6

Galerkin's Method

It is necessary to solve differential equations. (Newton)

Ideally, I'd like to be the eternal novice, for then only, the surprises would be endless. (Keith Jarret)

In Chapters 3 and 5, we discussed the numerical solution of the simple initial value problem $u'(x) = f(x)$ for $a < x \le b$ and $u(a) = u_0$, using piecewise polynomial approximation. In this chapter, we introduce *Galerkin's method* for solving a general differential equation, which is based on seeking an (approximate) solution in a (finite-dimensional) space spanned by a set of basis functions which are easy to differentiate and integrate, together with an orthogonality condition determining the coefficients or coordinates in the given basis. With a finite number of basis functions, Galerkin's method leads to a system of equations with finitely many unknowns which may be solved using a computer, and which produces an approximate solution. Increasing the number of basis functions improves the approximation so that in the limit the exact solution may be expressed as an infinite series. In this book, we normally use Galerkin's method in the computational form with a finite number of basis functions. The basis functions may be global polynomials, piecewise polynomials, trigonometric polynomials or other functions. The finite element method in basic form is Galerkin's method with piecewise polynomial approximation. In this chapter, we apply Galerkin's method to two examples with a variety of basis functions. The first example is an initial value problem that models population growth and we use a global polynomial approximation. The second example is a boundary

value problem that models the flow of heat in a wire and we use piecewise polynomial approximation, more precisely piecewise linear approximation. This is a classic example of the *finite element method.* For the second example, we also discuss the *spectral method* which is Galerkin's method with trigonometric polynomials.

The idea of seeking a solution of a differential equation as a linear combination of simpler basis functions, is old. Newton and Lagrange used power series with global polynomials and Fourier and Riemann used Fourier series based on trigonometric polynomials. These approaches work for certain differential equations posed on domains with simple geometry and may give valuable qualitative information, but cannot be used for most of the problems arising in applications. The finite element method based on piecewise polynomials opens the possibility of solving general differential equations in general geometry using a computer. For some problems, combinations of trigonometric and piecewise polynomials may be used.

6.1. Galerkin's method with global polynomials

6.1.1. A population model

In the simplest model for the growth of a population, like the population of rabbits in West Virginia, the rate of growth of the population is proportional to the population itself. In this model we ignore the effects of predators, overcrowding, and migration, for example, which might be okay for a short time provided the population of rabbits is relatively small in the beginning. We assume that the time unit is chosen so that the model is valid on the time interval $[0,1]$. We will consider more realistic models valid for longer intervals later in the book. If $u(t)$ denotes the population at time t then the differential equation expressing the simple model is $\dot{u}(t) = \lambda u(t)$, where λ is a positive real constant and $\dot{u} = du/dt$. This equation is usually posed together with an initial condition $u(0) = u_0$ at time zero, in the form of an initial value problem:

$$\begin{cases} \dot{u}(t) = \lambda u(t) & \text{for } 0 < t \le 1, \\ u(0) = u_0. \end{cases} \tag{6.1}$$