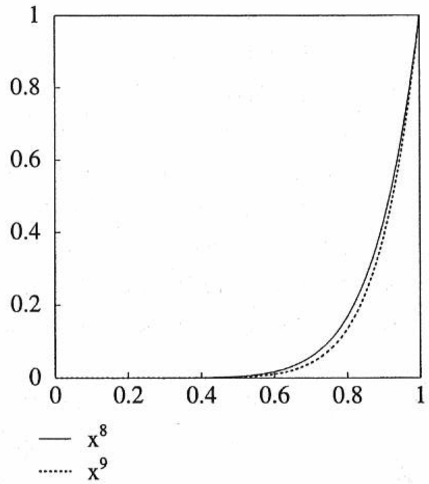

Figure 6.4: The basis functions t^8 and t^9.

In general, linear systems of algebraic equations obtained from the discretization of a differential equation tend to become ill-conditioned as the discretization is refined. This is understandable because refining the discretization and increasing the accuracy of the approximation makes it more likely that computing the residual error is influenced by the finite precision of the computer, for example. However, the degree of ill conditioning is influenced greatly by the differential equation and the choice of trial and test spaces, and even the choice of basis functions for these spaces. The standard monomial basis used above leads to an ill-conditioned system because the different monomials become very similar as the degree increases. This is related to the fact that the monomials are not an orthogonal basis. In general, the best results with respect to reducing the effects of ill-conditioning are obtained by using an orthogonal bases for the trial and test spaces. As an example, the Legendre polynomials, $\{\varphi_i(x)\}$, with $\varphi_0 \equiv 1$ and

$$\varphi_i(x) = (-1)^i \frac{\sqrt{2i+1}}{i!} \frac{d^i}{dx^i}\left(x^i(1-x)^i\right), \quad 1 \le i \le q,$$

form an orthonormal basis for $\mathcal{P}^q(0,1)$ with respect to the L_2 inner product. It becomes more complicated to formulate the discrete equations using this basis, but the effects of finite precision are greatly reduced.

Another possibility, which we take up in the second section, is to use piecewise polynomials. In this case, the basis functions are "nearly orthogonal".

Problem 6.6. (a) Show that φ_3 and φ_4 are orthogonal.

6.2. Galerkin's method with piecewise polynomials

We start by deriving the basic model of stationary heat conduction and then formulate a finite element method based on piecewise linear approximation.

6.2.1. A model for stationary heat conduction

We model heat conduction a thin heat-conducting wire occupying the interval $[0,1]$ that is heated by a *heat source* of intensity $f(x)$, see Fig. 6.5. We are interested in the stationary distribution of the temperature $u(x)$

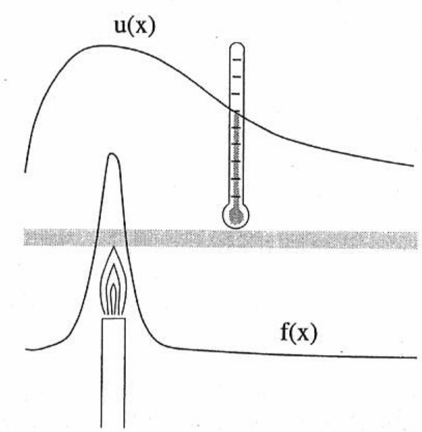

Figure 6.5: A heat conducting wire with a source $f(x)$.

in the wire. We let $q(x)$ denote the heat flux in the direction of the positive x-axis in the wire at $0 < x < 1$. Conservation of energy in a stationary case requires that the net heat flux through the endpoints of an arbitrary sub-interval (x_1, x_2) of $(0,1)$ be equal to the heat produced in (x_1, x_2) per unit time:

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} f(x)\, dx.$$