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Bubble stabilization of Discontinuous Galerkin methods

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Abstract — We analyze the stabilizing effect of the introduction of suitable bubble functions in DG formulations for linear second order elliptic problems, working, for the sake of simplicity, on Laplace operator. In particular we find that the addition of a single bubble per element can stabilize the non-symmetric formulation of Baumann-Oden

Keywords: Discontinuous Galerkin, Bubble stabilizations, Elliptic Problems, Baumann-Oden method

1. INTRODUCTION

Most commonly used DG methods need the addition of suitable stabilizing terms in order to provide good convergence properties. The typical stabilizing procedure consists in the introduction of *penalty terms* that penalize the jumps across neighboring elements. Sometimes, in hyperbolic or in convection dominated problems, one also use *upwind techniques*, consisting in replacing the *average* $((u^+ + u^-)/2)$ on an internal edge with the upwind value (that is, u^+ or u^-, according with the direction of the "wind"). This however, in most cases, can be seen again as a jump stabilization ([15], [13], [10]).

Another possible way of stabilizing DG methods consists in the addition of suitable terms (this time, internal to each element) of the so-called *Hughes-Franca* type: in general, the integral of the original equation (or one of the original equations), written in strong form *inside each element* in terms of the finite element unknowns (= trial functions), multiplied by a similar expression acting on the test functions. The most famous stabilization of this type, for standard Galerkin methods, is surely the SUPG stabilization of convection dominated equations [12]. A typical problem, in these cases, is the choice of the proper *stabilization coefficient* to be put in front of the stabilizing term.

In a recent paper (see [6]) we pointed out that, in DG methods, the jumps are themselves to be regarded as "equations", so that jump stabilizations (and hence *upwind*) could be regarded as Hughes-Franca stabilizations as well. And, indeed, the optimal choice of the coefficient in a jump-stabilization term is still a subject that might need a further investigation.

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In standard Galerkin methods (for instance in Stokes problem or in advection-diffusion problems) one of the possible ways of stabilizing an unstable formulation is to add one or more *bubble function* per element. We recall that a bubble function is, by definition, a function whose support is contained in a single element. The bubble stabilization, in its turn, can also be seen as a Hughes-Franca stabilization after eliminating the bubbles by static condensation. This has the effect of shifting the problem of choosing the optimal coefficient into the problem of choosing the optimal shape of the bubble (see e.g. [5], [2]). This last problem can however be solved, in some cases, with the use of Residual Free Bubbles (see [11], [14]), or Pseudo Residual Free Bubbles (see [8], [9]).

When using a discontinuous method the addition of bubble functions does not mean much, as all the basis functions already have support in a single element (hence, in a sense, they are *all*, already, bubbles). We could therefore consider that for DG methods adding bubbles is just the same as augmenting the finite element space, in an arbitrary way. For instance, in two dimensions, shifting from linear discontinuous elements to quadratic discontinuous elements could be seen as adding three bubbles per element (corresponding to x^2, y^2, and xy). The same is obviously true for any other increase of the local polynomial degree.

The problem whether the addition of bubbles could provide some additional stability for DG methods has therefore a rather academic nature. However, it is intellectually tackling to check whether and when a suitable (and possibly minimal) increase in the finite element space can turn an unstable formulation into a stable one. And, possibly, any discovery in this direction can provide some additional understanding of the underlying nature of DG methods.

Here we consider as a model (toy) problem the Poisson problem in a polygonal domain, and we address our attention to the so-called Baumann-Oden DG formulation ([3], [4], [16], [17], and many other papers). In particular we consider the (unstable) choice of piecewise linear discontinuous elements. This case is particularly easy, for our purposes, since we already know that the corresponding choice of piecewise quadratic elements (always for the Baumann-Oden formulation) *is* indeed stable ([17]). Hence we know already that, in some sense, adding three bubbles per element can stabilize the problem. What we address here is therefore the question whether it would be possible to stabilize the piecewise linear Baumann-Oden formulation *adding less than three bubbles per element*. Indeed we prove that the addition of *one* bubble per element can lead to a stable and converging method.

The practical impact of our investigation is surely questionable, although the possibility of avoiding the jump stabilization for linear elements is surely appealing, as it leads to a more “natural”choice of the interelement fluxes. Moreover we believe that our analysis provides a better understanding of some basic aspects and mechanisms related to DG methods, that might be of some help in designing new future methods. And as such, it might interest several curious scientists, as for instance Yuri Kuznetsov, to whom this little paper is dedicated.

An outline of the paper is as follows. In the next section we recall some notation on DG methods, and the Baumann-Oden formulation for Poisson problem. Then we introduce the bubble stabilization and prove stability of the augmented formulation, and optimal error estimates.

2. THE MODEL PROBLEM AND THE BAUMANN-ODEN METHOD

Let Ω be a convex polygonal domain, with boundary $\partial\Omega$. For every f, say, in $L^2(\Omega)$ we consider the model problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

It is well known that problem (2.1) has a unique solution, that belongs to $H^2(\Omega) \cap H^1_0(\Omega)$.

Let $\mathscr{T}_h$ be a decomposition of Ω into triangles T, with the usual *minimum angle* condition, let $\mathscr{E}_h^0$ be the set of internal edges of $\mathscr{T}_h$, and $\mathscr{E}_h$ the set of all the edges. We consider first the (infinite dimensional) space $V(\mathscr{T}_h)$ defined as

$$V(\mathscr{T}_h) = \{v \in L^2(\Omega) \text{ such that } v_{|T} \in H^2(T)\ \forall T \in \mathscr{T}_h\}. \tag{2.2}$$

Elements $v \in V(\mathscr{T}_h)$ will, in general, be discontinuous when passing from one element to a neighboring one. As usual in DG methods we have therefore to introduce boundary operators as *averages* and *jumps*. As we shall deal also with vector-valued functions which are smooth in each triangle but discontinuous from one triangle to another, we shall introduce these boundary operators for scalar and for vector-valued functions. Following [1] we set (see Figure 1):

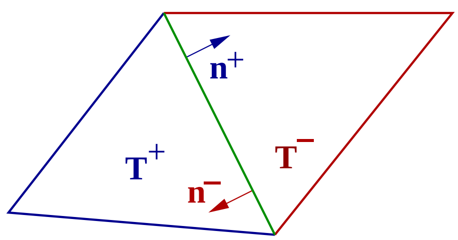

Figure 1. Two neighboring triangles and their normals.

$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+n^+ + v^-n^- \qquad \text{for all internal edges}$$
$$\{\tau\} = \frac{\tau^+ + \tau^-}{2}; \qquad [\![\tau]\!] = \tau^+n^+ + \tau^-n^- \qquad \text{for all internal edges}$$

On the boundary edges we define $[\![v]\!] = vn; \qquad \{\tau\} = \tau.$

We introduce now some further notation. For functions in $V(\mathscr{T}_h)$ we first introduce the elementwise gradient ∇_h, and then for u and v in $V(\mathscr{T}_h)$ we set

$$(\nabla_h u, \nabla_h v) := \sum_{T\in\mathscr{T}_h} \int_T \nabla u \cdot \nabla v \mathrm{d}x, \quad < \{\nabla_h u\}, [\![v]\!] > := \sum_{e\in\mathscr{E}_h} \int_e \{\nabla_h u\} \cdot [\![v]\!] \mathrm{d}s.$$

Setting, for u and v in $V(\mathscr{T}_h)$

$$a(u,v) := (\nabla_h u, \nabla_h v) - < \{\nabla_h u\}, [\![v]\!] > + < \{\nabla_h v\}, [\![u]\!] >, \tag{2.3}$$

the Baumann-Oden “continuous”formulation of (2.1) is now

$$\begin{cases} \text{Find } u \in V(\mathscr{T}_h) \text{ such that, } \forall v \in V(\mathscr{T}_h) : \\ a(u,v) = (f,v). \end{cases} \tag{2.4}$$

In $V(\mathscr{T}_h)$ we define the jump seminorm

$$\|v\|_j^2 = \sum_{e\in\mathscr{E}_h} \frac{1}{|e|} \int_e |[\![v]\!]|^2 \mathrm{d}s, \tag{2.5}$$

and the norm

$$\|v\|^2_{V(\mathscr{T}_h)} := \sum_{T\in\mathscr{T}_h} \left(\|\nabla_h v\|^2_{0,T} + h_T^2 \, |v|^2_{2,T} \right) + \|v\|_j^2. \tag{2.6}$$

We recall now the following useful result due to Agmon (see, e.g., [1]):

$$\forall T,\ \forall e \in \partial T,\ \forall v \in H^1(T) : \quad \int_e v^2 \mathrm{d}s \leqslant C_a (h_T^{-1} \|v\|^2_{0,T} + h_T |v|^2_{1,T}), \tag{2.7}$$

with C_a only depending on the minimum angle of T. Hence we have

$$\begin{aligned} | < \{\tau\}, [\![v]\!] > | &= \sum_{e\in\mathscr{E}_h} \int_e \{\tau\} \cdot [\![v]\!] \mathrm{d}s \\ &\leqslant C \left[\sum_T (\|\tau\|^2_{0,T} + h_T^2 |\tau|^2_{1,T}) \right]^{1/2} \left[\sum_{e\in\mathscr{E}_h} h_e^{-1} \int_e |[\![v]\!]|^2 \mathrm{d}s \right]^{1/2}, \end{aligned} \tag{2.8}$$

for all τ that are in $(H^1(T))^2$ for every T, and for all $v \in V(\mathscr{T}_h)$. From (2.8) we easily deduce the following proposition.

Proposition 2.1. *There exist a constant, that we again denote by C_{cont}, depending only on the minimum angle in $\mathscr{T}_h$, such that*

$$a(u,v) \leqslant C_{cont} \|u\|_{V(\mathscr{T}_h)} \|v\|_{V(\mathscr{T}_h)} \qquad \forall u,v \in V(\mathscr{T}_h). \tag{2.9}$$

3. APPROXIMATION

We set, for every element T,

$$
\begin{aligned}
V(T) &:= \{v| \quad v = a + bx + cy + d(x^2+y^2)\},\\
\Sigma(T) &:= \nabla(V(T)) = RT_0(T),
\end{aligned}
$$

where $RT_0(T)$ denotes the lowest order Raviart-Thomas space over the element T. We then extend our spaces to the whole Ω setting

$$
V_h := \prod_T V(T), \qquad \Sigma_h := \prod_T \Sigma(T).
$$

The discrete problem is then:

$$
\begin{cases}
\text{Find } u_h \in V_h \text{ such that, } \forall v \in V_h: \\
(\nabla_h u_h, \nabla_h v) - <\{\nabla_h u_h\}, [\![v]\!]> + <\{\nabla_h v\}, [\![u_h]\!]> = (f, v),
\end{cases}
\tag{3.1}
$$

where, here and in all the rest of the paper, the bilinear form $a(u,v)$ is defined in (2.3).

In the finite element space V_h we introduce the usual DG norm

$$
|||v|||^2 = |v|^2_{1,h} + \|v\|^2_j, \tag{3.2}
$$

and we note immediately that, with a simple use of the inverse inequality, we have that *on V_h the DG norm* (3.2) *is equivalent to the norm* (2.6) *originally introduced in $V(\mathscr{T}_h)$*. In particular we have

$$
\|v_h\|_{V(\mathscr{T}_h)} \leqslant C_{inv} |||v_h||| \leqslant C_{inv} \|v_h\|_{V(\mathscr{T}_h)} \qquad v_h \in V_h. \tag{3.3}
$$

Hence (2.8) can be simplified to

$$
\begin{aligned}
| <\{\tau\}, [\![v]\!]> | &\leqslant C \left[\sum_T (\|\tau\|^2_{0,T} + h_T^2 |\tau|^2_{1,T}) \right]^{1/2} \left[\sum_{e \in \mathscr{E}_h} h_e^{-1} \int_e |[\![v]\!]|^2 \mathrm{d}s \right]^{1/2} \\
&\leqslant C_s \|\tau\|_0 \|v\|_j
\end{aligned}
\tag{3.4}
$$

for all $\tau \in \Sigma_h$ and for all $v \in V_h$. Hence we immediately have the following result.

Proposition 3.1. *There exist a constant C_{cont}, depending only on the minimum angle in $\mathscr{T}_h$, such that*

$$
a(u_h, v_h) \leqslant C_{cont} |||u_h||| \, |||v_h||| \qquad \forall u_h, v_h \in V_h. \tag{3.5}
$$

Our main task will now to prove *stability* of the bilinear form $a(u,v)$ in the DG norm (3.2). This however will not be done by showing *ellipticity* of the bilinear form a, but rather by proving that there exists a mapping $S : V_h \to V_h$ such that

$$\sup_v \frac{a(u,v)}{|||v|||} \geqslant \frac{a(u,S(u))}{|||S(u)|||} \geqslant K\,|||u||| \qquad \forall u \in V_h \tag{3.6}$$

for a suitable constant K depending only on the minimum angle of $\mathscr{T}_h$. The target (3.6) will be reached by constructing an operator S which is *bounded*

$$|||S(u)||| \leqslant K_2 |||u|||, \tag{3.7}$$

and *bounding*

$$a(u,S(u)) \geqslant K_1 |||u|||^2, \tag{3.8}$$

so that (3.6) will follow with $K = K_1/K_2$. The construction of the operator S is the main difficulty of this paper, and it will be done in several steps.

To start with, for every element T and every $\tau \in \Sigma(T)$ we define its *potential* $p(\tau)$ by

$$\nabla p(\tau) = \tau \quad \text{ and } \quad \int_T p(\tau) = 0.$$

Note that p is *one-to-one* from $\Sigma(T)$ to the subset of $V(T)$ of functions having zero mean value on T.

We then extend the above definitions globally, defining $p : \Sigma_h \to V_h$ in the (obvious) element by element way, and we note that every $v \in V_h$ can be split in a *unique* way as

$$v = v_0 + v_1 \quad \textit{with} \quad v_0 = \text{ piecewise constant } \quad \textit{and } v_1 = p(\nabla v). \tag{3.9}$$

We shall now prove the boundedness of the p operator.

Proposition 3.2. *There exists a constant C_p, depending only on the minimum angle of the decomposition $\mathscr{T}_h$, such that*

$$|||p(\tau)||| \leqslant C_p \|\tau\|_{0,\Omega} \tag{3.10}$$

for all $\tau \in \Sigma_h$.

Proof. We first note that, since $p(\tau)$ has zero mean value in each triangle, we have

$$\forall T,\, \forall \tau \in \Sigma(T) : \;\; \|p(\tau)\|^2_{0,T} \leqslant C_1 h_T^2 |p(\tau)|^2_{1,h} = C_1 h_T^2 \|\tau\|^2_0.$$

Hence, using again the Agmon inequality (2.7) we deduce that

$$\begin{aligned}
\forall \tau \in \Sigma_h : \quad |||p(\tau)|||^2 &= \|\tau\|_0^2 + \|p(\tau)\|_j^2 \\
&\leqslant \|\tau\|_0^2 + CC_a \sum_T \frac{1}{h_T}(h_T^{-1}\|p(\tau)\|^2_{0,T} + h_T|p(\tau)|^2_{1,T}) \\
&\leqslant (1 + CC_aC_1 + CC_a)\|\tau\|_0^2,
\end{aligned}$$

and the result (3.10) follows immediately. $\square$

Next, we construct a mapping L from the space of piecewise constant scalars to the space Σ_h. For v_0 piecewise constant, we define

$$\tau = L(v_0) \qquad \text{iff} \qquad \tau \cdot n_e = \frac{1}{|e|}[\![v_0]\!] \cdot n_e \qquad \forall \text{ edge } e \in \mathscr{E}_h, \tag{3.11}$$

n_e being one of the two normal directions to e. The following two properties of the map L will play an important role in our analysis. The first is immediate, but important, and we state it as a proposition.

Proposition 3.3. *Let L be the operator defined in* (3.11)*. Then for every piecewise constant v_0 we have*

$$\sum_{e \in \mathscr{E}_h} \int_e L(v_0) \cdot [\![v_0]\!] \mathrm{d}s = |||v_0|||^2. \tag{3.12}$$

Proof. Equality (3.12) follows immediately from the definitions (3.11) (of L) and (3.2) (of the DG norm), taking into account that for a piecewise constant v_0 we have $|v_0|_{1,h} = 0$. □

The second property expresses the continuity (uniform in h) of the mapping $v_0 \to p(L(v_0))$.

Proposition 3.4. *Let L be the operator defined in* (3.11)*. Then there exists a constant γ, depending only on the minimum angle of the decomposition, such that*

$$\|p(L(v_0))|||^2_{0,\Omega} \leqslant \gamma^2 |||v_0|||^2 \tag{3.13}$$

for every piecewise constant v_0.

Proof. The proof will follow easily if we show that

$$\|L(v_0)\|^2_{0,\Omega} \leqslant C_i{}^2 |||v_0|||^2 = C_i{}^2 \|v_0\|^2_j \tag{3.14}$$

for all piecewise constant v_0, with a constant C_i depending only on the minimum angle in the decomposition. Indeed, if we have (3.14) then the required (3.13) will follow using (3.10):

$$\|p(L(v_0))|||^2_{0,\Omega} \leqslant C_p^2 \|L(v_0)\|^2_{0,\Omega} \leqslant C_p^2 C_i^2 |||v_0|||^2 =: \gamma^2 |||v_0|||^2. \tag{3.15}$$

Hence we have just to prove (3.14). This can be easily done by a usual *scaling argument*. We hint the procedure for convenience of the interested readers. For every triangle T we consider a "reference"triangle $\hat{T}$, homothetic to T and with diameter equal to 1. Then for every piecewise constant v_0 we define $\hat{\tau} \in \Sigma(\hat{T})$ by

$$\hat{\tau} \cdot n_e = \frac{1}{|\hat{e}|} \widehat{[\![v_0]\!]} \cdot n_e \qquad \forall \hat{e} \in \partial \hat{T}. \tag{3.16}$$

Note that we did not distinguish between n_e and $\hat{n}_{\hat{e}}$ since the two triangles T and $\hat{T}$ are homothetic. From (3.16), denoting by τ the restriction to T of $L(v_0)$, we have

$$\tau = h_T^{-1}\hat{\tau},$$

since, from the homothety, $|e| = h_T\,|\hat{e}|$ for two corresponding edges e and $\hat{e}$. Moreover we have $|\hat{T}| = h_T^{-2}|T|$, so that

$$\|L(v_0)\|^2_{0,T} = \|\hat{\tau}\|^2_{0,\hat{T}}. \tag{3.17}$$

It is also clear that on $\hat{T}$ there will be a constant $C_{\hat{T}}$, depending only on the minimum angle of $\hat{T}$, such that

$$\|\hat{\tau}\|^2_{0,\hat{T}} \leqslant C_{\hat{T}} \sum_{\hat{e}\in\partial\hat{T}} \frac{\|\widehat{[\![v_0]\!]}\|^2_{0,\hat{e}}}{|\hat{e}|}. \tag{3.18}$$

Finally we have easily

$$\sum_{\hat{e}\in\partial\hat{T}} \frac{\|\widehat{[\![v_0]\!]}\|^2_{0,\hat{e}}}{|\hat{e}|} = \sum_{e\in\partial T} \frac{\|[\![v_0]\!]\|^2_{0,e}}{|e|}. \tag{3.19}$$

Combining (3.17)-(3.19) we have

$$\|L(v_0)\|^2_{0,T} = \|\hat{\tau}\|^2_{0,\hat{T}} \leqslant C_{\hat{T}} \sum_{\hat{e}\in\partial\hat{T}} \frac{\|\widehat{[\![v_0]\!]}\|^2_{0,\hat{e}}}{|\hat{e}|} = C_{\hat{T}} \sum_{e\in\partial T} \frac{\|[\![v_0]\!]\|^2_{0,e}}{|e|}. \tag{3.20}$$

A usual continuity argument (for the dependence of $C_{\hat{T}}$ on the *shape* of $\hat{T}$) allows to use the "compactness of the *shape* of $\hat{T}$", due to the minimum angle assumption. The argument is further simplified by the fact that Σ is rotation-invariant, so that we can assume that one vertex of $\hat{T}$ is in the origin, and that the longest edge of $\hat{T}$ is a subset of the positive horizontal axes (see Figure 2). The shape of $\hat{T}$ will then depend only on the position of the last vertex P that, due to the minimum angle condition, can only vary in a closed bounded set. Hence there exists a $\hat{C}$, depending only on the minimum angle, such that

$$\|L(v_0)\|^2_{0,T} \leqslant \hat{C} \sum_{e\in\partial T} \frac{1}{|e|} \|[\![v_0]\!]\|^2_{0,e} \tag{3.21}$$

for every piecewise constant function v_0 and for every triangle $T \in \mathscr{T}_h$. Summing over the triangles we conclude the proof. □

The mapping $v_0 \to L(p(v_0))$ will be *the* crucial ingredient of our target operator S. Let us see some of its properties. As a first step,

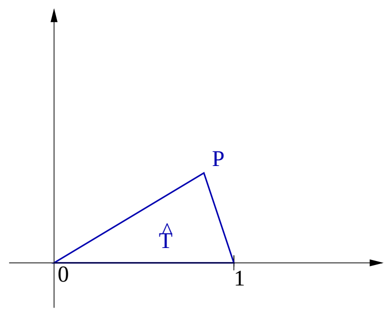

Figure 2. The reference triangle $\hat{T}$

Proposition 3.5. *There exists a constant K_0, depending only on the minimum angle in $\mathscr{T}_h$, such that*

$$a(u, p(L(u_0))) \geqslant |||u_0|||^2 - K_0 \, |u|_{1,h} \, |||u_0||| \tag{3.22}$$

for all $u \in V_h$, where u_0 is obtained from u through the splitting (3.9).

Proof. For $u \in V_h$, with $u = u_0 + u_1 = u_0 + p(\nabla_h u)$ as in (3.9), we have

$$\begin{aligned}
&a(u, p(L(u_0))) = (\nabla_h u, L(u_0)) - < \{\nabla_h u\}, [\![p(L(u_0))]\!] > + < \{L(u_0))\}, [\![u]\!] > \\
&= (\nabla_h u, L(u_0)) - < \{\nabla_h u\}, [\![p(L(u_0))]\!] > + < \{L(u_0))\}, [\![p(\nabla_h u)]\!] > + < \{L(u_0))\}, [\![u_0]\!] > \\
&= (\nabla_h u, L(u_0)) - < \{\nabla_h u\}, [\![p(L(u_0))]\!] > + < \{L(u_0))\}, [\![p(\nabla_h u)]\!] > + |||u_0|||^2.
\end{aligned}$$

Using this, Cauchy-Schwarz inequality, (3.4), (3.13)-(3.14), and finally (3.10) we then have

$$\begin{aligned}
&|||u_0|||^2 = a(u, p(L(u_0))) - (\nabla_h u, L(u_0)) + < \{\nabla_h u\}, [\![p(L(u_0))]\!] > - < \{L(u_0))\}, [\![p(\nabla_h u)]\!] > \\
&\leqslant a(u, p(L(u_0))) + |u|_{1,h} \, \|L(u_0)\|_{0,\Omega} + C_s \, |u|_{1,h} \, |||p(L(u_0))||| + C_s \, \|L(u_0)\|_{0,\Omega} |||p(\nabla_h u)||| \\
&\leqslant a(u, p(L(u_0))) + |u|_{1,h} C_i \, |||u_0||| + C_s \, |u|_{1,h} \, \gamma |||u_0||| + C_s C_i |||u_0||| \, C_1 |u|_{1,h} \\
&= a(u, p(L(u_0))) + K_0 \, |u|_{1,h} \, |||u_0|||,
\end{aligned}$$

which is inequality (3.22). □

The operator S will then be constructed as

$$S(u) := u + \alpha p(L(u_0) \tag{3.23}$$

for a suitable choice of α. It is clear that S, constructed as in (3.23), will be bounded. In particular we shall have

$$\begin{aligned}
|||u + \alpha p(L(u_0))|||^2 &\leqslant 2(|||u|||^2 + |||\alpha p(L(u_0))|||^2) \leqslant 2(|||u|||^2 + \gamma^2 \alpha^2 |||u_0|||^2) \\
&\leqslant {K_2}^2 |||u|||^2,
\end{aligned} \tag{3.24}$$

that is precisely the *boundedness* property (3.7), with K_2 depending only on α (still to be chosen) and on the minimum angle of the decomposition. Let us see that the *bounding* property (3.8) is also verified, for α small enough. Indeed, we remark first from the definition (2.3) of the bilinear form a that for all $v \in V_h$

$$a(v,v) = |v|^2_{1,h} \tag{3.25}$$

which is indeed the nicest feature of the Baumann-Oden formulation, compared with other DG formulations. Then we choose $\alpha := 2/(1+K_0^2)$, and we have

$$\begin{aligned}
&a(u,u+\alpha p(L(u_0))) = |u|^2_{1,h} + a(u,\alpha p(L(u_0)))\\
&\geqslant |u|^2_{1,h} + \alpha(|||u_0|||^2 - K_0\,|u|_{1,h}\,|||u_0|||)\\
&= \frac{\alpha}{2}(|u|^2_{1,h} + |||u_0|||^2) + (1-\frac{\alpha}{2})|u|^2_{1,h} + \frac{\alpha}{2}|||u_0|||^2 - \alpha K_0\,|u|_{1,h}\,|||u_0|||\\
&= \frac{1}{1+K_0^2}\,(|u|^2_{1,h} + |||u_0|||^2) + \frac{1}{1+K_0^2}(K_0|u|_{1,h} - |||u_0|||)^2\\
&\geqslant \frac{1}{1+K_0^2}\,(|u|^2_{1,h} + |||u_0|||^2).
\end{aligned} \tag{3.26}$$

On the other hand, using (3.9) and then (3.10) we have

$$|||u|||^2 \leqslant 2(|||u_0|||^2 + |||p(\nabla u)|||)^2 \leqslant 2(|||u_0|||^2 + C_p^2|u|^2_{1,h}), \tag{3.27}$$

which combined with (3.26) gives

$$a(u,u+\alpha p(L(u_0))) \geqslant K_1|||u|||^2, \tag{3.28}$$

with K_1 depending only on K_0 and C_p, that is (3.8).

We summarize the result in the following theorem

Theorem 3.1. *There exists a constant K, depending only on the minimum angle of $\mathscr{T}_h$, such that: for every $u_h \in V_h$ there exists a v_h (= $S(u_h)$) in V_h, different from zero, such that*

$$a(u_h,v_h) \geqslant K|||u_h|||\,|||v_h|||. \tag{3.29}$$

Now it is classical to deduce the error estimate.

Theorem 3.2. *In the above assumptions, for every $f \in L^2(\Omega)$ the discrete problem* (3.1) *has a unique solution u_h. Moreover the distance between u_h and the solution u of* (2.1) *can be estimated as*

$$\|u-u_h\|_{V(\mathscr{T}_h)} \leqslant C\,h\,|u|_{2,\Omega}. \tag{3.30}$$

where C is a constant depending only on the minimum angle of $\mathscr{T}_h$.

Proof. The proof is now classical. We start by defining u_I as the piecewise linear interpolant of u. Then we use (3.3), then (3.29), then Galerkin orthogonality, then (2.9), then again (3.3) to obtain

$$\begin{aligned}
\|u_h-u_I\|_{V(\mathscr{T}_h)} \leqslant C_{inv}\,|||u_h-u_I||| &\leqslant \frac{C_{inv}}{K}\frac{a(u_h-u_I,S(u_h-u_I))}{|||S(u_h-u_I)|||}\\
&= \frac{C_{inv}}{K}\frac{a(u-u_I,S(u_h-u_I))}{|||S(u_h-u_I)|||}\\
&\leqslant \frac{C_{inv}}{K}\frac{C_{cont}\|u-u_I\|_{V(\mathscr{T}_h)}\|S(u_h-u_I))\|_{V(\mathscr{T}_h)}}{|||S(u_h-u_I)|||}\\
&\leqslant \frac{C_{inv}}{K}\frac{C_{cont}\|u-u_I\|_{V(\mathscr{T}_h)}C_{inv}|||S(u_h-u_I))|||}{|||S(u_h-u_I)|||}\\
&= \frac{C_{inv}^2C_{cont}}{K}\|u-u_I\|_{V(\mathscr{T}_h)}
\end{aligned}\tag{3.31}$$

and the result follows from usual interpolation estimates. □

Remark 3.1. We want to point out that our choice of the bubble $(=d(x^2+y^2))$ was made in order to simplify the analysis. The nice feature of this bubble is that $\nabla V(T)\equiv RT_0(T)$, and we can then use all the well known properties of the lowest order Raviart-Thomas elements. Clearly other bubbles could do the job, possibly with some additional work. For instance, any bubble such that the average of its normal derivative is positive on each edge should work.

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