CCM, Part II (2)

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March 2005

Elliptic PDE's

One dimensional model problem ($\Omega =]a, b[$)

$$\begin{cases} -u''(x) = f(x) & \text{in } \Omega \\ u(a) = u(b) = 0 \end{cases}$$

Boundary value problem (other boundary conditions possible)

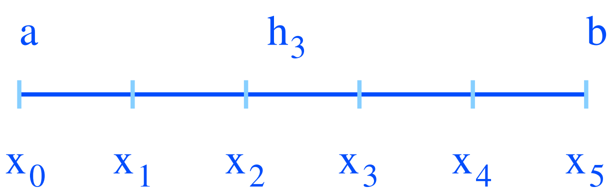
Generalization to $\Omega \in \mathbb{R}^d$ with boundary $\partial\Omega$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Theorem: well-posedness (existence, uniqueness, stability)

Finite differences

Summary: easy to design (approximate derivatives with difference quotients), easy to implement, very hard extension to general domains and boundary conditions



Here $N = 5$, $x_0 = a$, $x_i = a + \sum_{j=1}^{i} h_j$, $i = 1, \ldots, N$

Denoting $u_i = u(x_i)$, $u'_i = u'(x_i)$, first finite difference is

$$u'_i \simeq \frac{u_{i+1} - u_{i-1}}{h_i + h_{i+1}}$$ second order accurate in h (consistent)

Finite differences (cont'ed)

Approximation of second derivative

$$u_i'' \simeq \frac{u'_{i+1/2} - u'_{i-1/2}}{\frac{h_i + h_{i+1}}{2}} \simeq \frac{\frac{u_{i+1}-u_i}{h_{i+1}} - \frac{u_i - u_{i-1}}{h_i}}{\frac{h_i + h_{i+1}}{2}}$$

If $h_i = h$ (constant mesh size), simpler expression

$$u_i'' \simeq \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \qquad \text{second order consistent}$$

Our approximate equation at x_i reads

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i, \qquad i = 1, \ldots, N-1$$

Finite differences (cont'd)

Putting things together we are led to the linear system

$$\begin{cases} u_0 = 0 \\ \dots \\ \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i \\ \dots \\ u_N = 0 \end{cases}$$

$$AU = F \qquad A = [\text{tridiag}(-1, 2, -1)]/h^2$$

Weak formulations

Need for more general formulations.

Let's consider space $V = H_0^1(a,b)$ consisting of continuous functions on $[a,b]$, piecewise differentiable with bounded derivative, and vanishing at endpoints.

Generalization to 2D requires Lebesgue integral and Hilbert spaces

$$H^1(\Omega) = \{v \in L^2(\Omega) \text{ s.t. } \vec{\text{grad}}\, v \in L^2(\Omega)\}$$

where

$$L^2(\Omega) = \left\{ v : \Omega \to \mathbb{R} \text{ integrable s.t. } \int_\Omega v^2 < \infty \right\}$$

Weak formulations (cont'd)

Take our model equation, multiply by a generic $v \in V$ (test function), and integrate over (a, b)

$$-\int_a^b u''(x)v(x)\,dx = \int_a^b f(x)v(x)\,dx$$

Integrating by parts gives

$$\int_a^b u'(x)v'(x)\,dx = \int_a^b f(x)v(x)\,dx$$

$a : V \times V \to \mathbb{R}$, $F \in V^*$

$$a(u, v) = \int_a^b u'(x)v'(x)\,dx, \quad F(v) = \int_a^b f(x)v(x)\,dx$$

Weak formulations (cont'd)

Lax–Milgram Lemma

Find $u \in V$ such that $a(u,v) = F(v) \quad \forall v \in V$

This problem is well posed (exist., uniq., and stab.) provided

1. V Hilbert space

2. a bilinear, continuous, F linear, continuous

3. a *coercive*, that is there exists $\alpha > 0$ s.t.

$$a(v,v) \geq \alpha \|v\|_V^2, \quad \forall v \in V$$

$$\|u\|_V \leq \frac{1}{\alpha}\|F\|_{V^*}$$ Stability estimate

Weak formulations (cont'ed)

In our case hypotheses of LM Lemma OK (Poincaré inequality)

Theorem If f is smooth enough, the unique solution to weak formulation solves the original equation as well (strong solution)

More general situation

$$\begin{cases} -\operatorname{div}(\varepsilon\, \vec{\operatorname{grad}}\, u) + \vec{\beta} \cdot \vec{\operatorname{grad}}\, u + \sigma u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$a(u,v) = \int_\Omega \varepsilon\, \vec{\operatorname{grad}}\, u \cdot \vec{\operatorname{grad}}\, v\, d\vec{x} + \int_\Omega v \vec{\beta} \cdot \vec{\operatorname{grad}}\, u\, d\vec{x} + \int_\Omega \sigma u v\, d\vec{x}$$

Weak formulations (cont'ed)

In general problem in weak form, when a is symmetric, is equivalent to the following variational problem:

Find $u \in V$ such that

$$J(u) = \min_{v \in V} J(v), \quad J(v) = \frac{1}{2} a(v, v) - F(v)$$

In the one dimensional model problem, we have

$$J(v) = \frac{1}{2} \int_a^b (v'(x))^2 \, dx - \int_a^b f(x) v(x) \, dx$$

Finite elements (Galerkin method)

Consider a finite dimensional subspace $V_h \subset V$ (h refers to a mesh parameter).

Find $u_h \in V_h$ such that $a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

Problem is solvable by Lax–Milgram

Suppose that $V_h = \mathrm{span}\{\varphi_1, \ldots, \varphi_N(h)\}$, so $u_h = \sum_{j=1}^{N} u_j \varphi_j$

Problem can be written: find $\mathbf{u} = \{u_j\}$ s.t. for any i

$$a\Big(\sum_{j=1}^{N} u_j \varphi_j, \varphi_i\Big) = F(\varphi_i)$$

Galerkin method (cont'ed)

Bilinearity of a gives

$$\sum_{j=1}^{N} u_j a(\varphi_j, \varphi_i) = F(\varphi_i), \quad i = 1, \ldots, N$$

Let's denote by A the *stiffness* matrix $A_{ij} = a(\varphi_j, \varphi_i)$ and by b the *load* vector $b_i = F(\varphi_i)$. Then we have the matrix form of discrete problem

$$A\mathbf{u} = b$$

a symmetric and coercive implies A symmetric positive definite

Galerkin method (cont'ed)

Existence and uniqueness (Lax–Milgram)

Convergence = Consistency + Stability

Stability:

$$\|u_h\|_V \leq \frac{1}{\alpha}\|F\|_{V^*}$$

Strong consistency

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

Galerkin method (cont'ed)

Error estimate (Céa's Lemma)

$$\alpha\|u-u_h\|_V^2 \le a(u-u_h,u-u_h) = a(u-u_h,u-v_h)$$
$$\le M\|u-u_h\|_V\|u-v_h\|_V$$

$$\|u-u_h\|_V \le \frac{M}{\alpha}\inf_{v\in V_h}\|u-v_h\|_V$$

Error bounded by best approximation

Need for good choice of V_h!

Galerkin method (cont'ed)

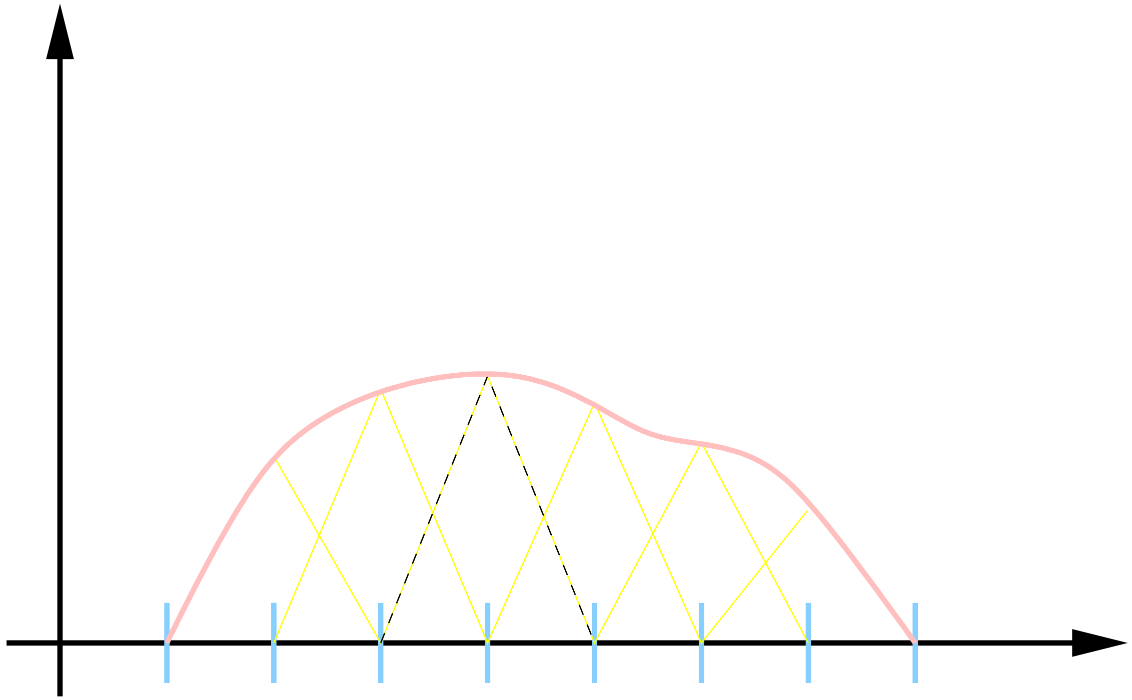
Moreover, when a is symmetric, we have the variational property

$$J(u_h) = \min_{v_h \in V_h} J(v_h)$$

Since $V_h \subset V$, in particular, we have

$$J(u) \leq J(u_h)$$

Finite elements



One dimensional p/w linear approximation.
Shape (or basis) functions: hat functions.

A finite element is defined by:
1) a domain (interval, triangle, tetrahedron,. . .),
2) a finite dimensional (polynomial) space,
3) a set of degrees of freedom.

Finite elements (cont'ed)

One dimensional finite elements

1) domain: interval
2) space: $\mathbb{P}_p$
3) d.o.f.'s: depend on polynomial order

linear element: endpoints (2)
quadratic element: endpoints + midpoint (3)
. . .

Set $\{a_j\}_{j=1}^N$ of degrees of freedom is *unisolvent*, that is, given N numbers $\alpha_1, \ldots, \alpha_N$, there exists a unique polynomial φ in $\mathbb{P}_p$ s. t.

$$\varphi(a_j) = \alpha_j, \quad j = 1, \ldots, N$$

Finite elements (cont'ed)

Approximation properties of one dimensional finite elements

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^k} \leq C h^{p+1-k} |u|_{H^{p+1}} \quad k = 0, 1$$

Remark on hp FEM

- Refine in h where solution is singular
- Refine in p where solution is regular

End of Part Two