Mathematics and Complexity in Life and Human Sciences

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The study of complex systems, namely systems of many individuals interacting in a non-linear manner, has received in recent years a remarkable increase of interest among applied mathematicians, physicists as well as researchers in various other fields as economy or social sciences.

The collective overall behavior of large systems of interacting individuals is obviously determined by the dynamics of their interactions. On the other hand, a traditional modelling of individual dynamics does not lead in a straightforward way to a mathematical description of collective emerging behaviors. In particular it is very difficult to understand and model these systems based on the sole description of the dynamics and interactions of a few individual entities localized in space and time.

This interest is however fuelled by the raising awareness that many real systems are of this kind, and by the practical experience that modelling them by traditional methods does not yield a satisfactory account of all the relevant phenomena that are observed in nature. Therefore, mathematicians and physicists are challenged by the need to look for new methodological approaches to complexity.

Furthermore, it appears that an increasing number of applications in technology, economy and social sciences can borrow techniques from complex systems with great benefit. This is due to a large number of entities involved in the dynamics and to the complex interactions among them. An additional difficulty characterizes living systems where interacting entities have the ability to organize specific strategies that depend not only on the state of the entities, but also to their number and localization.

These elements motivate the initiative of this special issue. It is devoted to collecting mathematical contributions in various fields of life and human sciences. The Editors do not conceal the hope that some general ideas and tools will emerge in the future. This volume is a contribution in this direction by bringing together various approaches. In this spirit, we hope to analyze some common features of complex systems and some of the difficulties encountered in their study. Naturally, we are far from providing an exhaustive overview. We rather select some topics which we feel are promising and provide good case studies from which one can gather further ideas.

Statistical physics has emerged as a discipline in physics in order to describe the macroscopic level from the properties at the microscopic level. It is thus not a surprise that tools taken from this field are being used by both physicists and mathematicians for the modelling of complex systems.

Obviously, modelling can be developed at various levels from the microscopic to the macroscopic scales. We claim that all these levels shed some light and have to be seen as complementary between themselves.

Regarding the microscopic scale, the methods developed in physics for the study of *disordered systems* are of particular interest. These are systems with heterogeneous interactions such as *spin glasses*. Many results obtained by physicists - and which can be taken as conjectures or *oracles* - have now been proved, see e.g. Talagrand[12]. Yet, the equilibrium and dynamical properties of such systems are far from being fully understood from a mathematical point of view. Undoubtedly, progress in this area would be very useful for the modelling of complex systems in social science. At another scale, generalizations of the mathematical kinetic theory (or of master equation methods for multi-agent systems) can be developed by assigning the description of the overall state of the system by a probability distribution over the microscopic state of the interacting entities.

Many further models of nonlinear partial differential equations are useful for macroscopic descriptions of phenomena such as diffusion, propagation, pattern formation, control etc. An important research objective, in the modelling of large living systems by PDEs, consists in the derivation of models at the macroscopic scale from the underlying description at the microscopic scale. It is a challenging target for applied mathematicians that, at present, has been solved only in the case of very particular models.

In any case one should keep in mind that the above representations are only partial. Indeed the number of interacting entities may not be large enough to justify either the continuum mechanics approximation, or the statistical representation by continuous probability distribution functions over the microscopic states according to the methods of kinetic theory. On the other hand, when the number of interacting entities is too large the approach at the microscopic scale can lead to highly challenging computational problem.

In some instances of models at the microscopic scale, one should take into account the fact that the strategy developed by interacting individuals depends also on their number within the influence area of interaction and on their localization. For instance, recent studies[2] conjecture, on the basis of empirical data, that some groups of animals develop a common strategy based on interactions depending on topological rather than metric distances. This opens some very interesting avenues in modelling.

An additional and much relevant problem that one has to deal with is heterogeneity. The ability of individuals to organize interactions according to well defined objectives or strategies often exhibits this feature. The self-organizing ability is not the same for all individuals, and may be regarded as a random variable linked to a probability distribution. Furthermore, this distribution may have a local nature and may be modified by several types of interactions at the microscopic level. Therefore, modelling that includes heterogeneity and stochastic features are also quite useful. From this point of view, one is often led to consider interaction models that do not follow the laws of classical mechanics, but include the effects of the different strategies developed by the interacting individuals.

The above mentioned characteristics already appear in a 2008-issue[3] devoted to traffic, crowds, and swarms. These are specific systems that show several common features belonging to the realm of complex systems. The papers published in that issue use an array of different approaches. For instance the modelling approach may possibly use stochastic differential equations[1] or master equations obtained by suitable limits[7]. Both papers by Albeverio and al.[1] and Degond et al.[7] deal with the modelling of aggregation phenomena of interacting individuals. Methods of continuum physics, although less used than methods of statistical physics, can be, in some cases, properly developed to describe complex behaviors such as clustering of criminality[4]. As already mentioned the identification of general ideas and methods represent one of the most challenging objectives of the research in the mathematics of complex systems.

As we already said this special issue is focused on some complex systems in life and human sciences. Six research articles cover the following topics: decision making among heterogenous and interacting agents, social dynamics, spread of epidemics in the case of virus mutations, clustering phenomena in flocks, multiscale biological systems.

The paper by Cucker and Dong[5] provides a mathematical analysis of the critical exponent introduced in a model of flocks under hierarchical leadership by Cucker and Smale[6]. A proof of convergence had already been given in the original paper. This result is improved here by showing that unconditional convergence holds indeed for weak interactions, extending the previous result about strong interactions.

The paper by De Lillo, Delitala, and Salvatori[8] concerns the onset and the spread of epidemics in the case of epidemics involving a virus that mutates in time due to a Darwinian selection. The approach is based on the generalized kinetic theory for active particles. It includes the heterogeneous distribution of the degree of malignancy of the affection and a detailed analysis of the role of the immune system. Numerical simulations allow one to visualize the emerging collective behavior and to shed light on the role of therapeutical actions.

Generalizing econometric discrete choice models, statistical mechanics models of choice under social influence have been developed in the recent years by both physicists and economists, making use of the framework of Ising spin models (also called Markov random fields in the mathematics literature), with heterogeneous interactions and/or local fields. Two papers in this issue make use of this framework. First, the paper by Gallo, Barra and Contucci[9] proposes a parameter evaluation procedure. It also provides a test by fitting the model against three families of data coming from different cases. The final result provide a quantitative confirmation of the peer imitation behavior found in social psychology.

Second, the contribution by Gordon, Nadal, Phan and Semeshenko[10] analyzes the case of heterogenous idiosyncratic preferences and positive interactions (cases of strategic complementarity) which are responsible for the appearance of multiple equilibria in collective behavior. This paper exhibits generic properties of the boundaries limiting the regions (in the parameters space) where the system presents different types of equilibria (unique or multiple).

The paper by Herrero, Köhn, and Pérez-Pomares[11] deals with the modelling of complex biological processes focused on blood vessel development. More precisely, this work summarizes their current state of knowledge about vasculogenesis, that is the formation of the primary vasculature (including arteries and veins). This process occurs mainly at the embryonic stage, while angiogenesis (the formation of new vessels out of a preexisting vasculature) continues to hold during all life. A relevant complexity source consists in linking macroscopic behaviors to the underlying description at the microscopic scale. An additional difficulty consists in capturing by mathematical equations the complex geometrical patterns generated by morphological growth. The authors provide a critical analysis of a large literature that can be a useful reference for applied mathematicians interested in the modelling of multiscale biological phenomena.

The paper by Zhao, Kirou, Ruszczycki, and Johnson[13] is focused on the modelling of coalescence and fragmentation phenomena related to social systems in which a typical fragmentation process corresponds to an entire group breaking up, as opposed to the typical binary splitting studied in physics and biology. Specific applications concern financial markets. A new model is proposed for financial market dynamics based on the combination of internal clustering (i.e. herding) dynamics with human decision-making. Resulting fluctuations in price movements are close to what is observed empirically. This leads us to speculate that a combination of dynamical clustering and decision-making may be a key element for developing quantitative models for the dynamics of social phenomena.

Our hope is that this issue, as well as the forthcoming one (also about complex systems) will contribute to a deeper understanding of the mathematical strategies to be used in modelling large complex living systems.

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