CCM, Part II

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Complexity and its Interdisciplinary Applications

Elliptic PDE's

One dimensional model problem ($\Omega =]a,b[$)

$$\begin{cases} -u''(x) = f(x) & \text{in } \Omega \\ u(a) = u(b) = 0 \end{cases}$$

Boundary value problem (other boundary conditions possible)

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Theorem: well-posedness (existence, uniqueness, stability)

Finite differences

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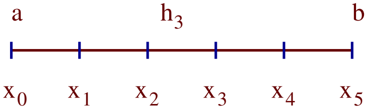

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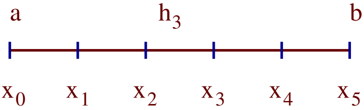

Here $N = 5$, $x_0 = a$, $x_i = a + \sum_{j=1}^{i} h_j$, $i = 1, \dots, N$

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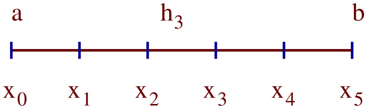

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Denoting $u_i = u(x_i)$, $u'_i = u'(x_i)$, first finite difference is

$$u'_i \simeq \frac{u_{i+1} - u_{i-1}}{h_i + h_{i+1}} \qquad \text{second order accurate in } h \text{ (consistent)}$$

Finite differences (cont'ed)

Approximation of second derivative

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If $h_i = h$ (constant mesh size), simpler expression

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Our approximate equation at x_i reads

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i, \qquad i = 1, \dots, N-1$$

Finite differences (cont'ed)

Putting things together we are led to the linear system

$$\begin{cases} u_0 = 0 \\ \dots \\ \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i \\ \dots \\ u_N = 0 \end{cases}$$

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$$AU = F \qquad A = [\mathrm{tridiag}(-1, 2, -1)]/h^2$$

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Generalization to 2D requires Lebesgue integral and Hilbert spaces

$$H^1(\Omega) = \{v \in L^2(\Omega) \text{ s.t. } \mathrm{grad}\, v \in L^2(\Omega)\}$$

where

$$L^2(\Omega) = \left\{ v : \Omega \to \mathbb{R} \text{ integrable s.t. } \int_\Omega v^2 < \infty \right\}$$

Weak formulations (cont'ed)

Take our model equation, multiply by a generic $v \in V$ (test function), and integrate over (a, b)

$$-\int_a^b u''(x)v(x)\,dx = \int_a^b f(x)v(x)\,dx$$

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$a : V \times V \to \mathbb{R}$, $F \in V^*$

$$a(u, v) = \int_a^b u'(x)v'(x)\,dx, \quad F(v) = \int_a^b f(x)v(x)\,dx$$

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This problem is well posed (exist., uniq., and stab.) provided

1. V Hilbert space
2. a bilinear, continuous, F linear, continuous
3. a *coercive*, that is there exists $\alpha > 0$ s.t.

$$a(v, v) \geq \alpha \|v\|_V^2, \quad \forall v \in V$$

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$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V^*} \qquad \text{Stability estimate}$$

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More general situation

$$\begin{cases} -\operatorname{div}(\varepsilon \operatorname{grad} u) + \vec{\beta} \cdot \operatorname{grad} u + \sigma u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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$$a(u, v) = \int_\Omega \varepsilon \operatorname{grad} u \cdot \operatorname{grad} v \, d\mathbf{x} + \int_\Omega v \vec{\beta} \cdot \operatorname{grad} u \, d\mathbf{x} + \int_\Omega \sigma u v \, d\mathbf{x}$$

Weak formulations (cont'd)

In general problem in weak form, when a is symmetric, is equivalent to the following variational problem:

Find $u \in V$ such that

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In the one dimensional model problem, we have

$$J(v) = \frac{1}{2} \int_a^b (v'(x))^2 \, dx - \int_a^b f(x) v(x) \, dx$$

Finite elements (Galerkin method)

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Problem can be written: find $\mathbf{u} = \{u_j\}$ s.t. for any i

$$a\Big(\sum_{j=1}^{N} u_j \varphi_j, \varphi_i\Big) = F(\varphi_i)$$

Galerkin method (cont'ed)

Bilinearity of a gives

$$\sum_{j=1}^{N} u_j a(\varphi_j, \varphi_i) = F(\varphi_i), \quad i = 1, \dots, N$$

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Let's denote by A the *stiffness* matrix $A_{ij} = a(\varphi_j, \varphi_i)$ and by b the *load* vector $b_i = F(\varphi_i)$. Then we have the matrix form of discrete problem

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a *symmetric and coercive* implies A *symmetric positive definite*

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Convergence = Consistency + Stability

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Stability:

$$\|u_h\|_V \leq \frac{1}{\alpha}\|F\|_{V^*}$$

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Strong consistency

$$a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$$

Galerkin method (cont'ed)

Error estimate (Céa's Lemma)

$$\begin{aligned}\alpha\|u-u_h\|_V^2 &\le a(u-u_h,u-u_h)=a(u-u_h,u-v_h)\\ &\le M\|u-u_h\|_V\|u-v_h\|_V\end{aligned}$$

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Error bounded by best approximation
Need for good choice of V_h!

Galerkin method (cont'ed)

Moreover, when a is symmetric, we have the variational property

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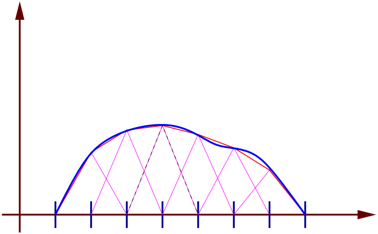
Since $V_h \subset V$, in particular, we have

$$J(u) \leq J(u_h)$$

Finite elements



One dimensional p/w linear approximation. Shape (or basis) functions: hat functions.

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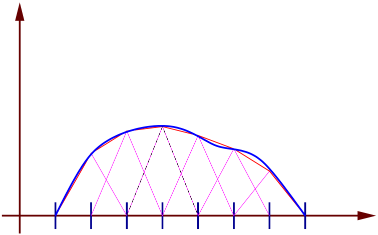

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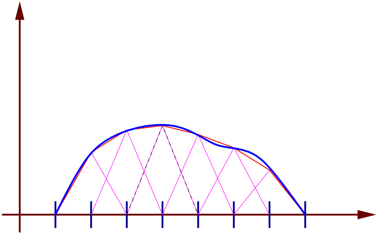

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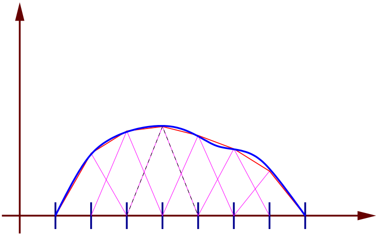

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2. a finite dimensional (polynomial) space,
3. a set of degrees of freedom.

Finite elements (cont'ed)

One dimensional finite elements

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linear element: endpoints (2)
quadratic element: endpoints + midpoint (3)
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Set $\{a_j\}_{j=1}^N$ of degrees of freedom is *unisolvent*, that is, given N numbers $\alpha_1, \dots, \alpha_N$, there exists a unique polynomial φ in $\mathcal{P}_p$ s. t.

$$\varphi(a_j) = \alpha_j, \quad j = 1, \dots, N$$

Finite elements (cont'ed)

Approximation properties of one dimensional finite elements

Finite elements (cont'ed)

Approximation properties of one dimensional finite elements

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^k} \le C h^{p+1-k} |u|_{H^{p+1}} \quad k = 0, 1$$

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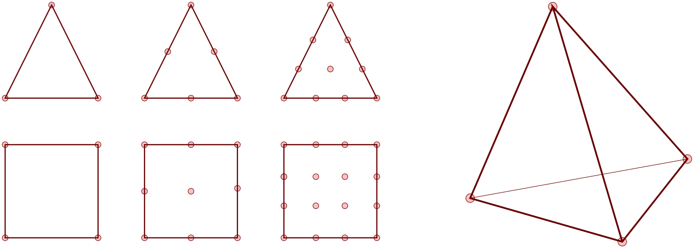
Remark on *hp* FEM

- Refine in h where solution is singular
- Refine in p where solution is regular

Finite elements (cont'ed)

Generalization to more space dimensions

Example of unisolvent degrees of freedom



Finite elements (cont'ed)

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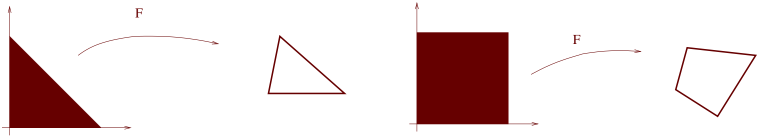
In general one considers reference elements and mappings to actual elements



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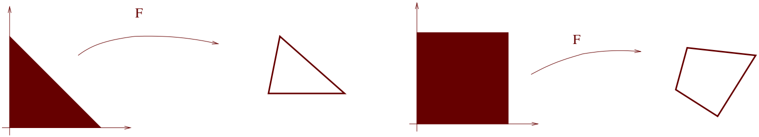

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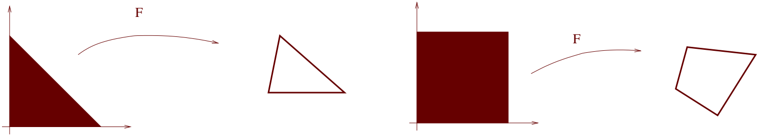

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$\hat{\varphi}_1, \dots \hat{\varphi}_N$ reference shape functions;
$\varphi_1, \dots \varphi_N$ actual shape functions

Finite elements (cont'ed)

How to map the shape functions

$$F_K : \hat{K} \to K, \quad \vec{x} = F(\hat{\vec{x}})$$

$$\varphi(\vec{x}) = \hat{\varphi}(F^{-1}(\vec{x}))$$

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Example of computation of *local* stiffness matrix (one dimensional)

$$A_{ji} = a(\varphi_i, \varphi_j) = \int_a^b \varphi_i'(x)\varphi_j'(x)\,dx = \sum_K \int_K \varphi_i'(x)\varphi_j'(x)\,dx$$

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$$\int_K \varphi_i'(x)\varphi_j'(x)\,dx = \int_{\hat{K}} \frac{\hat{\varphi}_i'(\hat{x})}{F'(\hat{x})}\frac{\hat{\varphi}_j'(\hat{x})}{F'(\hat{x})}F'(\hat{x})\,d\hat{x} = \int_{\hat{K}} \frac{\hat{\varphi}_i'(\hat{x})\hat{\varphi}_j'(\hat{x})}{F'(\hat{x})}\,d\hat{x}$$

Finite elements (cont'ed)

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In more space dimensions, F is affine for most popular elements.

$$\int_K \mathrm{grad}\,\varphi_i(\vec{x}) \cdot \mathrm{grad}\,\varphi_j(\vec{x})\, d\vec{x} = ?$$

Finite elements (cont'ed)

General strategy for assembling stiffness matrix and load vector

Finite elements (cont'ed)

General strategy for assembling stiffness matrix and load vector

- Loop over elements $ie = 1, \dots, ne$
- Compute local stiffness matrix $A^{loc}_{ji} = a(\varphi_i, \varphi_j)$, $i, j = 1, \dots, ndof$ and local load vector $F^{loc}_i = F(\varphi_i)$, $i = 1, \dots, ndof$
- Loop for $i, j = 1, \dots, ndof$ and assembly of global matrix

$$A_{iglob,jglob} = A_{iglob,jglob} + A^{loc}_{ij}$$

- Account for boundary conditions

Finite elements (cont'ed)

Some remarks on the discrete linear system

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End of part II