

The solution of (6.1), $u(t) = u_0 \exp(\lambda t)$, is a smooth increasing function when $\lambda > 0$.

6.1.2. Galerkin's method

We now show how to compute a polynomial approximation U of u in the set of polynomials $V^{(q)} = \mathcal{P}^q(0, 1)$ on $[0, 1]$ of degree at most q using Galerkin's method. We know there are good approximations of the solution u in this set, for example the Taylor polynomial and interpolating polynomials, but these require knowledge of u or derivatives of u at certain points in $[0, 1]$. The goal here is to compute a polynomial approximation of u using only the information that u solves a specified differential equation and has a specified value at one point. We shall see that this is precisely what Galerkin's method achieves. Since we already know the analytic solution in this model case, we can use this knowledge to evaluate the accuracy of the approximations.

Because $\{t^j\}_{j=0}^q$ is a basis for $V^{(q)}$, we can write $U(t) = \sum_{j=0}^q \xi_j t^j$ where the coefficients $\xi_j \in \mathbb{R}$ are to be determined. It is natural to require that $U(0) = u_0$, that is $\xi_0 = u_0$, so we may write

$$U(t) = u_0 + \sum_{j=1}^q \xi_j t^j,$$

where the “unknown part” of U , namely $\sum_{j=1}^q \xi_j t^j$, is in the subspace $V_0^{(q)}$ of $V^{(q)}$ consisting of the functions in $V^{(q)}$ that are zero at $t = 0$, i.e. in $V_0^{(q)} = \{v : v \in V^{(q)}, v(0) = 0\}$.

Problem 6.1. Prove that $V_0^{(q)}$ is a subspace of $V^{(q)}$.

We determine the coefficients by requiring U to satisfy the differential equation in (6.1) in a suitable “average” sense. Of course U can't satisfy the differential equation at every point because the exact solution is not a polynomial. In Chapter 4, we gave a concrete meaning to the notion that a function be zero on average by requiring the function to be orthogonal to a chosen subspace of functions. The Galerkin method is based on this idea. We define the *residual error* of a function v for the equation (6.1) by

$$R(v(t)) = \dot{v}(t) - \lambda v(t).$$

The residual error $R(v(t))$ is a function of t once v is specified. $R(v(t))$ measures how well v satisfies the differential equation at time t . If the residual is identically zero, that is $R(v(t)) \equiv 0$ for all $0 \leq t \leq 1$, then the equation is satisfied and v is the solution. Since the exact solution u is not a polynomial, the residual error of a function in $V^{(q)}$ that satisfies the initial condition is never identically zero, though it can be zero at distinct points.

The Galerkin approximation U is the function in $V^{(q)}$ satisfying $U(0) = u_0$ such that its residual error $R(U(t))$ is orthogonal to all functions in $V_0^{(q)}$, i.e.,

$$\int_0^1 R(U(t))v(t) dt = \int_0^1 (\dot{U}(t) - \lambda U(t))v(t) dt = 0 \quad \text{for all } v \in V_0^{(q)}. \quad (6.2)$$

This is the *Galerkin orthogonality* property of U , or rather of the residual $R(U(t))$. Since the coefficient of U with respect to the basis function 1 for $V^{(q)}$ is already known ($\xi_0 = u_0$), we require (6.2) to hold only for functions v in $V_0^{(q)}$. By way of comparison, note that the true solution satisfies a stronger orthogonality condition, namely

$$\int_0^1 (\dot{u} - \lambda u)v dt = 0 \quad \text{for all functions } v. \quad (6.3)$$

We refer to the set of functions where we seek the Galerkin solution U , in this case the space $V^{(q)}$ of polynomials w satisfying $w(0) = u_0$, as the *trial space* and the space of the functions used for the orthogonality condition, which is $V_0^{(q)}$, as the *test space*. In this case, the trial and test space are different because of the *non-homogeneous* initial condition $w(0) = u_0$ (assuming $u_0 \neq 0$), satisfied by the trial functions and the homogeneous boundary condition $v(0) = 0$ satisfied by the test functions $v \in V_0^{(q)}$. In general, different methods are obtained choosing the trial and test spaces in different ways.

6.1.3. The discrete system of equations

We now show that (6.2) gives an invertible system of linear algebraic equations for the coefficients of U . Substituting the expansion for U