

Classical computational methods, Part III

Daniele Boffi

Dipartimento di Matematica “F. Casorati”

Università di Pavia

`boffi@dimat.unipv.it`

`http://www-dimat.unipv.it/boffi/`

March 2004

Some examples

A second order ODE

$$-u''(x) = f(x)$$

Solution can be explicitly determined (closed form solution)

$$u'(x) = u'(x_0) + \int_{x(0)}^x u''(t) dt = u'(x_0) - \int_{x(0)}^x f(t) dt$$

$$u(x) = u(x_0) + \int_{x(0)}^x u'(t) dt$$

In general

$$u(x) = \alpha + \beta x - \int \int f(t) dt$$

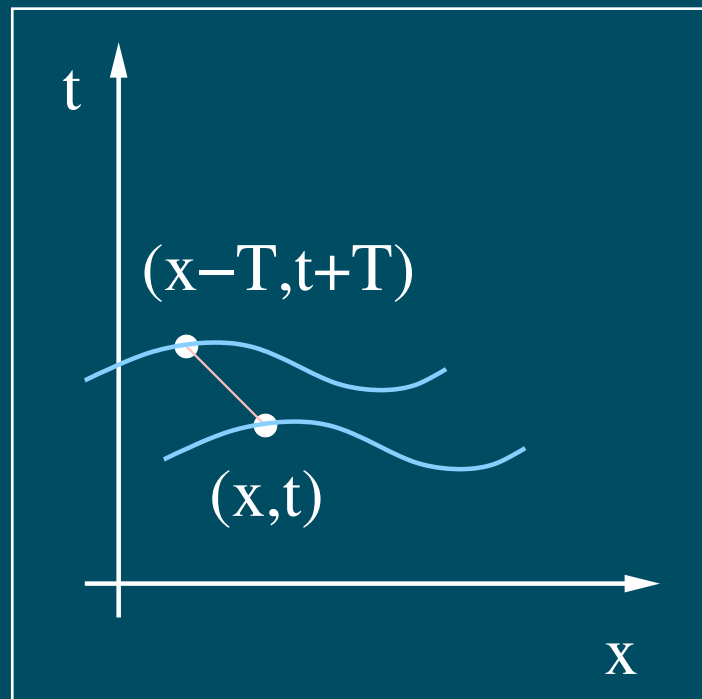
Some examples (cont'ed)

One dimensional convection equation

$$\frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial x}(x, t) = 0$$

Closed form solution

$$u(x, t) = w(x + t)$$



Some examples (cont'ed)

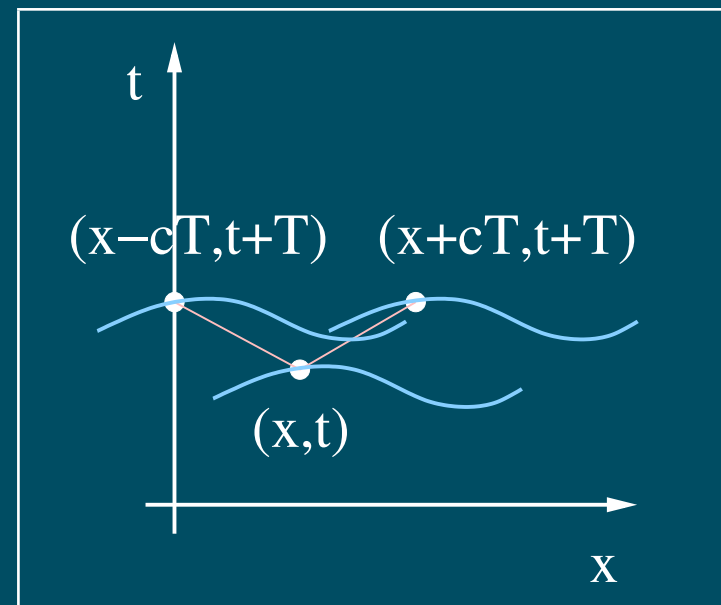
One dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}(x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = 0$$

Closed form solution

$$u(x, t) = w_1(x + ct) + w_2(x - ct)$$

Proof



Some examples (cont'ed)

One dimensional heat equation

$$\frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = 0, \quad x \in (0, 1), \quad t > 0$$

Closed form solution

$$u(x, t) = \sum_{j=1}^{\infty} u_{0,j} e^{-(j\pi)^2 t} \sin(j\pi x),$$

where $u_0(x) = u(x, 0)$ is the initial datum and

$$u_{0,j} = 2 \int_0^1 u_0(x) \sin(j\pi x) dx$$

Some examples (cont'ed)

Convection equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(\vec{\beta}u) = 0$$

First order linear equation.

N.B.: divergence operator $\operatorname{div} \vec{v} = \sum_{i=1}^d \frac{\partial v_i}{\partial x_i}$

This equation states the mass conservation of a body occupying a region $\Omega \in \mathbb{R}^d$, with density u and velocity $\vec{\beta}$

Some examples (cont'ed)

Laplace/Beltrami/Poisson equation

$$-\Delta u = f$$

Second order linear equation.

N.B.: Laplace operator $\Delta v = \sum_{i=1}^d \frac{\partial^2 v}{\partial x_i^2}$

This equation states the diffusion of a homogeneous and isotropic fluid occupying a region $\Omega \in \mathbb{R}^d$, as well as the vertical displacement of an elastic membrane. Fundamental equation for several models.

Some examples (cont'ed)

Heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f$$

Second order linear equation

Wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

Second order linear equation

Some examples (cont'ed)

Burgers equation ($d = 1$)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

First order quasi-linear equation

Viscous Burgers equation ($d = 1$)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad \varepsilon > 0$$

First order semi-linear equation

Plan of the Course

✎ Classification of Partial Differential Equations (PDE)

✎ Elliptic PDE's

- ▶ Finite differences
- ▶ Examples with Matlab
- ▶ Finite elements
- ▶ Where the theory is elegant and complete. . .

✎ From elliptic to hyperbolic PDE's

- ▶ Convection-diffusion equation
- ▶ Finite differences, upwind
- ▶ Integrating along the characteristics
- ▶ Stabilization of finite elements

👉 Parabolic equation

- ▶ Heat equation: space semidiscretization and evolution in time
- ▶ Stability of θ -method

👉 Conclusions and comments

👉 Questions and answers

Classification of (linear) PDE's

The case of two variables (can be generalized)

$$Lu \equiv \left(A \frac{\partial^2 u}{\partial x_1^2} + B \frac{\partial^2 u}{\partial x_1 \partial x_2} + C \frac{\partial^2 u}{\partial x_2^2} \right) + L.O.T.$$

Matrix associated to quadratic form

$$QF = \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix}$$

Note: A , B , and C might be functions themselves.

Classification of PDE's (cont'ed)

Compute eigenvalues λ_i of QF

- ▶ *Elliptic* equation: $\lambda_1 \lambda_2 > 0$
- ▶ *Parabolic* equation: $\lambda_1 \lambda_2 = 0$
- ▶ *Hyperbolic* equation: $\lambda_1 \lambda_2 < 0$

With the notation of quadratic forms: *definite* form, *semidefinite* form, *indefinite* form, respectively.

Classification of PDE's (cont'ed)

Consider operator

$$\mathcal{L}u \equiv A \frac{\partial^2 u}{\partial x_1^2} + B \frac{\partial^2 u}{\partial x_1 \partial x_2} + C \frac{\partial^2 u}{\partial x_2^2} = 0$$

and look for change of variables

$$\xi = \alpha x_2 + \beta x_1, \quad \eta = \gamma x_2 + \delta x_1$$

so that $\mathcal{L}u$ is a multiple of $\frac{\partial^2 u}{\partial \xi \partial \eta}$ (see wave equation)

$$\begin{aligned} \mathcal{L}u &= (A\beta^2 + B\alpha\beta + C\alpha^2) \frac{\partial^2 u}{\partial \xi^2} + (A\delta^2 + B\gamma\delta + C\gamma^2) \frac{\partial^2 u}{\partial \eta^2} \\ &\quad + (2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma) \frac{\partial^2 u}{\partial \xi \partial \eta} \end{aligned}$$

Classification of PDE's (cont'ed)

$$\mathcal{L}u = (A\beta^2 + B\alpha\beta + C\alpha^2)\frac{\partial^2 u}{\partial \xi^2} + (A\delta^2 + B\gamma\delta + C\gamma^2)\frac{\partial^2 u}{\partial \eta^2} \\ + (2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma)\frac{\partial^2 u}{\partial \xi \partial \eta}$$

If $A = C = 0$, trivial. Suppose $A \neq 0$; we want

$$A\beta^2 + B\alpha\beta + C\alpha^2 = 0, \quad A\delta^2 + B\gamma\delta + C\gamma^2 = 0$$

When $\alpha\gamma \neq 0$, divide first equation by α^2 , second one by γ^2 and solve for β/α and δ/γ , resp.

$$\beta/\alpha = (2A)^{-1}(-B \pm \sqrt{\Delta}), \quad \delta/\gamma = (2A)^{-1}(-B \pm \sqrt{\Delta})$$

$$\Delta = B^2 - 4AC$$

Classification of PDE's (cont'ed)

Hyperbolic case

$$\xi = \alpha x_2 + \beta x_1, \quad \eta = \gamma x_2 + \delta x_1$$

$$\beta/\alpha = (2A)^{-1}(-B \pm \sqrt{\Delta}), \quad \delta/\gamma = (2A)^{-1}(-B \pm \sqrt{\Delta})$$

For nonsingular change of variables, Δ must be positive

$$\alpha = \gamma = 2A, \quad \beta = -B + \sqrt{\Delta}, \delta = -B - \sqrt{\Delta}$$

$$\mathcal{L}u = -4A(B^2 - 4AC) \frac{\partial^2 u}{\partial \xi \partial \eta}$$

As before, solution has the form $u = p(\xi) + q(\eta)$ and the lines $\xi = \text{constant}$ and $\eta = \text{constant}$ are called *characteristics*.

Classification of PDE's (cont'ed)

Actually, when $x_1 = t$ and $x_2 = x$, the change of variables

$$x' = x - \frac{B}{2A}t, \quad t' = t$$

maps our hyperbolic operator ($A \neq 0$) to a multiple of wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}$$

Hence, $\mathcal{L}$ is a wave operator in a frameset moving at speed $-B/(2A)$.

Classification of PDE's (cont'ed)

Parabolic case

$$\begin{aligned}\mathcal{L}u = & (A\beta^2 + B\alpha\beta + C\alpha^2)\frac{\partial^2 u}{\partial \xi^2} + (A\delta^2 + B\gamma\delta + C\gamma^2)\frac{\partial^2 u}{\partial \eta^2} \\ & + (2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma)\frac{\partial^2 u}{\partial \xi \partial \eta}\end{aligned}$$

For $\beta/\alpha = -B/(2A)$ coefficient of $\frac{\partial^2 u}{\partial \xi^2}$ vanishes

But $B/(2A) = 2C/B$, so coefficient of $\frac{\partial^2 u}{\partial \xi \partial \eta}$ is zero as well

Everything can be written as a multiple of $\frac{\partial^2 u}{\partial \eta^2}$

Classification of PDE's (cont'ed)

In conclusion, in the parabolic case, the change of variables

$$\xi = 2Ax_2 - Bx_1, \quad \eta = x_1$$

maps the equation to

$$A \frac{\partial^2 u}{\partial \eta^2} = 0$$

which has the general solution

$$u = p(\xi) + \eta q(\xi)$$

One family of characteristics $\xi = \text{constant}$

Classification of PDE's (cont'ed)

Elliptic case

No choice of parameters makes coefficients of $\frac{\partial^2 u}{\partial \xi^2}$ and $\frac{\partial^2 u}{\partial \eta^2}$ vanish
In this case change of variables

$$\xi = \frac{2Ax_2 - Bx_1}{\sqrt{4AC - B^2}}, \quad \eta = x_1$$

maps equation to

$$A \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \right) = 0$$

No family of characteristics (infinite speed of propagation, no discontinuities allowed)

Classification of PDE's (cont'ed)

Final examples

- ▶ Laplace equation: elliptic
- ▶ Wave equation: hyperbolic
- ▶ Heat equation: parabolic
- ▶ Convection-diffusion equation:

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u + \operatorname{div}(\vec{\beta}u) = 0$$

parabolic, degenerating to hyperbolic as ε tends to zero.

End of Part I

Closed form of 1D wave equation solution

Change of variables

$$y = x + ct, \quad z = x - ct, \quad u(x, t) = w(y, z)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$\frac{\partial^2 w}{\partial y \partial z} = 0 \Rightarrow w = w_1(y) + w_2(z) = w_1(x + ct) + w_2(x - ct)$$

