CCM, Part III

Daniele Boffi

Dipartimento di Matematica, Università di Pavia
http://www-dimat.unipv.it/boffi

Complexity and its Interdisciplinary Applications

Convection diffusion equation

As usual... a one dimensional example

$$\begin{cases} -\varepsilon u''(x) + bu'(x) = 0 & 0 < x < 1 \\ u(0) = 0, \ u(1) = 1 \end{cases}$$

Non-homogeneous boundary conditions (!)

Convection diffusion equation

As usual... a one dimensional example

$$\begin{cases} -\varepsilon u''(x) + bu'(x) = 0 & 0 < x < 1 \\ u(0) = 0,\ u(1) = 1 \end{cases}$$

Non-homogeneous boundary conditions (!)
Péclet number $\mathcal{P} = |b|L/(2\varepsilon)$ ($L = 1$ in our case)

Convection diffusion equation

As usual... a one dimensional example

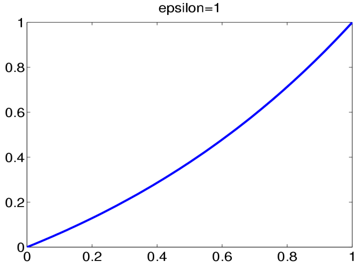
$$\begin{cases} -\varepsilon u''(x) + bu'(x) = 0 \qquad 0 < x < 1 \\ u(0) = 0,\ u(1) = 1 \end{cases}$$

Non-homogeneous boundary conditions (!)
Péclet number $\mathcal{P} = |b|L/(2\varepsilon)$ ($L = 1$ in our case)
Closed form solution can be explicitly computed

$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$

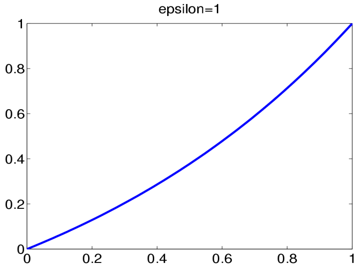
Convection diffusion equation (cont'ed)

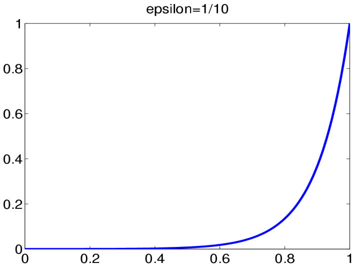
$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$



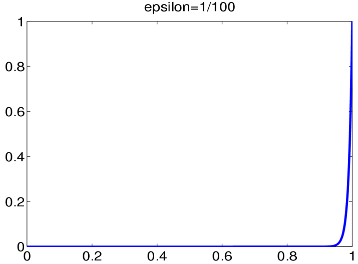
Convection diffusion equation (cont'ed)

$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$

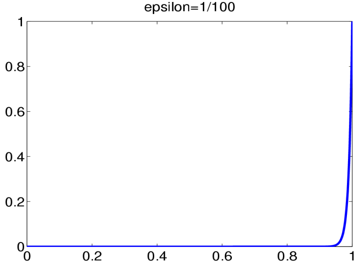




Convection diffusion equation (cont'ed)



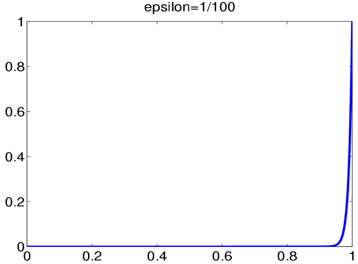
Convection diffusion equation (cont'ed)



$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$

If $b/\varepsilon \ll 1$ then $u(x) \simeq x$
If $b/\varepsilon \gg 1$ then $u(x) \simeq \exp(-b(1-x)/\varepsilon)$

Convection diffusion equation (cont'ed)



$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$

If $b/\varepsilon \ll 1$ then $u(x) \simeq x$
If $b/\varepsilon \gg 1$ then $u(x) \simeq \exp(-b(1-x)/\varepsilon)$
In the second case, *boundary layer* of size $\mathcal{O}(\varepsilon/b)$

Convection diffusion equation (cont'ed)

Approximation by finite elements

$$a(u,v) = \int_0^1 \left(\varepsilon u'(x)v'(x) + bu'(x)v(x)\right)\,dx$$

Convection diffusion equation (cont'ed)

Approximation by finite elements

$$a(u, v) = \int_0^1 \left(\varepsilon u'(x)v'(x) + bu'(x)v(x)\right)\, dx$$

After some computations... stiffness matrix is (uniform mesh):

$$\left(\frac{b}{2} - \frac{\varepsilon}{h}\right) u_{i+1} + \frac{2\varepsilon}{h} u_i + \left(-\frac{b}{2} - \frac{\varepsilon}{h}\right) u_{i-1}$$

Convection diffusion equation (cont'ed)

Approximation by finite elements

$$a(u,v) = \int_0^1 \left(\varepsilon u'(x)v'(x) + bu'(x)v(x)\right)\,dx$$

After some computations... stiffness matrix is (uniform mesh):

$$\left(\frac{b}{2} - \frac{\varepsilon}{h}\right) u_{i+1} + \frac{2\varepsilon}{h} u_i + \left(-\frac{b}{2} - \frac{\varepsilon}{h}\right) u_{i-1}$$

Local (discrete) Péclet number is $\mathcal{P}(h) = |b|h/(2\varepsilon)$, so that our system has the structure

$$(\mathcal{P}(h) - 1)u_{i+1} + 2u_i - (\mathcal{P}(h) + 1)u_{i-1} = 0$$

Convection diffusion equation (cont'ed)

$$(\mathcal{P}(h)-1)u_{i+1}+2u_i-(\mathcal{P}(h)+1)u_{i-1}=0$$

General solution

$$u_i=\frac{1-\left(\frac{1+\mathcal{P}(h)}{1-\mathcal{P}(h)}\right)^i}{1-\left(\frac{1+\mathcal{P}(h)}{1-\mathcal{P}(h)}\right)^N}\quad i=1,\dots,N$$

Convection diffusion equation (cont'ed)

$$(\mathcal{P}(h)-1)u_{i+1}+2u_i-(\mathcal{P}(h)+1)u_{i-1}=0$$

General solution

$$u_i=\frac{1-\left(\frac{1+\mathcal{P}(h)}{1-\mathcal{P}(h)}\right)^i}{1-\left(\frac{1+\mathcal{P}(h)}{1-\mathcal{P}(h)}\right)^N}\quad i=1,\dots,N$$

If $\mathcal{P}(h)>1$ solution oscillates!

Stabilization techiniques

- Upwind (finite differences)

Stabilization techiniques

- Upwind (finite differences)
- Artificial viscosity, streamline diffusion (loosing consistency)

Stabilization techiniques

- Upwind (finite differences)
- Artificial viscosity, streamline diffusion (loosing consistency)
- Petrov–Galerkin, SUPG (strongly consistent)

Hyperbolic equations

Let's consider the model problem (one dimensional convection equation)

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, & t > 0,\ x \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

Solution is a traveling wave $u(x,t) = u_0(x - at)$.

Hyperbolic equations

Let's consider the model problem (one dimensional convection equation)

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, & t > 0,\ x \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

Solution is a traveling wave $u(x,t) = u_0(x - at)$.
We consider a finite difference approximation.

Hyperbolic equations (cont'ed)

$u_j^n \simeq u(x_j, t_n)$

Hyperbolic equations (cont'ed)

$u_j^n \simeq u(x_j, t_n)$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(h_{j+1/2}^n - h_{j-1/2}^n)$$

where $h_{j+1/2} = h(u_j, u_{j+1})$ is a *numerical flux*

Hyperbolic equations (cont'ed)

$u_j^n \simeq u(x_j, t_n)$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(h_{j+1/2}^n - h_{j-1/2}^n)$$

where $h_{j+1/2} = h(u_j, u_{j+1})$ is a *numerical flux*
Indeed,

$$\frac{\partial}{\partial t}U_j = -\left((au)(x_{j+1/2}) - (au)(x_{j-1/2})\right)$$

with

$$U_j = \int_{x_{j-1/2}}^{x_{j+1/2}} u,\ dx$$

Hyperbolic equations (cont'ed)

Courant–Friedrichs–Lewy (CFL) condition

$$\left| a \frac{\Delta t}{\Delta x} \right| \leq 1$$

Hyperbolic equations (cont'ed)

Courant–Friedrichs–Lewy (CFL) condition

$$\left| a \frac{\Delta t}{\Delta x} \right| \leq 1$$

Very clear geometrical interpretation (see also multidimensional extension and generalization to systems)

Hyperbolic equations (cont'ed)

Courant–Friedrichs–Lewy (CFL) condition

$$\left| a \frac{\Delta t}{\Delta x} \right| \le 1$$

Very clear geometrical interpretation (see also multidimensional extension and generalization to systems)
Remark: implicit schemes (in time) don't have restrictions, but add artificial diffusion

Parabolic problems

Heat equation

$$\frac{\partial u(t)}{\partial t} - \Delta u(t) = f(t)$$

Parabolic problems

Heat equation

$$\frac{\partial u(t)}{\partial t} - \Delta u(t) = f(t)$$

Variational formulation: for each t, find $u(t) \in V = H^1_0(\Omega)$ s.t.

$$\left(\frac{\partial u(t)}{\partial t}, v \right) + a(u(t), v) = (f(t), v) \quad \forall v \in V$$

Parabolic problems

Heat equation

$$\frac{\partial u(t)}{\partial t} - \Delta u(t) = f(t)$$

Variational formulation: for each t, find $u(t) \in V = H^1_0(\Omega)$ s.t.

$$\left(\frac{\partial u(t)}{\partial t}, v\right) + a(u(t), v) = (f(t), v) \quad \forall v \in V$$

Space semidiscretization. Take $V_h \subset V$ and, for each t, look for $u_h(t) \in V_h$ such that

$$\left(\frac{\partial u_h(t)}{\partial t}, v_h\right) + a(u_h(t), v_h) = (f(t), v_h) \quad \forall v_h \in V_h$$

Parabolic problems (cont'ed)

Fully discretized problem

- Explicit Euler

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + a(u_h^n, v_h) = (f^n, v_h) \quad \forall v_h \in V_h$$

Parabolic problems (cont'ed)

Fully discretized problem

- Explicit Euler

$$\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h\right) + a(u_h^n, v_h) = (f^n, v_h) \quad \forall v_h \in V_h$$

- Implicit Euler

$$\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h\right) + a(u_h^{n+1}, v_h) = (f^{n+1}, v_h) \quad \forall v_h \in V_h$$

Parabolic problems (cont'ed)

θ-method (somewhat inbetween explicit and implicit)

Parabolic problems (cont'ed)

θ-method (somewhat inbetween explicit and implicit)

$0 \leq \theta \leq 1$

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h \right) + (1-\theta) a(u_h^n, v_h) + \theta a(u_h^{n+1}, v_h) =$$

$$(1-\theta)(f^n, v_h) + \theta (f^{n+1}, v_h) \quad \forall v_h \in V_h$$

Parabolic problems (cont'ed)

θ-method (somewhat inbetween explicit and implicit)
$0 \le \theta \le 1$

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + (1-\theta)a(u_h^n, v_h) + \theta a(u_h^{n+1}, v_h) = (1-\theta)(f^n, v_h) + \theta(f^{n+1}, v_h) \quad \forall v_h \in V_h$$

In one space dimension (finite differences, and $f = 0$)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{h^2}(1-\theta)(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \theta(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$

Parabolic problems (cont'ed)

If $u(0) = \sin(\pi x)$ then the solution in $[0, 1]$ with homogeneous Dirichlet boundary conditions is

$$u(t) = \sin(\pi x) \exp(-\pi^2 t)$$

In particular, it goes to zero as $t \to +\infty$

Parabolic problems (cont'ed)

If $u(0) = \sin(\pi x)$ then the solution in $[0, 1]$ with homogeneous Dirichlet boundary conditions is

$$u(t) = \sin(\pi x) \exp(-\pi^2 t)$$

In particular, it goes to zero as $t \to +\infty$
Study of discrete (absolute) stability
Discrete solution has the form

$$u_i^n = \alpha^n \sin(\pi i h)$$

Parabolic problems (cont'ed)

If $u(0) = \sin(\pi x)$ then the solution in $[0, 1]$ with homogeneous Dirichlet boundary conditions is

$$u(t) = \sin(\pi x) \exp(-\pi^2 t)$$

In particular, it goes to zero as $t \to +\infty$
Study of discrete (absolute) stability
Discrete solution has the form

$$u_i^n = \alpha^n \sin(\pi i h)$$

Stability condition $|\alpha| \leq 1$

Parabolic problems (cont'ed)

$$u_i^{n+1} - u_i^n = \frac{k}{h^2}(1-\theta)(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \theta(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$

$$u_i^n = \alpha^n \sin(\pi i h)$$

Parabolic problems (cont'ed)

$$u_i^{n+1} - u_i^n = \frac{k}{h^2}(1-\theta)(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \theta(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$

$$u_i^n = \alpha^n \sin(\pi i h)$$

Some trigonometry

$$\sin\pi(i+1)h - 2\sin(\pi i h) + \sin\pi(i-1)h = 2\sin(\pi i h)\cos(\pi h) - 2\sin(\pi i h) = \sin(\pi i h)(-4\sin^2(\pi h/2))$$

Parabolic problems (cont'ed)

$$u_i^{n+1} - u_i^n = \frac{k}{h^2}(1-\theta)(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \theta(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$

$$u_i^n = \alpha^n \sin(\pi i h)$$

Some trigonometry

$$\sin \pi(i+1)h - 2\sin(\pi i h) + \sin \pi(i-1)h = 2\sin(\pi i h)\cos(\pi h) - 2\sin(\pi i h) = \sin(\pi i h)(-4\sin^2(\pi h/2))$$

Hence

$$\alpha - 1 = \frac{k}{h^2}((1-\theta) + \theta\alpha)(-4\sin^2(\pi h/2))$$

Parabolic problems (cont'ed)

Finally

$$\alpha = \frac{1-(1-\theta)w}{1+\theta w} = 1 - \frac{w}{1+\theta w}$$

with $w = 4\frac{k}{h^2}\sin^2(\pi h/2) \geq 0$

Parabolic problems (cont'ed)

Finally

$$\alpha = \frac{1-(1-\theta)w}{1+\theta w} = 1 - \frac{w}{1+\theta w}$$

with $w = 4\frac{k}{h^2}\sin^2(\pi h/2) \geq 0$
Condition $|\alpha| \leq 1$ equivalent to

$$w(1-2\theta) \leq 2$$

Parabolic problems (cont'ed)

Finally

$$\alpha = \frac{1-(1-\theta)w}{1+\theta w} = 1 - \frac{w}{1+\theta w}$$

with $w = 4\frac{k}{h^2}\sin^2(\pi h/2) \geq 0$
Condition $|\alpha| \leq 1$ equivalent to

$$w(1-2\theta) \leq 2$$

- $1/2 \leq \theta \leq 1$ inconditionally stable
- $0 \leq \theta < 1/2$ stability condition $\dfrac{k}{h^2} \leq \dfrac{1}{2(1-2\theta)}$

Parabolic problems (cont'ed)

Finally

$$\alpha = \frac{1-(1-\theta)w}{1+\theta w} = 1 - \frac{w}{1+\theta w}$$

with $w = 4\frac{k}{h^2}\sin^2(\pi h/2) \geq 0$
Condition $|\alpha| \leq 1$ equivalent to

$$w(1-2\theta) \leq 2$$

- $1/2 \leq \theta \leq 1$ inconditionally stable
- $0 \leq \theta < 1/2$ stability condition $\dfrac{k}{h^2} \leq \dfrac{1}{2(1-2\theta)}$

End of Part III