Classical computational methods, Part II

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Some examples

A second order ODE

$$-u''(x) = f(x)$$

Solution can be explicitly determined (closed form solution)

$$u'(x) = u'(x_0) + \int_{x(0)}^{x} u''(t)\,dt = u'(x_0) - \int_{x(0)}^{x} f(t)\,dt$$

$$u(x) = u(x_0) + \int_{x(0)}^{x} u'(t)\,dt$$

In general

$$u(x) = \alpha + \beta x - \int\int f(t)\,dt$$

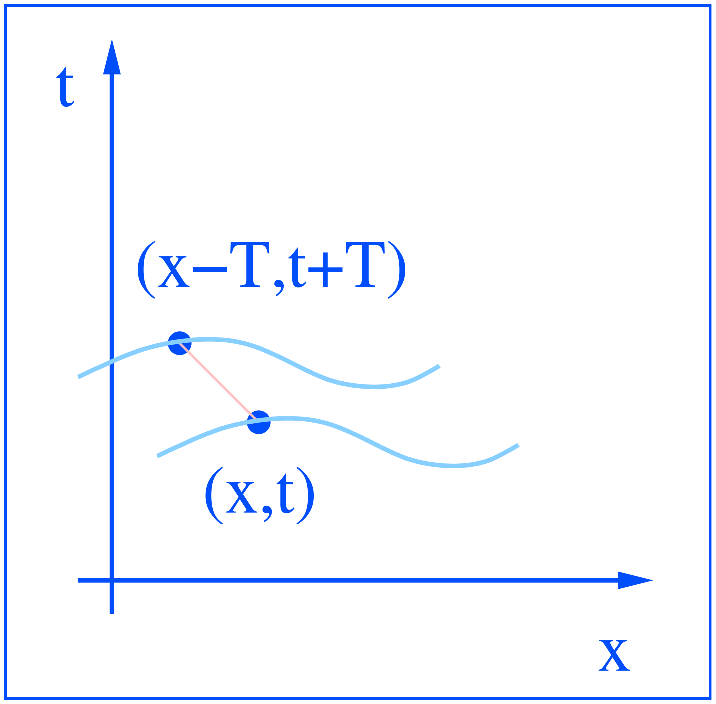
Some examples (cont'ed)

One dimensional convection equation (PDE)

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial u}{\partial x}(x,t) = 0$$

Closed form solution

$$u(x,t) = w(x+t)$$



Some examples (cont'ed)

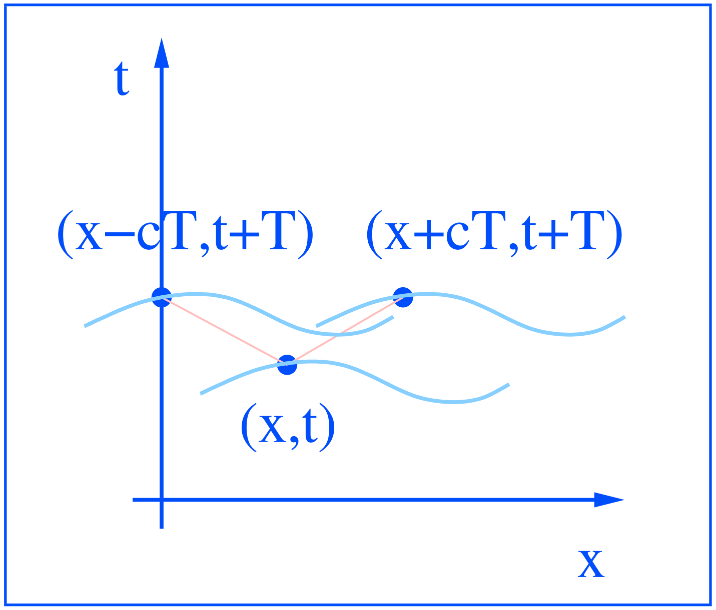
One dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

Closed form solution

$$u(x,t) = w_1(x+ct) + w_2(x-ct)$$

Proof



Some examples (cont'ed)

One dimensional heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \qquad x \in (0,1),\ t > 0$$

Closed form solution

$$u(x,t) = \sum_{j=1}^{\infty} u_{0,j} e^{-(j\pi)^2 t} \sin(j\pi x),$$

where $u_0(x) = u(x,0)$ is the initial datum and

$$u_{0,j} = 2 \int_0^1 u_0(x) \sin(j\pi x)\, dx$$

Some examples (cont'ed)

Convection equation

$$\frac{\partial u}{\partial t} + \mathrm{div}(\vec{\beta} u) = 0$$

First order linear equation.

N.B.: divergence operator $\mathrm{div}\,\vec{v} = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}$

This equation states the mass conservation of a body occupying a region $\Omega \in \mathbb{R}^d$, with density u and velocity $\vec{\beta}$

Some examples (cont'ed)

Laplace/Beltrami/Poisson equation

$$-\Delta u = f$$

Second order linear equation.

N.B.: Laplace operator $\Delta v = \sum_{i=1}^{d} \frac{\partial^2 v}{\partial x_i^2}$

This equation states the diffusion of a homogeneous and isotropic fluid occupying a region $\Omega \in \mathbb{R}^d$, as well as the vertical displacement of an elastic membrane. Fundamental equation for several models.

Some examples (cont'd)

Heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f$$

Second order linear equation

Wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

Second order linear equation

Some examples (cont'ed)

Burgers equation $(d = 1)$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

First order quasi-linear equation

Viscous Burgers equation $(d = 1)$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \qquad \varepsilon > 0$$

Second order semi-linear equation

Plan of the Course

- ☞ Classification of Partial Differential Equations (PDE)
- ☞ Elliptic PDE's
  - ▶ Finite differences
  - ▶ Finite elements
  - ▶ Where the theory is elegant and complete. . .
- ☞ From elliptic to hyperbolic PDE's
  - ▶ Convection-diffusion equation
  - ▶ Finite differences, upwind
  - ▶ Integrating along the characteristics
  - ▶ Stabilization of finite elements

- Parabolic equation
  - Heat equation: space semidiscretization and evolution in time
  - Stability of θ-method
- Examples with Matlab
- Conclusions and comments
- Questions and answers

Classification of (linear) PDE's

The case of two variables (can be generalized)

$$Lu \equiv \left(A\frac{\partial^2 u}{\partial x_1^2} + B\frac{\partial^2 u}{\partial x_1 \partial x_2} + C\frac{\partial^2 u}{\partial x_2^2} \right) + L.O.T.$$

Matrix associated to quadratic form

$$QF = \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix}$$

Note: A, B, and C might be functions themselves.

Classification of PDE's (cont'ed)

Compute eigenvalues λ_i of QF

- *Elliptic* equation: $\lambda_1\lambda_2 > 0$
- *Parabolic* equation: $\lambda_1\lambda_2 = 0$
- *Hyperbolic* equation: $\lambda_1\lambda_2 < 0$

With the notation of quadratic forms: *definite* form, *semidefinite* form, *indefinite* form, respectively.

Classification of PDE's (cont'ed)

Consider operator

$$\mathcal{L}u \equiv A\frac{\partial^2 u}{\partial x_1^2} + B\frac{\partial^2 u}{\partial x_1 \partial x_2} + C\frac{\partial^2 u}{\partial x_2^2} = 0$$

and look for change of variables

$$\xi = \alpha x_2 + \beta x_1, \quad \eta = \gamma x_2 + \delta x_1$$

so that $\mathcal{L}u$ is a multiple of $\frac{\partial^2 u}{\partial \xi \partial \eta}$ (see wave equation)

$$\mathcal{L}u = (A\beta^2 + B\alpha\beta + C\alpha^2)\frac{\partial^2 u}{\partial \xi^2} + (A\delta^2 + B\gamma\delta + C\gamma^2)\frac{\partial^2 u}{\partial \eta^2}$$
$$+ (2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma)\frac{\partial^2 u}{\partial \xi \partial \eta}$$

Classification of PDE's (cont'ed)

$$\mathcal{L}u = (A\beta^2 + B\alpha\beta + C\alpha^2)\frac{\partial^2 u}{\partial \xi^2} + (A\delta^2 + B\gamma\delta + C\gamma^2)\frac{\partial^2 u}{\partial \eta^2}$$
$$+ (2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma)\frac{\partial^2 u}{\partial \xi \partial \eta}$$

If $A = C = 0$, trivial. Suppose $A \neq 0$; we want

$$A\beta^2 + B\alpha\beta + C\alpha^2 = 0, \quad A\delta^2 + B\gamma\delta + C\gamma^2 = 0$$

When $\alpha\gamma \neq 0$, divide first equation by α^2, second one by γ^2 and solve for β/α and δ/γ, resp.

$$\beta/\alpha = (2A)^{-1}(-B \pm \sqrt{\Delta}), \quad \delta/\gamma = (2A)^{-1}(-B \pm \sqrt{\Delta})$$

$\Delta = B^2 - 4AC$

Classification of PDE's (cont'ed)

Hyperbolic case

$$\xi = \alpha x_2 + \beta x_1, \quad \eta = \gamma x_2 + \delta x_1$$

$$\beta/\alpha = (2A)^{-1}(-B \pm \sqrt{\Delta}), \quad \delta/\gamma = (2A)^{-1}(-B \pm \sqrt{\Delta})$$

For nonsingular change of variables, Δ must be positive

$$\alpha = \gamma = 2A, \quad \beta = -B + \sqrt{\Delta}, \delta = -B - \sqrt{\Delta}$$

$$\mathcal{L}u = -4A(B^2 - 4AC)\frac{\partial^2 u}{\partial \xi \partial \eta}$$

As before, solution has the form $u = p(\xi) + q(\eta)$ and the lines $\xi = constant$ and $\eta = constant$ are called *characteristics*.

Classification of PDE's (cont'ed)

Actually, when $x_1 = t$ and $x_2 = x$, the change of variables

$$x' = x - \frac{B}{2A}t, \quad t' = t$$

maps our hyperbolic operator $(A \neq 0)$ to a multiple of wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}$$

Hence, $\mathcal{L}$ is a wave operator in a frameset moving at speed $-B/(2A)$.

Classification of PDE's (cont'ed)

Parabolic case

$$\mathcal{L}u = (A\beta^2 + B\alpha\beta + C\alpha^2)\frac{\partial^2 u}{\partial \xi^2} + (A\delta^2 + B\gamma\delta + C\gamma^2)\frac{\partial^2 u}{\partial \eta^2}$$
$$+ (2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma)\frac{\partial^2 u}{\partial \xi \partial \eta}$$

For $\beta/\alpha = -B/(2A)$ coefficient of $\frac{\partial^2 u}{\partial \xi^2}$ vanishes
But $B/(2A) = 2C/B$, so coefficient of $\frac{\partial^2 u}{\partial \xi \partial \eta}$ is zero as well

Everything can be written as a multiple of $\frac{\partial^2 u}{\partial \eta^2}$

Classification of PDE's (cont'ed)

In conclusion, in the parabolic case, the change of variables

$$\xi = 2Ax_2 - Bx_1, \quad \eta = x_1$$

maps the equation to

$$A\frac{\partial^2 u}{\partial \eta^2} = 0$$

which has the general solution

$$u = p(\xi) + \eta q(\xi)$$

One family of characteristics $\xi = constant$

Classification of PDE's (cont'ed)

Elliptic case

No choice of parameters makes coefficients of $\frac{\partial^2 u}{\partial \xi^2}$ and $\frac{\partial^2 u}{\partial \eta^2}$ vanish
In this case change of variables

$$\xi = \frac{2Ax_2 - Bx_1}{\sqrt{4AC - B^2}}, \quad \eta = x_1$$

maps equation to

$$A\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) = 0$$

No family of characteristics (infinite speed of propagation, no discontinuities allowed)

Classification of PDE's (cont'ed)

Final examples

- Laplace equation: elliptic
- Wave equation: hyperbolic
- Heat equation: parabolic
- Convection-diffusion equation:

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u + \mathrm{div}(\vec{\beta} u) = 0$$

parabolic, degenerating to hyperbolic as ε tends to zero.

End of Part I

Closed form of 1D wave equation solution

Change of variables

$$y = x + ct, \quad z = x - ct, \qquad u(x,t) = w(y,z)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$\frac{\partial^2 w}{\partial y \partial z} = 0 \Rightarrow w = w_1(y) + w_2(z) = w_1(x+ct) + w_2(x-ct)$$

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