into (6.2) gives

$$\int_0^1 \left(\sum_{j=1}^q j\xi_j t^{j-1} - \lambda u_0 - \lambda \sum_{j=1}^q \xi_j t^j \right) v(t)\,dt = 0 \quad \text{for all } v \in V_0^{(q)}.$$

It suffices to insure that this equation holds for every basis function for $V_0^{(q)}$, yielding the set of equations:

$$\sum_{j=1}^q j\xi_j \int_0^1 t^{j+i-1}\,dt - \lambda \sum_{j=1}^q \xi_j \int_0^1 t^{j+i}\,dt = \lambda u_0 \int_0^1 t^i\,dt, \quad i = 1, ..., q,$$

where we have moved the terms involving the initial data to the right-hand side. Computing the integrals gives

$$\sum_{j=1}^q \left(\frac{j}{j+i} - \frac{\lambda}{j+i+1} \right) \xi_j = \frac{\lambda}{i+1} u_0, \quad i = 1, ..., q. \tag{6.4}$$

This is a $q \times q$ system of equations that has a unique solution if the matrix $A = (a_{ij})$ with coefficients

$$a_{ij} = \frac{j}{j+i} - \frac{\lambda}{j+i+1}, \quad i, j = 1, ..., q,$$

is invertible. It is possible to prove that this is the case, though it is rather tedious and we skip the details. In the specific case $u_0 = \lambda = 1$ and $q = 3$, the approximation is

$$U(t) \approx 1 + 1.03448t + .38793t^2 + .301724t^3,$$

which we obtain solving a 3×3 system.

Problem 6.2. Compute the Galerkin approximation for $q = 1, 2, 3$, and 4 assuming that $u_0 = \lambda = 1$.

Plotting the solution and the approximation for $q = 3$ in Fig. 6.1, we see that the two essentially coincide.

Since we know the exact solution u in this case, it is natural to compare the accuracy of U to other approximations of u in $V^{(q)}$. In Fig. 6.2, we plot the errors of U, the third degree polynomial interpolating u at $0, 1/3, 2/3$, and 1, and the third degree Taylor polynomial

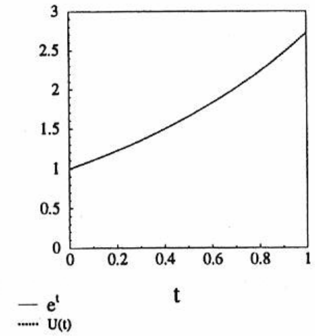

Figure 6.1: The solution of $\dot{u} = u$ and the third degree Galerkin approximation.

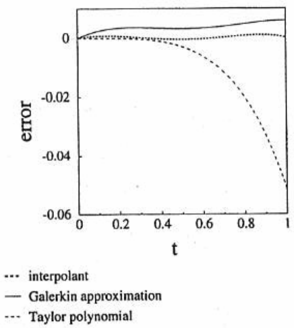

Figure 6.2: The errors of the third degree Galerkin approximation, a third degree interpolant of the solution, and the third degree Taylor polynomial of the solution.

of u computed at $t = 0$. The error of U compares favorably with the error of the interpolant of U and both of these are more accurate than the Taylor polynomial of u in the region near $t = 1$ as we would expect. We emphasize that the Galerkin approximation U attains this accuracy without any specific knowledge of the solution u except the initial data at the expense of solving a linear system of equations.

Problem 6.3. Compute the $L_2(0,1)$ projection into $\mathcal{P}^3(0,1)$ of the exact solution u and compare to U.

Problem 6.4. Determine the discrete equations if the test space is changed to $V^{(q-1)}$.