
[\varphi, \Pi^f_{\mathcal{E}} \mathbf{c}]_{\mathcal{E},f} &= -\sum_{e\in\partial f} \frac{1}{|e|}\, \varphi|_e \int_e \overline{\mathbf{c}}^{\perp}\cdot(\boldsymbol{\xi}-\boldsymbol{\xi}_f)\, \mathrm{d}s.
\end{aligned}
\tag{34}
$$

Finally, we require that the scalar products verify the following scaling properties: there are two constants c, C such that for all $P \in \mathcal{T}_h$

$$
\begin{aligned}
\mathsf{c}\, d(P)^3 \sum_{V\in\partial P} |\ u|_V|^2 \quad &\le [u,u]_{\mathcal{N},P} \le \quad \mathsf{C}\, d(P)^3 \sum_{V\in\partial P} |\ u|_V|^2 \quad \forall\, u \in \mathcal{N}_P \\
\mathsf{c}\, d(P) \sum_{e\in\partial P} |\ \varphi|_e|^2 \quad &\le [\varphi,\varphi]_{\mathcal{E},P} \le \quad \mathsf{C}\, d(P) \sum_{e\in\partial P} |\ \varphi|_e|^2 \quad \forall\, \varphi \in \mathcal{E}_P \\
\frac{\mathsf{c}}{d(P)} \sum_{f\in\partial P} |\ \sigma|_f|^2 \quad &\le [\sigma,\sigma]_{\mathcal{F},P} \le \quad \frac{\mathsf{C}}{d(P)} \sum_{f\in\partial P} |\ \sigma|_f|^2 \quad \forall\, \sigma \in \mathcal{F}_P,
\end{aligned}
\tag{35}
$$

where $d(P)$ denotes the diameter of the element P. Note that (33) and (35) do not fix uniquely the scalar product, but *any scalar product fulfilling* (33) ***and*** (35) *can be used to define our discrete formulations.* The problem of the "optimal" choice of scalar products (among the ones that satisfy (33) and (35)) remains an open question. From now on we suppose that we are given three scalar products fulfilling (33) and (35). We refer the reader to e.g., [11], [10] for their actual algebraic constructions.

So far, we considered only scalar products where the material properties were given by the identity tensor. In presence of material parameters, the requirements (32) must be modified, since we want to mimic a *weighted* L^2 scalar product and no longer the standard one. We concentrate on the cases of interests, which are $\mathcal{E}$ and $\mathcal{F}$. Let $\mathbb{K}$ denote a constant Hermitian, positive definite, 3×3 tensor on P. If we denote by $[\cdot,\cdot]^{\mathbb{K}}_{\mathcal{E},P}$ and $[\cdot,\cdot]^{\mathbb{K}}_{\mathcal{F},P}$ the weighted scalar products on $\mathcal{E}$ and $\mathcal{F}$ respectively, the requirements (32) modify in:

$$\begin{aligned} [\varphi, \Pi_{\mathcal{E}}\mathbf{c}]^{\mathbb{K}}_{\mathcal{E},P} = \int_P \mathbb{K}\, R_{\mathcal{E}}\varphi \cdot \overline{\mathbf{c}} \;\; &= [\varphi, \Pi_{\mathcal{E}}(\mathbb{K}\mathbf{c})]_{\mathcal{E},P} \qquad \forall\, \varphi \in \mathcal{E}_P, \\ [\sigma, \Pi_{\mathcal{F}}\,\mathbf{c}]^{\mathbb{K}}_{\mathcal{F},P} = \int_P \mathbb{K}\, R_{\mathcal{F}}\sigma \cdot \overline{\mathbf{c}} \;\; &= [\sigma, \Pi_{\mathcal{F}}(\mathbb{K}\mathbf{c})]_{\mathcal{F},P} \qquad \forall\, \sigma \in \mathcal{F}_P, \end{aligned} \tag{36}$$

and the formulae (33) modify accordingly in the sense that in the right hand sides, $\mathbf{c}$ must be replaced by $\mathbb{K}\mathbf{c}$ at each instance.

The global scalar products are then obtained by summing the contributions coming from each single element P. With a little abuse of notation, we denote now by $\mathbb{K}$ a uniformly positive and bounded, piecewise constant Hermitian 3×3 tensor, and we set $\mathbb{K}_P = \mathbb{K}|_P$. We define:

$$[\varphi, \psi]^{\mathbb{K}}_{\mathcal{E}} = \sum_P [\varphi, \psi]^{\mathbb{K}_P}_{\mathcal{E},P}\,, \qquad [\sigma, \tau]^{\mathbb{K}}_{\mathcal{F}} = \sum_P [\sigma, \tau]^{\mathbb{K}_P}_{\mathcal{F},P}\,. \tag{37}$$

The scaling requirements (35) also need to be modified, in a natural way, for instance assuming that in each P the constants c and C scale like the $trace(\mathbb{K}_P)$.

Similar requirements, with a similar notation, apply to the scalar products in $\mathcal{N}$ and in $\mathcal{P}$ in the presence of a (material dependent) uniformly positive and bounded piecewise constant α. For instance, for the scalar product in $\mathcal{N}$ we will have

$$[u, v]^{\alpha}_{\mathcal{N}} = \sum_P [u, v]^{\alpha_P}_{\mathcal{N},P}\,,$$

and each local scalar product will satisfy

$$[u, \Pi_{\mathcal{N}} c]^{\alpha_P}_{\mathcal{N},P} = \int_P R_{\mathcal{N}} u\; \alpha_P\, c = \sum_{f \in \partial P} [u|_f, \Pi^f_{\mathcal{N}}(\alpha_P\, \overline{c}(\mathbf{x} - \mathbf{x}_P) \cdot \mathbf{n}_f)]_{\mathcal{N},f}$$

together with the first of (35) with c and C scaling like α.

Finally, in $\mathcal{P}$ we will naturally have

$$[\rho,\eta]^{\alpha}_{\mathcal{P}} = \sum_P |P|\ \ \alpha_{|P}\rho_{|P}\,\overline{\eta}_{|P} = \int_P \alpha_{|P} R_{\mathcal{P}}(\rho_{|P})\,\overline{R_{\mathcal{P}}(\eta_{|P})}$$

where the reconstructions $R_{\mathcal{P}}$ are the natural ones already used in (31).

4 Mimetic finite difference discretizations

We propose here to use cochains as a discretization strategy for the differential problems mentioned in the Introduction. We denote by $\mathcal{N}_0$, $\mathcal{E}_0$, $\mathcal{F}_0$ the spaces where the natural boundary conditions are imposed; e.g., $\mathcal{N}_0$ will be the space of nodes unknowns which are zero on each node on the boundary of the computational domain Ω. Moreover, we suppose we are given piecewise constant approximations $\widetilde{\boldsymbol{\varepsilon}}$ and $\widetilde{\boldsymbol{\mu}}$ of the tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$, respectively. We suppose that:

$$\|\boldsymbol{\varepsilon} - \widetilde{\boldsymbol{\varepsilon}}\|_{L^\infty(P)} + \|\boldsymbol{\mu} - \widetilde{\boldsymbol{\mu}}\|_{L^\infty(P)} \le C d(P) \qquad \forall\, P \in \mathcal{T}_h \tag{38}$$

where C does not depend on P.

The discretization of (10) and of the corresponding variation formulation (18) will then be: *Find* $p_h \in \mathcal{N}_0$ *such that for all* $q_h \in \mathcal{N}_0$

$$[\mathcal{GRAD}\, p_h, \mathcal{GRAD}\, q_h]^{\widetilde{\boldsymbol{\varepsilon}}}_{\mathcal{E}} = [\Pi_{\mathcal{N}}\rho, q_h]_{\mathcal{N}}. \tag{39}$$

A discretization related to (39) has been studied in [8]. Indeed, in [8], the authors consider a type of consistency constraint which is different from (32) and is, in principle, weaker.

The discretization of (9) and, consequently, of (17) will in turn be: *Find* $\mathbf{D}_h$ *in* $\mathcal{F}$*, and* p_h *in* $\mathcal{P}$ *such that for all* $\mathbf{F}_h \in \mathcal{F}$*, and* $q_h \in \mathcal{P}$

$$\begin{aligned} [\mathbf{D}_h, \mathbf{F}_h]^{\widetilde{\boldsymbol{\varepsilon}}^{-1}}_{\mathcal{F}} - [p_h, \mathcal{DIV}\,\mathbf{F}_h]_{\mathcal{P}} &= 0 \\ [\mathcal{DIV}\,\mathbf{D}_h, q_h]_{\mathcal{P}} &= [\Pi_{\mathcal{P}}\rho, q_h]_{\mathcal{P}} \end{aligned} \tag{40}$$

This problem has been studied first in [9] (see also [10] and [11]).

We now turn to magnetostatics. Let $\mathcal{F}^0 = \{\varphi \in \mathcal{F} \ :\ \mathcal{DIV}\,\varphi = 0\}$. The MFD discretization of (19) reads: *Find* $\mathbf{H}_h \in \mathcal{E}$*,* $\mathbf{u}_h \in \mathcal{F}^0$ *such that*

$$\begin{aligned} [\mathbf{H}_h, \delta\mathbf{H}_h]^{\widetilde{\boldsymbol{\mu}}}_{\mathcal{E}} - [\mathbf{u}_h, \mathcal{CURL}\,\delta\mathbf{H}_h]_{\mathcal{F}} &= 0 && \forall\, \delta\mathbf{H}_h \in \mathcal{E} \\ [\delta\mathbf{u}_h, \mathcal{CURL}\,\mathbf{H}_h]_{\mathcal{F}} &= [\delta\mathbf{u}_h, \Pi_{\mathcal{F}}\mathbf{J}]_{\mathcal{F}} && \forall\, \delta\mathbf{u}_h \in \mathcal{F}^0. \end{aligned} \tag{41}$$

Note that, thanks to the commuting property (30), $\operatorname{div}\,\mathbf{J} = 0$ implies $\operatorname{div}\Pi_{\mathcal{F}}\mathbf{J} = 0$, and hence $\Pi_{\mathcal{F}}\mathbf{J} \in \mathcal{F}^0$.

The discretization of (20) is instead: *Find* $\mathbf{u}_h \in \mathcal{E}_0$ *and* $p_h \in \mathcal{N}_0$ *such that for all* $\delta\mathbf{u}_h \in \mathcal{E}_0$ *and* $\delta p_h \in \mathcal{N}_0$

$$\begin{aligned} [\mathcal{CURL}\,\mathbf{u}_h, \mathcal{CURL}\,\delta\mathbf{u}_h]_{\mathcal{F}}^{\widetilde{\boldsymbol{\mu}}^{-1}} - [\mathcal{GRAD}\,p_h, \delta\mathbf{u}_h]_{\mathcal{E}} &= [\widetilde{\mathbf{J}}, \delta\mathbf{u}_h]_{\mathcal{E}} \\ [\mathbf{u}_h, \mathcal{GRAD}\,\delta p_h]_{\mathcal{E}} &= 0. \end{aligned} \tag{42}$$

where $\widetilde{\mathbf{J}}$ denotes an approximation of $\mathbf{J}$ in $\mathcal{E}$ (for instance $\widetilde{\mathbf{J}} = \Pi_{\mathcal{E}}\mathbf{J}$, or $\widetilde{\mathbf{J}} = \Pi_{\mathcal{P}}\mathbf{J}$).

The convergence analysis for (41) and (42) is beyond the scope of this paper and is object of ongoing studies. It should be said, however, that the main building block for the well-posedness of problems (41) and (42) is the validity of a commuting diagram property (see e.g., [1]) which is valid also in our context as it is expressed in (30). These discretizations are also related to the one proposed in [13].

Finally, the discretization of problem (21) and the corresponding eigenproblem (22) can be easily written along the same lines. For example, the discretization of the source problem (21) is: *Find* $\mathbf{E}_h \in \mathcal{E}_0$ *such that for all* $\delta\mathbf{E}_h \in \mathcal{E}_0$ *it holds*

$$[\mathcal{CURL}\,\mathbf{E}_h, \mathcal{CURL}\,\delta\mathbf{E}_h]_{\mathcal{F}}^{\widetilde{\boldsymbol{\mu}}^{-1}} - \omega^2[\mathbf{E}_h, \delta\mathbf{E}_h]_{\mathcal{E}}^{\widetilde{\boldsymbol{\varepsilon}}} = i\omega[\widetilde{\mathbf{J}}, \delta\mathbf{E}_h]_{\mathcal{E}}. \tag{43}$$

As for (42) the full convergence analysis for this problem is left for the future, but it is worth saying that the building blocks for its analysis are already contained in [7] and [9].

5 Conclusion

In this paper we presented a whole set of instruments that in our opinion are capable to deal with the most common electromagnetic problems. Their main advantage is that the fundamental physical problems of the differential operators are reproduced exactly. An additional advantage is also given by their adaptability to very general geometries, meaning that essentially every polyhedral decomposition (non degenerated, in a reasonable sense) can be employed. The methods allow a certain freedom in the choice of the inner products for cochain spaces. The best use of such freedom is still an open problem. On the other hand, the numerical experiments made so far indicate that very simple and cheap choices provide already quite satisfactory results. Part of the convergence analysis has been made already, part is under study, and part will be the object of future works. However the *commuting diagram* properties constitute a very reliable promise.

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