Theorem 4.4. *If the restriction of a to any $T \in \mathcal{T}_h$ is constant, there exists a constant C_4 such that, for any $T \in \mathcal{T}_h$,*

$$\eta_{sc,T} \leq C_4\{|T|^{\frac{1}{2}}\|\mathbf{u} - \mathbf{u}_h\|_{L^2(D_T)} + \|p - p_h\|_{L^2(D_T)}\}. \qquad (78)$$

Proof. This result is an immediate consequence of Lemmas 4.4 and 4.5.

Putting together the results for both estimators we have the following a posteriori error estimate for the mixed finite element approximation.

We define

$$\eta_T^2 = \eta_{vect,T}^2 + \eta_{sc,T}^2 \quad \text{and} \quad \eta^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2.$$

Theorem 4.5. *If Ω is simply connected and the restriction of a to any $T \in \mathcal{T}_h$ is constant, there exist constants C_5 and C_6 such that*

$$\eta_T \leq C_5\{\|\mathbf{u} - \mathbf{u}_h\|_{L^2(D_T)} + \|p - p_h\|_{L^2(D_T)}\}$$

and

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq C_6\{\eta + h\|f - P_h f\|_{L^2(\Omega)}\}.$$

Proof. This result is an immediate consequence of Theorems 4.1, 4.2, 4.3 and 4.4.

5 The General Abstract Setting

The problem considered in the previous sections is a particular case of a general class of problems that we are going to analyze in this section. The theory presented here was first developed by Brezzi [13]. Some of the ideas were also introduced for particular problems by Babuška [9] and by Crouzeix and Raviart [22]. We also refer the reader to [32, 31] and to the books [17, 45, 37].

Let V and Q be two Hilbert spaces and suppose that $a(\ ,\ )$ and $b(\ ,\ )$ are continuous bilinear forms on $V \times V$ and $V \times Q$ respectively, i.e.,

$$|a(u, v)| \leq \|a\|\|u\|_V\|v\|_V \quad \forall u \in V, \forall v \in V$$

and

$$|b(v, q)| \leq \|b\|\|v\|_V\|q\|_Q \quad \forall v \in V,\ \forall q \in Q.$$

Consider the problem: given $f \in V'$ and $g \in Q'$ find $(u, p) \in V \times Q$ solution of

$$\begin{cases} a(u, v) + b(v, p) = \langle f, v\rangle & \forall v \in V \\ \qquad\qquad\ b(u, q) = \langle g, q\rangle & \forall q \in Q \end{cases} \qquad (79)$$

where $\langle\, .\, ,\, .\, \rangle$ denotes the duality product between a space and its dual one.

For example, the mixed formulation of second order elliptic problems considered in the previous sections can be written in this way with

$$V = H(\text{div}, \Omega), \quad Q = L^2(\Omega)$$

and

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mu \mathbf{u} \cdot \mathbf{v} \, dx, \quad b(\mathbf{v}, p) = \int_\Omega p \, \text{div} \, \mathbf{v} \, dx \, .$$

The general problem (79) can be written in the standard way

$$c((u,p),(v,q)) = \langle f, v \rangle + \langle g, q \rangle \;\; \forall (v,q) \in V \times Q \tag{80}$$

where c is the continuous bilinear form on $V \times Q$ defined by

$$c((u,p),(v,q)) = a(u,v) + b(v,p) + b(u,q).$$

However, the bilinear form is not coercive and therefore the usual finite element error analysis can not be applied.

We will give sufficient conditions (indeed, they are also necessary although we are not going to prove it here, we refer to [17, 37]) on the forms a and b for the existence and uniqueness of a solution of problem (79). Below, we will also show that their discrete version ensures the stability and optimal order error estimates for the Galerkin approximations. These results were obtained by Brezzi [13] (see also [17] were more general results are proved).

Introducing the continuous operators $A : V \to V'$, $B : V \to Q'$ and its adjoint $B^* : Q \to V'$ defined by,

$$\langle Au, v \rangle_{V' \times V} = a(u,v)$$

and

$$\langle Bv, q \rangle_{Q' \times Q} = b(v,q) = \langle v, B^* q \rangle_{V \times V'}$$

problem (79) can also be written as

$$\begin{cases} Au + B^* p = f & \text{in } V' \\ Bu = g & \text{in } Q'. \end{cases} \tag{81}$$

Let us introduce $W = KerB \subset V$ and, for $g \in Q'$,

$$W(g) = \{ v \in V : \; Bv = g \}.$$

Now, if $(u,p) \in V \times Q$ is a solution of (79) then, it is easy to see that u is a solution of the problem

$$u \in W(g), \quad a(u,v) = \langle f, v \rangle \quad \forall v \in W. \tag{82}$$

We will find conditions under which both problems (79) and (82) are equivalent, in the sense that for a solution $u \in W(g)$ of (82) there exists a unique $p \in Q$ such that (u,p) is a solution of (79).

In what follows we will use the following well-known result of functional analysis. Given a Hilbert space V and $S \subset V$ we define $S^0 \subset V'$ by

$$S^0 = \{ L \in V' : \langle L \, , \, v \rangle = 0, \;\; \forall v \in S \}.$$

Theorem 5.1. *Let V_1 and V_2 be Hilbert spaces and $A : V_1 \to V_2'$ be a continuous linear operator. Then,*

$$(Ker\,A)^0 = \overline{Im\,A^*} \tag{83}$$

and

$$(Ker\,A^*)^0 = \overline{Im\,A}. \tag{84}$$

Proof. It is easy to see that $Im\,A^* \subset (Ker\,A)^0$ and that $(Ker\,A)^0$ is a closed subspace of V_1'. Therefore

$$\overline{Im\,A^*} \subset (Ker\,A)^0\,.$$

Suppose now that there exists $L_0 \in V_1'$ such that $L_0 \in (Ker\,A)^0 \setminus \overline{Im\,A^*}$. Then, by the Hahn-Banach theorem there exists a linear continuous functional defined on V_1' which vanishes on $\overline{Im\,A^*}$ and is different from zero on L_0. In other words, using the standard identification between V_1'' and V_1, there exists $v_0 \in V_1$ such that

$$\langle L_0, v_0\rangle \neq 0 \quad \text{and} \quad \langle L, v_0\rangle = 0 \quad \forall L \in \overline{Im\,A^*}.$$

In particular, for all $v \in V_2$

$$\langle Av_0, v\rangle = \langle v_0, A^*v\rangle = 0$$

and so $v_0 \in Ker\,A$ which, since $L_0 \in (Ker\,A)^0$, contradicts $\langle L_0, v_0\rangle \neq 0$. Therefore, $(Ker\,A)^0 \subset \overline{Im\,A^*}$ and so (83) holds. Finally, (84) is an immediate consequence of (83) because $(A^*)^* = A$.

Lemma 5.1. *The following properties are equivalent:*

(a) There exists $\beta > 0$ such that

$$\sup_{v\in V} \frac{b(v,q)}{\|v\|_V} \geq \beta\|q\|_Q \quad \forall q \in Q. \tag{85}$$

(b) B^* is an isomorphism from Q onto W^0 and,

$$\|B^*q\|_{V'} \geq \beta\|q\|_Q \quad \forall q \in Q. \tag{86}$$

(c) B is an isomorphism from $W^\perp$ onto Q' and,

$$\|Bv\|_{Q'} \geq \beta\|v\|_V \quad \forall v \in W^\perp. \tag{87}$$

Proof. Assume that (a) holds then, (86) is satisfied and so B^* is injective. Moreover $Im\,B^*$ is a closed subspace of V', indeed, suppose that $B^*q_n \to w$ then, it follows from (86) that

$$\|B^*(q_n - q_m)\|_{V'} \geq \beta\|q_n - q_m\|_Q$$

and, therefore, $\{q_n\}$ is a Cauchy sequence and so it converges to some $q \in Q$ and, by continuity of B^*, $w = B^*q \in Im\,B^*$. Consequently, using (83) we obtain that $Im\,B^* = W^0$ and therefore (b) holds.

Now, we observe that W^0 can be isometrically identified with $(W^\perp)'$. Indeed, denoting with $P^\perp : V \to W^\perp$ the orthogonal projection, for any $g \in (W^\perp)'$ we define $\tilde{g} \in W^0$ by $\tilde{g} = g \circ P^\perp$ and it is easy to check that $g \to \tilde{g}$ is an isometric bijection from $(W^\perp)'$ onto W^0 and then, we can identify these two spaces. Therefore (b) and (c) are equivalent.

Corollary 5.1. *If the form* b *satisfies (85) then, problems (79) and (82) are equivalent, that is, there exists a unique solution of (79) if and only if there exists a unique solution of (82).*

Proof. If (u, p) is a solution of (79) we know that u is a solution of (82). It rests only to check that for a solution u of (82) there exists a unique $p \in Q$ such that $B^*p = f - Au$ but, this follows from (b) of the previous lemma since, as it is easy to check, $f - Au \in W^0$.

Now we can prove the fundamental existence and uniqueness theorem for problem (79).

Lemma 5.2. *If there exists* $\alpha > 0$ *such that* a *satisfies*

$$\sup_{v \in W} \frac{a(u,v)}{\|v\|_V} \geq \alpha \|u\|_V \quad \forall u \in W \tag{88}$$

$$\sup_{u \in W} \frac{a(u,v)}{\|u\|_V} \geq \alpha \|v\|_V \quad \forall v \in W \tag{89}$$

then, for any $g \in W'$ *there exists* $w \in W$ *such that*

$$a(w, v) = \langle g, v \rangle \qquad \forall v \in W$$

and moreover

$$\|w\|_W \leq \frac{1}{\alpha} \|g\|_{W'}. \tag{90}$$

Proof. Considering the operators

$$A : W \to W' \ \text{ and } \ A^* : W \to W'$$

defined by

$$\langle Au, v \rangle_{W' \times W} = a(u, v) \ \text{ and } \ \langle u, A^*v \rangle_{W \times W'} = a(u, v),$$

conditions (88) and (89) can be written as

$$\|Au\|_{W'} \geq \alpha \|u\|_W \quad \forall u \in W \tag{91}$$

and

$$\|A^*v\|_{W'} \geq \alpha \|v\|_W \quad \forall v \in W \tag{92}$$

respectively. Therefore, it follows from (89) that

$$Ker\, A^* = \{0\}.$$

Then, from (84), we have

$$(Ker\, A^*)^0 = \overline{Im\, A}$$

and so

$$\overline{Im\, A} = W'.$$

Using now (91) and the same argument used in (85) to prove that $Im\, B^*$ is closed, we can show that $Im\, A$ is a closed subspace of W' and consequently $Im\, A = W'$ as we wanted to show. Finally (90) follows immediately from (91).

Theorem 5.2. *If a satisfies (88) and (89), and b satisfies (85) then, there exists a unique solution $(u, p) \in V \times Q$ of problem (79) and moreover,*

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{Q'} \tag{93}$$

and,

$$\|p\|_Q \leq \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|f\|_{V'} + \frac{\|a\|}{\beta^2}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{Q'}. \tag{94}$$

Proof. First we show that there exists a solution u of problem (82). Since (85) holds we know from Lemma 5.1 that there exists a unique $u_0 \in W^\perp$ such that $Bu_0 = g$ and

$$\|u_0\|_V \leq \frac{1}{\beta}\|g\|_{Q'} \tag{95}$$

then, the existence of u solution of (82) is equivalent to the existence of $w = u - u_0 \in W$ such that

$$a(w, v) = \langle f, v\rangle - a(u_0, v) \ \ \forall v \in W$$

but, from Lemma 5.2, it follows that such a w exists and moreover,

$$\|w\|_V \leq \frac{1}{\alpha}\{\|f\|_{V'} + \|a\|\|u_0\|_V\} \leq \frac{1}{\alpha}\{\|f\|_{V'} + \frac{\|a\|}{\beta}\|g\|_{Q'}\}$$

where we have used (95).

Therefore, $u = w + u_0$ is a solution of (82) and satisfies (93).

Now, from Corollary 5.1 it follows that there exists a unique $p \in Q$ such that (u, p) is a solution of (79). On the other hand, from Lemma 5.1 it follows that (86) holds and using it, it is easy to check that

$$\|p\|_Q \leq \frac{1}{\beta}\{\|f\|_{V'} + \|a\|\|u\|_V\}$$

which combined with (93) yields (94). Finally, the uniqueness of solution follows from (93) and (94).

Assume now that we have two families of subspaces $V_h \subset V$ and $Q_h \subset Q$. The Galerkin approximation $(u_h, p_h) \in V_h \times Q_h$ to the solution $(u, p) \in V \times Q$ of problem (79), is defined by

$$\begin{cases} a(u_h, v) + b(v, p_h) = \langle f, v \rangle & \forall v \in V_h \\ b(u_h, q) = \langle g, q \rangle & \forall q \in Q_h. \end{cases} \tag{96}$$

For the error analysis it is convenient to introduce the associated operator $B_h : V_h \to Q'_h$ defined by

$$\langle B_h v, q \rangle_{Q'_h \times Q_h} = b(v, q)$$

and the subsets of V_h, $W_h = Ker\, B_h$ and

$$W_h(g) = \{v \in V_h \ : B_h v = g \ \text{ in } \ Q'_h\}$$

where g is restricted to Q_h.

In order to have the Galerkin approximation well defined we need to know that there exists a unique solution $(u_h, p_h) \in V_h \times Q_h$ of problem (96). In view of Theorem 5.2, this will be true if there exist $\alpha^* > 0$ and $\beta^* > 0$ such that

$$\sup_{v \in W_h} \frac{a(u, v)}{\|v\|_V} \geq \alpha^* \|u\|_V \quad \forall u \in W_h \tag{97}$$

$$\sup_{u \in W_h} \frac{a(u, v)}{\|u\|_V} \geq \alpha^* \|v\|_V \quad \forall v \in W_h \tag{98}$$

and,

$$\sup_{v \in V_h} \frac{b(v, q)}{\|v\|_V} \geq \beta^* \|q\|_Q \quad \forall q \in Q_h. \tag{99}$$

In fact, (98) follows from (97) since W_h is finite dimensional.

Now, we can prove the fundamental general error estimates due to Brezzi [13].

Theorem 5.3. *If the forms a and b satisfy (97), (98) and (99), problem (96) has a unique solution and there exists a constant C, depending only on α^*, β^*, $\|a\|$ and $\|b\|$ such that the following estimates hold. In particular, if the constants α^* and β^* are independent of h then, C is independent of h.*

$$\|u - u_h\|_V + \|p - p_h\|_Q \leq C\{\inf_{v \in V_h} \|u - v\|_V + \inf_{q \in Q_h} \|p - q\|_Q\} \tag{100}$$

and, when $Ker\, B_h \subset Ker\, B$,

$$\|u - u_h\|_V \leq C \inf_{v \in V_h} \|u - v\|_V. \tag{101}$$

Proof. From Theorem 5.2, there exists a unique solution $(u_h, p_h) \in V_h \times Q_h$ of (96).

On the other hand, given $(v, q) \in V_h \times Q_h$, we have

$$a(u_h - v, w) + b(w, p_h - q) = a(u - v, w) + b(w, p - q) \quad \forall w \in V_h \tag{102}$$

and

$$b(u_h - v, r) = b(u - v, r) \quad \forall r \in Q_h. \tag{103}$$

Now, for fixed (v, q) , the right hand sides of (102) and (103) define linear functionals on V_h and Q_h which are continuous with norms bounded by

$$\|a\|\|u - v\|_V + \|b\|\|p - q\|_Q \quad \text{and} \quad \|b\|\|u - v\|_V$$

respectively. Then, it follows from Theorem 5.2 that, for any $(v, q) \in V_h \times Q_h$,

$$\|u_h - v\|_V + \|p_h - q\|_Q \leq C\{\|u - v\|_V + \|p - q\|_Q\}$$

and therefore (100) follows by the triangular inequality.

On the other hand, we know that $u_h \in W_h(g)$ is a solution of

$$a(u_h, v) = \langle f, v\rangle \quad \forall v \in W_h \tag{104}$$

and, since $W_h \subset W$, subtracting (104) from (82) we have,

$$a(u - u_h, v) = 0 \quad \forall v \in W_h. \tag{105}$$

Now, for $w \in W_h(g)$, $u_h - w \in W_h$ and so from (97) and (105) we have

$$\alpha^* \|u_h - w\|_V \leq \sup_{v \in W_h} \frac{a(u_h - w, v)}{\|v\|_V} = \sup_{v \in W_h} \frac{a(u - w, v)}{\|v\|_V} \leq \|a\|\|u - w\|_V$$

and therefore,

$$\|u - u_h\|_V \leq \left(1 + \frac{\|a\|}{\alpha^*}\right) \inf_{w \in W_h(g)} \|u - w\|_V.$$

To conclude the proof we will see that, if (99) holds then,

$$\inf_{w \in W_h(g)} \|u - w\|_V \leq \left(1 + \frac{\|b\|}{\beta^*}\right) \inf_{v \in V_h} \|u - v\|_V. \tag{106}$$

Given $v \in V_h$, from Lemma 5.1 we know that there exists a unique $z \in W_h^\perp$ such that

$$b(z, q) = b(u - v, q) \quad \forall q \in Q_h$$

and

$$\|z\|_V \leq \frac{\|b\|}{\beta^*}\|u - v\|_V$$

thus, $w = z + v \in V_h$ satisfies $B_h w = g$, that is, $w \in W_h(g)$. But

$$\|u - w\|_V \leq \|u - v\|_V + \|z\|_V \leq (1 + \frac{\|b\|}{\beta^*})\|u - v\|_V$$

and so (106) holds.

In the applications, a very useful criterion to check the inf–sup condition (99) is the following result due to Fortin [32].

Theorem 5.4. *Assume that (85) holds. Then, the discrete inf–sup condition (99) holds with a constant* $\beta^* > 0$ *independent of* h*, if and only if, there exists an operator*

$$\Pi_h : V \to V_h$$

such that

$$b(v - \Pi_h v, q) = 0 \quad \forall v \in V \,,\; \forall q \in Q_h \tag{107}$$

and,

$$\|\Pi_h v\|_V \leq C\|v\|_V \quad \forall v \in V \tag{108}$$

with a constant $C > 0$ *independent of* h.

Proof. Assume that such an operator Π_h exists. Then, from (107), (108) and (85) we have, for $q \in Q_h$,

$$\beta\|q\|_Q \leq \sup_{v\in V} \frac{b(v,q)}{\|v\|_V} = \sup_{v\in V} \frac{b(\Pi_h v,q)}{\|v\|_V} \leq C \sup_{v\in V} \frac{b(\Pi_h v,q)}{\|\Pi_h v\|_V}$$

and therefore, (99) holds with $\beta^* = \beta/C$.

Conversely, suppose that (99) holds with β^* independent of h. Then, from (87) we know that for any $v \in V$ there exists a unique $v_h \in W_h^\perp$ such that

$$b(v_h, q) = b(v, q) \quad \forall q \in Q_h$$

and,

$$\|v_h\|_V \leq \frac{\|b\|}{\beta^*}\|v\|_V \,.$$

Therefore, $\Pi_h v = v_h$ defines the required operator.

Remark 5.1. In practice, it is sometimes enough to show the existence of the operator Π_h on a subspace $S \subset V$, where the exact solution belongs, verifying (107) and (108) for $v \in S$ and the norm on the right hand side of (108) replaced by a strongest norm (that of the space S). This is in some cases easier because the explicit construction of the operator Π_h requires regularity assumptions which do not hold for a general function in V. For example, in the problem analyzed in the previous sections we have constructed this operator on a subspace of $V = H(\text{div}, \Omega)$ because the degrees of freedom defining the operator do not make sense in $H(\text{div}, T)$, indeed, we need more regularity for $\mathbf{v}$ (for example $\mathbf{v} \in H^1(T)^n$) in order to have the integral of the normal component of $\mathbf{v}$ against a polynomial on a face F of T well defined. It is possible to show the existence of Π_h defined on $H(\text{div}, \Omega)$ satisfying (107) and (108) (see [32, 46]). However, as we have seen, this is not really necessary to obtain optimal error estimates.

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