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COMPUTATIONAL MODELS OF ELECTROMAGNETIC RESONATORS: ANALYSIS OF EDGE ELEMENT APPROXIMATION*

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Abstract. The purpose of this paper is to address some difficulties which arise in computing the eigenvalues of Maxwell's system by a finite element method. Depending on the method used, the spectrum may be polluted by spurious modes which are difficult to pick out among the approximations of the physically correct eigenvalues. Here we propose a criterion to establish whether or not a finite element scheme is well suited to approximate the eigensolutions and, in the positive case, we estimate the rate of convergence of the eigensolutions. This criterion involves some properties of the finite element space and of a suitable Fortin operator. The lowest-order edge elements, under some regularity assumptions, give an example of space satisfying the required conditions. The construction of such a Fortin operator in very general geometries and for any order edge elements is still an open problem.

Moreover, we give some justification for the spectral pollution which occurs when nodal elements are used. Results of numerical experiments confirming the theory are also reported.

Key words. spurious eigenvalues, finite element methods, Maxwell's equations, edge elements

AMS subject classifications. 65N25, 65N30

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1. Introduction.

Computational models of electromagnetic cavity resonators are of basic importance in designing resonant structures for advanced applications such as particle acceleration for high energy physics experiments or synchrotron radiation production. The same can be said of advanced waveguide design, which exploits very similar models, sharing the same main features. The usual way of modeling a cavity resonator (or a waveguide) leads to a system where the divergence-free constraint is a consequence of the other equations in the case of nonzero frequencies. Since this constraint may be difficult to impose on the numerical approximation, it is often dropped, giving rise to a seemingly simple differential eigenvalue problem. A major difficulty is actually hidden in this procedure: a physically meaningless infinite-dimensional eigenspace related to the zero eigenvalue is added to the spectrum. The first researchers who tried to solve this problem were not fully aware of this fact and its mathematical implications. In particular, they did not suspect that in this case, as the resolvent operator is noncompact, a framework different than usual is needed to study the convergence of a finite element method [18, 19].

Early attempts to solve this eigenvalue problem by finite element methods (e.g., [14, 17]) were done, with each field component represented by nodal elements extending in the most straightforward way the methods that had been successful with scalar problems. It soon became apparent that the computed spectrum was affected by the occurrence of modes lacking any physical meaning interspersed among the

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correct ones (e.g., [14, 17]). In engineering literature, these modes are usually thought to be produced by a bad approximation of the zero frequency infinite-dimensional eigenspace and are usually called "spurious." This name is questionable: in this context, we prefer to call a "nonphysical" mode any (good or bad) approximation of a zero frequency eigensolution and to regard as spurious only those discrete eigensolutions produced by the numerical method.

Efforts to avoid nonphysical modes by forcing the solution to be divergence free (e.g., [28]) or to push their frequencies toward the higher part of the spectrum by a penalty method (e.g., [33]) have been only partially successful. Practical experience has shown that these modes may appear just at zero frequency or may pollute the whole numerical spectrum, depending on the finite element basis used in the approximation. In the latter case, it is very difficult to distinguish physical modes from others. Moreover, although a particular nonphysical mode occurring at a positive frequency approaches the zero frequency when the mesh is refined, new ones enter the spectrum from above. This behavior of nonphysical modes at positive frequency makes their occurrence a very annoying phenomenon [33]. One more reason to avoid them is the following: in order to deal with steady-state and transient problems, finite element bases which confine the nonphysical modes exactly at zero frequency are required [11]. Now, it is well known that edge elements enjoy this property (e.g., [9, 35, 20]), while nodal elements, on general meshes, do not (e.g., [33, 14, 17]). However, in our opinion, a fully satisfactory theory about the discrete nonphysical eigensolutions still needs to be developed [20, 13].

In previous papers, the so-called "inclusion condition" [22, 15] was believed to be the reason why some finite elements well approximate the infinite-dimensional eigenspace corresponding to the zero frequency. However, it has been proved recently that the inclusion condition, although sufficient for that, is too strong [13] and not satisfied by edge elements [13, 12].

Another common belief (e.g., [34]), now recognized as false, was that edge elements work well because they implicitly enforce the divergence-free constraint. This is evidently wrong; indeed, even though the lowest-order edge elements are divergence free inside each element, a nonzero divergence in the sense of distributions arises from the discontinuities of the normal component at the interelement boundaries [20, 22].

Also, the fact that the lowest-order edge element space includes the gradients of the linear nodal elements on the same mesh was indicated as the reason for the good behavior of edge element approximations [9]: the proof given is far from convincing; therefore this point should be thoroughly investigated.

A proof of convergence for the lowest-order edge element method in the very similar case of the waveguide problem has been given in [3]. That proof, which makes use of a theory for the approximation of the spectrum of a noncompact operator [18, 19] and of the discrete compactness property in [27], could be adapted to our case.

In this paper, which extends [8], we study the approximation of the eigenproblem associated with Maxwell's equation by finite element methods. Our purpose is to give abstract conditions on the choice of finite element spaces in order to avoid the occurrence of spectral pollution. Our argument is quite easy and relies upon an equivalent mixed formulation for which the associated operator is compact. Next we state sufficient conditions which guarantee that the discrete eigenmodes of the mixed formulation converge to those of the continuous problem. In this framework, using the results of [5] we show under suitable assumptions that, using lowest-order edge elements, all the nonphysical modes are confined to zero frequency. This approach is completely different from that proposed in [3] and does not require the theory

developed in [18, 19]. Moreover, our theory can be applied to every choice of finite element basis in order to state whether positive nonphysical and spurious modes can enter the discrete spectrum.

The outline of the paper is the following. The original problem, the equivalent mixed formulation, and their discrete approximations are introduced in section 2. Section 3 recalls the relevant theory, which is then applied in section 4 to show that the lowest-order edge element approximation converges and to give a partial explanation of the spectral pollution occurring with nodal elements. In section 5 very general situations covering most of the applications are considered: the performance of lowest-order edge elements in this case relies on a regularity result which is still an open problem. In section 6 results of numerical experiments confirming the theory are reported.

2. A model eigenvalue problem. In this section we introduce the eigenvalue problem we shall deal with. It can be deduced from Maxwell's system when a cavity resonator loaded with different homogeneous materials is modeled under the simplifying assumptions that the materials are loss free (i.e., no energy dissipation takes place in them) and that the cavity walls are perfectly conducting.

Let $\Omega \subset \mathbb{R}^3$ be an open Lipschitz bounded domain with outward normal unit vector $\underline{n}$. For the sake of simplicity we assume that Ω is simply connected. For a generalization of this hypothesis, see section 5. As usual, we denote by $\underline{E} : \Omega \to \mathbb{C}^3$ and $\underline{H} : \Omega \to \mathbb{C}^3$ the electric field phasor and magnetic field phasor, respectively, with the understanding that the observable fields are $\mathrm{Re}(\underline{E}\mathrm{e}^{j\omega t})$ and $\mathrm{Re}(\underline{H}\mathrm{e}^{j\omega t})$, with $\omega \in \mathbb{R}$ being the angular frequency. Let ε and μ be the electric permittivity and the magnetic permeability, respectively.

Moreover, we assume ε and μ to be real scalar functions satisfying

$$0 < \varepsilon_* \le \varepsilon(\underline{x}) \le \varepsilon^*, \qquad 0 < \mu_* \le \mu(\underline{x}) \le \mu^* \tag{1}$$

for almost every $\underline{x}$ in Ω.

By writing Maxwell's system for vector phasors we get

$$\begin{cases} \text{(i)} & \operatorname{curl} \underline{E} = -j\omega\mu\underline{H} & \text{in} & \Omega, \\ \text{(ii)} & \operatorname{div}(\varepsilon\underline{E}) = 0 & \text{in} & \Omega, \\ \text{(iii)} & \operatorname{curl} \underline{H} = j\omega\varepsilon\underline{E} & \text{in} & \Omega, \\ \text{(iv)} & \operatorname{div}(\mu\underline{H}) = 0 & \text{in} & \Omega, \\ \text{(v)} & \underline{E} \times \underline{n} = 0 & \text{on} & \partial\Omega, \\ \text{(vi)} & (\mu\underline{H}) \cdot \underline{n} = 0 & \text{on} & \partial\Omega. \end{cases} \tag{2}$$

Hence we look for a triplet $(\omega, \underline{E}, \underline{H})$ such that $\underline{E}$ and $\underline{H}$ do not vanish identically. Notice that the zero frequency $\omega = 0$ is not a solution of this problem. In fact, with a formal argument, we show that, if $\omega = 0$, problem (2) has the trivial solution only. Since (2i) becomes $\operatorname{curl} \underline{E} = 0$ and Ω is simply connected, we can write $\underline{E} = \underline{\nabla}\varphi$. Substituting it in (2ii) and (2v), we get that $\operatorname{div}(\varepsilon \underline{\nabla}\varphi) = 0$ in Ω and that φ is constant on $\partial\Omega$, from which follows that φ is constant in Ω and, thus, $\underline{E} \equiv 0$ in Ω. Similarly one also obtains $\underline{H} \equiv 0$.

It is classical to reduce the problem by eliminating one of the two fields. For instance, we eliminate the magnetic field and obtain the equivalent eigenvalue problem for the electric field (now denoted by $\underline{u}$),

$$\begin{cases} \text{(i)} & \operatorname{curl}(\mu^{-1} \operatorname{curl} \underline{u}) = \omega^2 \varepsilon \underline{u} & \text{in } \Omega, \\ \text{(ii)} & \operatorname{div}(\varepsilon\underline{u}) = 0 & \text{in } \Omega, \\ \text{(iii)} & \underline{u} \times \underline{n} = \underline{0} & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Before deducing a variational formulation of problem (3), we introduce some Hilbert spaces:

$$
\begin{aligned}
H^1(\Omega) &= \left\{v \in L^2(\Omega) : \underline{\nabla}\, v \in [L^2(\Omega)]^3\right\},\\
H_0^1(\Omega) &= \left\{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\right\},\\
H(\mathrm{curl};\Omega) &= \left\{\underline{v} \in [L^2(\Omega)]^3 : \mathrm{curl}\,\underline{v} \in [L^2(\Omega)]^3\right\},\\
H_0(\mathrm{curl};\Omega) &= \left\{\underline{v} \in H(\mathrm{curl};\Omega) : \underline{v} \times \underline{n}|_{\partial\Omega} = 0\right\},\\
H_0(\mathrm{curl}^0;\Omega) &= \left\{\underline{v} \in H_0(\mathrm{curl};\Omega) : \mathrm{curl}\,\underline{v} = 0\right\},\\
H(\mathrm{div};\Omega) &= \left\{\underline{v} \in [L^2(\Omega)]^3 : \mathrm{div}\,\underline{v} \in L^2(\Omega)\right\},\\
H(\mathrm{div}^0;\Omega) &= \left\{\underline{v} \in [L^2(\Omega)]^3 : \mathrm{div}\,\underline{v} = 0\right\},\\
H_0(\mathrm{div}^0;\Omega) &= \left\{\underline{v} \in H(\mathrm{div}^0;\Omega) : \underline{v} \cdot \underline{n}|_{\partial\Omega} = 0\right\},\\
H(\mathrm{div}^0;\Omega;w) &= \left\{\underline{v} \in [L^2(\Omega)]^3 : \mathrm{div}(w\underline{v}) = 0\right\},\\
H_0(\mathrm{div}^0;\Omega;w) &= \left\{\underline{v} \in H(\mathrm{div}^0;\Omega;w) : (w\underline{v}) \cdot \underline{n}|_{\partial\Omega} = 0\right\},\\
H(\mathrm{curl};\Omega;w) &= \{\underline{v} \in [L^2(\Omega)]^3 : \mathrm{curl}(w\underline{v}) \in [L^2(\Omega)]^3\}.
\end{aligned}
\tag{4}
$$

The symbol w in the last two spaces of (4) denotes some scalar-valued function as well as some tensor-valued function defined in Ω. Let us define the norms which will be useful in the following:

$$
\begin{aligned}
\|v\|_1^2 &= \|v\|_0^2 + \|\underline{\nabla}\, v\|_0^2,\\
\|v\|_{\mathrm{curl}}^2 &= \|v\|_0^2 + \|\mathrm{curl}\, v\|_0^2,\\
\|v\|_{\mathrm{div}}^2 &= \|v\|_0^2 + \|\mathrm{div}\, v\|_0^2,
\end{aligned}
\tag{5}
$$

where $\|\cdot\|_0$ denotes the usual norm of $[L^2(\Omega)]^d$; every integer $d \geq 1$; $(\cdot,\cdot)$ will denote the inner product of $[L^2(\Omega)]^d$.

The variational form of (3) then reads as follows:

$$
\begin{aligned}
&\text{find } \omega \in \mathbb{R} \text{ s.t. } \exists 0 \not\equiv \underline{u} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega;\varepsilon) :\\
&\quad (\mu^{-1}\,\mathrm{curl}\,\underline{u}, \mathrm{curl}\,\underline{v}) = \omega^2(\varepsilon\underline{u},\underline{v}) \qquad \forall \underline{v} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega;\varepsilon).
\end{aligned}
\tag{6}
$$

Thanks to the fact that the bilinear form $(\mu^{-1}\,\mathrm{curl}\,\underline{u}, \mathrm{curl}\,\underline{v})$ is symmetric, continuous, and coercive on the space $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega;\varepsilon)$ and to the compact embedding of $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega;\varepsilon)$ in $[L^2(\Omega)]^3$ (see [24, 21]), the operator associated with problem (6) is compact and self-adjoint. Hence problem (6) admits a countable set of real and positive eigenvalues and each eigenspace is finite-dimensional. Moreover, all the eigenfunctions can be chosen to be real, so that the whole analysis can be performed using real spaces.

We observe that $\omega^2 = 0$ is not an eigenvalue of problem (6). In fact, taking $\underline{v} = \underline{u}$ in (6) we have $(\mu^{-1}\,\mathrm{curl}\,\underline{u}, \mathrm{curl}\,\underline{u}) = 0$; that is, $\mathrm{curl}\,\underline{u} = 0$, due to (1). Moreover, since $\underline{u} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega;\varepsilon)$ and Ω is simply connected, $\mathrm{curl}\,\underline{u} = 0$ implies $\underline{u} \equiv 0$ (see [24], for instance).

Notice also that if $\omega^2 \neq 0$, then (3ii) is a consequence of (3i). One could drop the constraint (3ii) and, equivalently, seek $\underline{u}$ in $H_0(\mathrm{curl};\Omega)$ in problem (6). However, in this way, $\omega^2 = 0$ is added to the spectrum of the operator; the corresponding eigenspace is infinite-dimensional and coincides with $H_0(\mathrm{curl}^0;\Omega) = \underline{\nabla}\, H_0^1(\Omega)$. The compactness of the resolvent operator is lost, of course. Moreover, this new eigensolution lacks any physical meaning. On the other hand, the eigensolutions corresponding to $\omega^2 \neq 0$ remain unchanged.

In a finite element approximation of (6), however, there are two good reasons to drop condition (3ii) and discard a posteriori the zero frequency modes arising from

the meaningless infinite-dimensional eigenspace. First, the numerical treatment of this constraint is difficult (see, e.g., [33, 28, 10]); second, the same approach does not extend to steady-state and transient problems, and uneven approaches lead to difficulties in comparing the results of different analyses of the same electromagnetic device. In fact, since the solutions of steady-state and transient problems are not necessarily divergence free, they may have components in $H_0(\mathrm{curl}^0;\Omega)$ and, thus, the eigenfunctions corresponding to $\omega^2 = 0$ are required to expand them, in general, although they are physically meaningless by themselves. Hence the infinite-dimensional eigenspace $H_0(\mathrm{curl}^0;\Omega)$ must be accurately approximated rather than excluded from the problem (see [11]).

As a matter of fact, finite element approximations of resonant cavity problems are very often based on the variational problem

$$\begin{aligned} &\text{find } \omega \in \mathbb{R} \text{ s.t. } \exists 0 \not\equiv \underline{u} \in H_0(\mathrm{curl};\Omega) : \\ &\qquad (\mu^{-1}\,\mathrm{curl}\,\underline{u}, \mathrm{curl}\,\underline{v}) = \omega^2(\varepsilon\underline{u},\underline{v}) \qquad \forall \underline{v} \in H_0(\mathrm{curl};\Omega). \end{aligned} \tag{7}$$

Let Σ_h be a finite-dimensional subspace of $H_0(\mathrm{curl};\Omega)$; then an approximation of problem (7) reads

$$\begin{aligned} &\text{find } \omega_h \in \mathbb{R} \text{ s.t. } \exists 0 \not\equiv \underline{u}_h \in \Sigma_h : \\ &\qquad (\mu^{-1}\,\mathrm{curl}\,\underline{u}_h, \mathrm{curl}\,\underline{v}_h) = \omega_h^2(\varepsilon\underline{u}_h,\underline{v}_h) \qquad \forall \underline{v}_h \in \Sigma_h. \end{aligned} \tag{8}$$

Let W_h denote the following finite element subspace of $H_0(\mathrm{div}^0;\Omega;\mu^{1/2})$:

$$W_h = \mu^{-1/2}\,\mathrm{curl}(\Sigma_h). \tag{9}$$

The following results state when a finite element method is well suited in order to discretize problem (7); in particular, this requires that all the vanishing eigenvalues are well separated and that no spurious mode polluting the spectrum is generated by the numerical scheme.

Consider the following mixed discrete eigenproblem:

$$\begin{aligned} &\text{find } \omega_h \in \mathbb{R} \text{ s.t. } \exists \underline{0} \not\equiv (\underline{\alpha}_h, \underline{p}_h) \in \Sigma_h \times W_h: \\ &\quad \begin{cases} (\varepsilon\underline{\alpha}_h, \underline{\tau}_h) - (\mu^{-1/2}\,\mathrm{curl}\,\underline{\tau}_h, \underline{p}_h) = 0 & \forall \underline{\tau}_h \in \Sigma_h, \\ (\mu^{-1/2}\,\mathrm{curl}\,\underline{\alpha}_h, \underline{q}_h) = \omega_h^2(\underline{p}_h, \underline{q}_h) & \forall \underline{q}_h \in W_h. \end{cases} \end{aligned} \tag{10}$$

Notice that all the eigenvalues of problem (10) are strictly positive (take $\underline{\tau}_h = \underline{\alpha}_h$ in the first equation, $\underline{q}_h = \underline{p}_h$ in the second, sum the two equations, and use (9)); moreover, the following theorem establishes that they coincide with the nonvanishing eigenvalues of (8).

THEOREM 2.1. *If $(\omega_h, \underline{u}_h) \in \mathbb{R} \times \Sigma_h$, with $\omega_h \neq 0$, is an eigensolution of problem (8), then there exists $\underline{p}_h \in W_h$ such that $(\omega_h, \underline{\alpha}_h, \underline{p}_h)$ is an eigensolution of problem (10) with $\underline{\alpha}_h = \underline{u}_h$. Conversely, if $(\omega_h, \underline{\alpha}_h, \underline{p}_h) \in \mathbb{R} \times \Sigma_h \times W_h$ is an eigensolution of problem (10), then $(\omega_h, \underline{u}_h)$ is an eigensolution of problem (8) with $\underline{u}_h = \underline{\alpha}_h$.*

Proof. We present a proof which has been suggested by a referee and which is simpler than the one proposed by the authors.

Let $\omega_h \neq 0$ be an eigenvalue of (8) and $\underline{u}_h$ be the corresponding eigenfunction. Define $\underline{\alpha}_h = \underline{u}_h$ and $\underline{p}_h = \omega_h^{-2}\mu^{-1/2}\,\mathrm{curl}\,\underline{u}_h$; then $(\omega_h, \underline{\alpha}_h, \underline{p}_h)$ solves (10).

Conversely, the second equation of (10) implies $\underline{p}_h = \omega_h^{-2}\mu^{-1/2}\,\mathrm{curl}\,\underline{\alpha}_h$ and this, together with the first equation, yields (8) with $\underline{u}_h = \underline{\alpha}_h$. ∎

Remark 2.1. If $(0, \underline{u}_h)$ is an eigensolution of (8), then $\underline{u}_h$ can be characterized as the gradient of an element of $H_0^1(\Omega)$. This follows directly by taking $\underline{v}_h = \underline{u}_h$ in (8).

We notice that (10) is nothing other than a discretization of the following mixed eigenvalue problem:

$$\begin{aligned} &\text{find } \omega \in \mathbb{R} \text{ s.t. } \exists \underline{0} \not\equiv (\underline{\alpha}, \underline{p}) \in H_0(\mathrm{curl};\Omega) \times H_0(\mathrm{div}^0;\Omega;\mu^{1/2}) : \\ &\quad \begin{cases} (\varepsilon\underline{\alpha}, \underline{\tau}) - (\mu^{-1/2}\,\mathrm{curl}\,\underline{\tau}, \underline{p}) = 0 & \forall \underline{\tau} \in H_0(\mathrm{curl};\Omega), \\ (\mu^{-1/2}\,\mathrm{curl}\,\underline{\alpha}, \underline{q}) = \omega^2(\underline{p}, \underline{q}) & \forall \underline{q} \in H_0(\mathrm{div}^0;\Omega;\mu^{1/2}), \end{cases} \end{aligned} \tag{11}$$

which is equivalent to the original problem (6) with $\underline{u} = \underline{\alpha}$. The proof is similar to that of Theorem 2.1; notice that $\underline{\alpha} = \varepsilon^{-1}\,\mathrm{curl}(\mu^{-1/2}\underline{p})$; hence $\underline{\alpha} \in H(\mathrm{div}^0;\Omega;\varepsilon)$ and $\underline{p} \in H(\mathrm{curl};\Omega;\mu^{-1/2})$. The equivalence to problem (6) implies that the continuous problems (11) and (7) are related in the same way as the discrete problems (10) and (8).

3. Theoretical framework. In order to analyze the convergence of the eigensolutions of (10) to those of (11), we recall some abstract results which can be deduced from [2, 26, 7, 6]. For more general results we refer to the original papers and the references therein.

Let Σ, W be Hilbert spaces and $a(\cdot,\cdot) : \Sigma \times \Sigma \to \mathbb{R}$, $b(\cdot,\cdot) : \Sigma \times W \to \mathbb{R}$ be continuous bilinear forms. We assume that $a(\cdot,\cdot)$ is symmetric and that it verifies

$$\begin{aligned} a(\tau,\tau) &\geq 0 && \forall \tau \in \Sigma, \\ a(\tau,\tau) &\geq \alpha\|\tau\|_\Sigma^2 && \forall \tau \in \mathbb{K}, \end{aligned} \tag{12}$$

where $\mathbb{K} = \{\tau \in \Sigma : b(\tau, q) = 0 \ \forall q \in W\}$.

We consider the following eigenvalue problem:

$$\begin{aligned} &\text{find } \lambda \in \mathbb{C} \text{ s.t. } \exists \underline{0} \not\equiv (\sigma, p) \in \Sigma \times W : \\ &\quad \begin{cases} a(\sigma,\tau) + b(\tau,p) = 0 & \forall \tau \in \Sigma, \\ b(\sigma,q) = -\lambda(p,q)_W & \forall q \in W, \end{cases} \end{aligned} \tag{13}$$

where $(\cdot,\cdot)_W$ is the inner product in W.

Let us introduce, as usual, the self-adjoint operator T, which associates with every $f \in W$ the solution $Tf \in W$ of the following source problem:

$$\begin{cases} a(\sigma,\tau) + b(\tau, Tf) = 0 & \forall \tau \in \Sigma, \\ b(\sigma,q) = -(f,q)_W & \forall q \in W. \end{cases} \tag{14}$$

In particular, we assume that (14) is uniquely solvable for every $f \in W$ and that the a priori estimate $\|(\sigma, Tf)\|_{\Sigma\times W} \leq C\|f\|_W$ holds true.

Moreover, the operator $S : W \to \Sigma$ which maps f into σ (see (14)) will be useful.

We make the following assumption:

$$T \text{ is compact.} \tag{15}$$

The hypotheses on the form $a(\cdot,\cdot)$ imply that all eigenvalues of (13) are real and positive. Moreover, as a consequence of (15), problem (13) admits a countable set of eigenvalues (each of them with multiplicity m_i), which can be ordered in an increasing divergent sequence

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots. \tag{16}$$

By choosing finite-dimensional subspaces $\Sigma_h \subseteq \Sigma$ and $W_h \subseteq W$, we obtain a finite element discretization of problem (13)

$$\text{find } \lambda_h \in \mathbb{R} \text{ s.t. } \exists \underline{0} \not\equiv (\sigma_h, p_h) \in \Sigma_h \times W_h : \quad \begin{cases} a(\sigma_h, \tau_h) + b(\tau_h, p_h) = 0 & \forall \tau_h \in \Sigma_h, \\ b(\sigma_h, q_h) = -\lambda_h (p_h, q_h)_W & \forall q_h \in W_h. \end{cases} \tag{17}$$

We denote by $\lambda_{h,1} \le \lambda_{h,2} \le \cdots \le \lambda_{h,N_h}$ the eigenvalues of problem (17), N_h being the dimension of W_h.

As before, we consider the source problem associated with (17); the discrete operator $T_h : W \to W_h \subset W$ is defined in the following way:

$$\begin{cases} a(\sigma_h, \tau_h) + b(\tau_h, T_h f) = 0 & \forall \tau_h \in \Sigma_h, \\ b(\sigma_h, q_h) = -(f, q_h)_W & \forall q_h \in W_h. \end{cases} \tag{18}$$

We shall use also the operator $S_h : W \to \Sigma_h$ which maps f into σ_h (see (18)).

Problem (18) is uniquely solvable if the following assumptions are fulfilled:

$$a(\sigma_h, \sigma_h) \ge \alpha_h \|\sigma_h\|_\Sigma^2 \qquad \forall \sigma_h \in \mathbb{K}_h, \tag{19}$$

$$\inf_{q_h \in W_h} \sup_{\tau_h \in \Sigma_h} \frac{|b(\tau_h, q_h)|}{\|q_h\|_W \|\tau_h\|_\Sigma} \ge \beta_h > 0, \tag{20}$$

where $\mathbb{K}_h = \{\tau_h \in \Sigma_h : b(\tau_h, q_h) = 0 \ \forall q_h \in W_h\}$. It is well known (see [10]) that if

$$\alpha_h \ge \alpha_0 > 0, \tag{21}$$

$$\beta_h \ge \beta_0 > 0, \tag{22}$$

uniformly in h, and if

$$\lim_{h \to 0} \left(\inf_{\tau_h \in \Sigma_h} \|\sigma - \tau_h\|_\Sigma + \inf_{q_h \in W_h} \|p - q_h\|_W \right) = 0 \qquad \forall \sigma \in \Sigma, \ p \in W, \tag{23}$$

then the solution of problem (18) converges to the solution of problem (14).

Remark 3.1. We observe that in this case the abstract theory presented in [29] cannot be applied, since this would require stronger compactness hypotheses than our assumption (15). For a deeper discussion on this subject, see [7], as well as sections 11.1, 11.2, and Remark 13.4 of [2].

In order to obtain the convergence of eigenvalues and eigenfunctions, we assume also that the discrete operator T_h converges uniformly to the continuous one, that is,

$$\|T - T_h\|_{\mathcal{L}(W)} \to 0 \quad \text{for} \quad h \to 0; \tag{24}$$

we notice that, since the range of each T_h is finite-dimensional, this hypothesis implies (15).

The following perturbation property of the spectrum of a self-adjoint compact operator will be useful (see [26, pp. 290–291]).

THEOREM 3.1. *Let $\mathcal{K}$ be a real compact subset of the resolvent set of T. Under assumption* (24) *there exists h_0 such that for every $0 < h < h_0$ all the eigenvalues of T_h lie outside $\mathcal{K}$.*

In other words, given a real compact set $\mathcal{K}$ (which may be a finite union of bounded closed intervals), which does not intersect the set $\{\lambda_i, i = 1, 2, \ldots\}$, there exists h_0 such that for every $0 < h < h_0$ all the discrete eigenvalues do not belong to $\mathcal{K}$.

Regarding the rate of convergence of eigenvalues or eigenvectors, we refer to the classical results contained in [2] (see, in particular, Theorems 11.1 and 11.2).

Recently, in [6] sufficient and necessary conditions for obtaining the uniform convergence (24) have been introduced in an abstract setting. In the following we recall some assumptions which imply (24). They can be deduced from the sufficient conditions stated in a quite general framework in [6].

H1. The *ellipticity on the discrete kernel* is fulfilled if there exists $\alpha_0 > 0$ such that (19) and (21) hold true.

H2. Let W_0 be the range of the operator T endowed with its natural norm. The *approximability of* W_0 is satisfied if there exists $\rho_1(h)$ tending to zero as h goes to zero such that for every $p \in W_0$ there exists $q_h \in W_h$ verifying

$$\|p - q_h\|_W \leq \rho_1(h)\|p\|_{W_0}. \tag{25}$$

H3. Let Σ_0 be the space containing all the solutions σ of problem (14) with f varying in W. We denote by $\|\cdot\|_{\Sigma_0}$ its natural norm. An operator $\Pi_h : \Sigma_0 \to \Sigma_h$ is called a *Fortin operator* (see [23]) if there exists C independent of h such that for every $\sigma \in \Sigma_0$ it holds that

$$\begin{aligned} &b(\sigma - \Pi_h\sigma, q_h) = 0 \quad \forall q_h \in W_h, \\ &\|\Pi_h\sigma\|_\Sigma \leq C\|\sigma\|_{\Sigma_0}. \end{aligned} \tag{26}$$

H4. A Fortin operator is said to *converge uniformly to the identity* if there exists $\rho_2(h)$ tending to zero as h goes to zero such that

$$\|\sigma - \Pi_h\sigma\|_a \leq \rho_2(h)\|\sigma\|_{\Sigma_0} \quad \forall \sigma \in \Sigma_0, \tag{27}$$

where $\|\cdot\|_a$ denotes the seminorm associated with the bilinear form $a(\cdot,\cdot)$.

For the reader's convenience we write the natural norms of the spaces W_0 and Σ_0, respectively:

$$\begin{aligned} \|q\|_{W_0} &:= \inf\{\|\eta\|_W,\ T\eta = q\}, \\ \|\tau\|_{\Sigma_0} &:= \inf\{\|\eta\|_W,\ S\eta = \tau\}. \end{aligned} \tag{28}$$

The following theorem holds true (see [6]).

THEOREM 3.2. *Assume that the hypotheses introduced in* H1–H4 *are valid. Then the uniform convergence* (24) *holds true.*

4. Applications. In this section we apply the results of the previous ones to two particular finite element methods: edge and nodal elements. Our analysis is based on the equivalence stated in Theorem 2.1 between the eigensolutions of problem (8), corresponding to nonzero eigenvalues, and the ones of problem (10). Since the same kind of equivalence holds between the continuous problems (7) and (11), we study the convergence of the eigensolutions of problem (10) to those of problem (11). Next, we return to problem (8) and deduce how its eigensolutions approximate those of problem (7).

In this section we assume the following additional regularity hypotheses on the domain Ω and on the coefficients μ and ε:

$$\begin{aligned} &\Omega \text{ is a convex polyhedron,} \\ &\mu \text{ and } \varepsilon \text{ are Lipschitz continuous.} \end{aligned} \tag{29}$$

In section 5 we shall treat the problem in a more general framework.

Problem (11) fits into the setting of section 3 by choosing

$$\begin{aligned} \Sigma &= H_0(\mathrm{curl};\Omega), \\ W &= H_0(\mathrm{div}^0;\Omega;\mu^{1/2}), \\ a(\underline{\sigma},\underline{\tau}) &= (\varepsilon\underline{\sigma},\underline{\tau}), \\ b(\underline{\sigma},\underline{p}) &= -(\mu^{-1/2}\,\mathrm{curl}\,\underline{\sigma},\underline{p}). \end{aligned} \tag{30}$$

We observe that the natural norm of W reduces to the L^2-norm. Moreover, the seminorm associated with the bilinear form $a(\cdot,\cdot)$ is equivalent to the L^2-norm for vector-valued functions thanks to (1).

It is not difficult to see that problem (14) is well-posed. Regarding hypothesis (15), we introduce, as in (14), the operator T, which is continuous from W to $H_0(\mathrm{div}^0;\Omega;\mu^{1/2}) \cap H(\mathrm{curl};\Omega;\mu^{-1/2})$. Thanks to (29), this last space is contained in $H^1(\Omega)$ and hence is compactly embedded in $[L^2(\Omega)]^3$; thus we can conclude the compactness of T.

In the following subsections we shall consider the discretization (10) of problem (11) by means of edge and nodal finite elements. We shall see that both edge and nodal approximations satisfy H1 and H2; the lowest-order edge elements also satisfy H3 and H4 under hypothesis (29).

4.1. Edge elements. Let us describe the so-called face and edge element spaces of order $k \geq 0$ (see [30] for details). Given a regular family $\{\mathcal{T}_h\}$ of triangulations of the domain Ω, let K be a tetrahedron of $\mathcal{T}_h$. We define

$$\begin{aligned} \mathcal{D}^k(K) &= [P_k(K)]^3 \oplus \tilde{P}_k(K)\,\underline{r}, \\ \mathcal{R}^k(K) &= [P_k(K)]^3 \oplus \underline{r} \times [\tilde{P}_k(K)]^3, \end{aligned} \tag{31}$$

where $P_k(K)$ is the space of polynomials of degree less than or equal to k on K, $\tilde{P}_k(K)$ is the space of homogeneous polynomials of degree k on K, and $\underline{r}$ is the vector $\underline{r} = (x, y, z)$. Using this notation, we introduce the so-called "face" and "edge" element spaces which approximate $H_0(\mathrm{div};\Omega)$ and $H_0(\mathrm{curl};\Omega)$, respectively:

$$\begin{aligned} \mathcal{D}^k &= \left\{\underline{v} \in H_0(\mathrm{div};\Omega) : \underline{v}|_K \in \mathcal{D}^k(K)\ \forall K \in \mathcal{T}_h\right\}, \\ \mathcal{R}^k &= \left\{\underline{v} \in H_0(\mathrm{curl};\Omega) : \underline{v}|_K \in \mathcal{R}^k(K)\ \forall K \in \mathcal{T}_h\right\}. \end{aligned} \tag{32}$$

We notice that the elements of $\mathcal{D}^k$ (respectively, $\mathcal{R}^k$) have a normal (respectively, tangential) component which is continuous across the interelement boundaries. Moreover, there exists a positive constant C independent of h, such that the interpolation errors in $\mathcal{D}^k$ and $\mathcal{R}^k$ can be estimated as follows:

$$\begin{aligned} &\inf_{\underline{q}_h \in \mathcal{D}^k} \|\underline{q} - \underline{q}_h\|_0 \leq Ch^m |\underline{q}|_m \qquad \forall \underline{q} \in [H^m(\Omega)]^3, \\ &\inf_{\underline{v}_h \in \mathcal{R}^k} \|\underline{v} - \underline{v}_h\|_0 \leq Ch^m |\underline{v}|_m \qquad \forall \underline{v} \in [H^m(\Omega)]^3, \end{aligned} \tag{33}$$

where $1 \leq m \leq k+1$, and $|\cdot|_m$ stands for the seminorm in $H^m(\Omega)$.

Moreover, the following estimate will be useful (see [30, 24]):

$$\inf_{\underline{v}_h \in \mathcal{R}^k} \|\,\mathrm{curl}\,\underline{v} - \mathrm{curl}\,\underline{v}_h\|_0 \leq Ch^m |\,\mathrm{curl}\,\underline{v}|_m \quad \forall \underline{v} \text{ such that } \mathrm{curl}\,\underline{v} \in [H^m(\Omega)]^3, \tag{34}$$

with $m \leq k+1$.

Let us now come to the discretized problem (8). We take $\Sigma_h = \mathcal{R}^k$; it is well known (see [30]) that $\mathrm{curl}\,\mathcal{R}^k = \mathcal{D}^k \cap H(\mathrm{div}^0;\Omega)$; hence we set $W_h = \mu^{-1/2}(\mathcal{D}^k \cap H(\mathrm{div}^0;\Omega))$.

Using the theoretical framework introduced in section 3, we study the convergence of the eigenpairs of problem (10) to the ones of the continuous problem (11). We start by showing that the discrete operator T_h is well defined.

We observe that in our case (19) and (21) are straightforward consequences of the fact that $\mathbb{K}_h \subseteq \mathbb{K}$ (in fact the operator "$\mu^{-1/2}\,\mathrm{curl}$" is subjective from Σ_h onto W_h). Hence assumption H1 holds true. Moreover, since $[\mathcal{D}(\Omega)]^3$ is dense in $H_0(\mathrm{curl};\Omega)$, then (23) follows from (33) and (34).

The following proposition implies that the inf-sup condition, (20) and (22), is satisfied.

PROPOSITION 4.1. *For every $\underline{q}_h \in W_h$ there exists $\underline{\alpha}_h \in \Sigma_h$ such that*

$$\mu^{-1/2}\,\mathrm{curl}\,\underline{\alpha}_h = \underline{q}_h \tag{35}$$

and

$$\|\underline{\alpha}_h\|_{\mathrm{curl}} \le C\|\underline{q}_h\|_0 \tag{36}$$

with C independent of h.

Proof. Let $\underline{q}_h$ be given in W_h and let $\underline{\varphi}_h = \mu^{1/2}\underline{q}_h \in \mathcal{D}^k \cap H(\mathrm{div}^0;\Omega)$. The inf-sup condition between $\mathcal{R}^k$ and $\mathcal{D}^k \cap H(\mathrm{div}^0;\Omega)$, holding for a convex domain (see Theorem 1 of [30]), implies that there exists $\underline{\alpha}_h \in \mathcal{R}^k$ such that

$$\mathrm{curl}\,\underline{\alpha}_h = \underline{\varphi}_h \tag{37}$$

and

$$\|\underline{\alpha}_h\|_{\mathrm{curl}} \le C\|\underline{\varphi}_h\|_0. \tag{38}$$

Hence we get

$$\mu^{-1/2}\,\mathrm{curl}\,\underline{\alpha}_h = \mu^{-1/2}\underline{\varphi}_h = \underline{q}_h \tag{39}$$

and, thanks also to (1),

$$\|\underline{\alpha}_h\|_{\mathrm{curl}} \le C\|\underline{\varphi}_h\|_0 \le C(\mu^*)^{1/2}\|\underline{q}_h\|_0. \qquad \square \tag{40}$$

Proposition 4.1, together with the validity of H1, implies the following pointwise convergence (see, e.g., [10]).

PROPOSITION 4.2. *The operator T_h* (see (18)) *is defined, and for every $f \in W$ $T_h f$ converges to Tf.*

In order to prove the uniform convergence (24) we limit ourselves to the case of lowest-order edge elements, that is, $k = 0$.

Following the setting of the previous section, we check the validity of hypotheses H1–H4 which imply the uniform convergence (24) (see Theorem 3.2).

Hypothesis H1 has already been shown to hold true. Regarding hypothesis H2, recall that the norm of W coincides with the $L^2(\Omega)$-norm, so that H2 is implied by the compact embedding of W_0 in $[L^2(\Omega)]^3$. It is not difficult to see that, thanks to (29), one can take $\rho_1(h) = Ch$.

Hypotheses H3 and H4 are stated in the following theorem proved in [5].

THEOREM 4.3. *Under hypothesis* (29), *for every $\underline{\tau} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega;\varepsilon)$ there exists $\Pi_h\underline{\tau} \in \Sigma_h$ such that*

$$\begin{aligned} (\mu^{-1/2}\,\mathrm{curl}(\underline{\tau} - \Pi_h\underline{\tau}), \underline{q}_h) = 0 \qquad & \forall \underline{q}_h \in W_h, \\ \|\Pi_h\underline{\tau}\|_\Sigma \le C\|\underline{\tau}\|_{\Sigma_0}. & \end{aligned} \tag{41}$$

Moreover, there exists $\rho_2(h)$ tending to zero as h goes to zero such that

$$\|\underline{\tau} - \Pi_h\underline{\tau}\|_0 \le \rho_2(h)\|\underline{\tau}\|_{\Sigma_0}. \tag{42}$$

Hence the hypotheses of Theorem 3.2 are satisfied. We can then apply the results of section 3, together with those of [2], to obtain the following theorem.

THEOREM 4.4. *Let ω_i^2 be an eigenvalue of problem* (11)*, with multiplicity m_i, and denote by E_i the corresponding eigenspace. Then exactly m_i eigenvalues of problem* (10) *$\omega_{h,i_1}^2, \dots, \omega_{h,i_{m_i}}^2$ converge to ω_i^2. Denoting by $E_{h,i}$ the direct sum of the eigenspaces corresponding to $\omega_{h,i_1}^2, \dots, \omega_{h,i_{m_i}}^2$, we have that there exists h_0 such that for $0 < h < h_0$ the following inequalities hold:*

$$\begin{aligned} |\omega_i^2 - \omega_{h,i_j}^2| &\le C\,(\rho_2(h)+h)^2 \qquad \forall j = 1, \dots, m_i, \\ \delta(E_i, E_{h,i}) &\le C\,(\rho_2(h)+h), \end{aligned} \tag{43}$$

where C is a constant independent of h and $\delta(E_i, E_{h,i})$ denotes the gap between E_i and $E_{h,i}$.

Moreover, let $\mathcal{K}$ be a real compact subset of the resolvent set of the operator associated with problem (11)*. Then there exists h_0 such that for every $0 < h < h_0$ all the nonzero eigenvalues of problem* (10) *lie outside $\mathcal{K}$.*

Proof. Let us assume for the sake of simplicity that $m_i = 1$. Let $(\underline{\sigma}, \underline{p})$ be the eigenfunction associated with ω_i^2. Let $\omega_{h,i}^2$ be the corresponding discrete eigenvalue and let S, S_h, T, T_h be defined as in section 3. We set $\underline{f} = \omega_i^{-2}\underline{p}$ and suppose that $\|\underline{f}\|_0 = 1$. It follows that $S\underline{f} = \underline{\sigma}$ and $T\underline{f} = \underline{p}$. As a consequence of the normalization and of (29), there exists a constant C not depending on $\underline{f}$ such that

$$\|S\underline{f}\|_1 + \|T\underline{f}\|_2 \le C. \tag{44}$$

Combining Theorem 11.2 of [2] and Theorem 4 of [6], using the ellipticity on the discrete kernel H1 and (44), it is not difficult to estimate the gap between the continuous and the discrete eigenspaces to obtain the second inequality of (43).

Regarding the first estimate of (43), Theorem 11.1 of [2] implies that

$$|\omega_i^2 - \omega_{h,i}^2| \le C\left\{\|S\underline{f} - S_h\underline{f}\|_0^2 + \|S\underline{f} - S_h\underline{f}\|_{\mathrm{curl}}\|T\underline{f} - T_h\underline{f}\|_0 + \|T\underline{f} - T_h\underline{f}\|_0^2\right\}.$$

Using the ellipticity in the kernel H1, estimate (44), and the fact that $\operatorname{curl} S_h\underline{f}$ is the projection of $\operatorname{curl} S\underline{f}$ onto W_h (cf. the second equations of (14) and (18)), we finally obtain the desired estimate. ∎

Remark 4.1. Regarding the number of zero eigenvalues of problem (8), we can observe that Remark 2.1 can be made more precise in this case. Namely, let us consider the conforming subspace $\mathcal{L}_{k+1}^1 \cap H_0^1(\Omega)$ of $H_0^1(\Omega)$, whose elements are continuous piecewise polynomials of degree at most $k+1$. It is well known that $\underline{\nabla}(\mathcal{L}_{k+1}^1 \cap H_0^1(\Omega)) \subseteq \mathcal{R}^k$. Conversely, it can be shown that all the functions of $\mathcal{R}^k \cap H_0(\mathrm{curl}^0; \Omega)$ belong to $\underline{\nabla}(\mathcal{L}_{k+1}^1 \cap H_0^1(\Omega))$. Hence problem (8) admits a number of zero eigenvalues equal to the number of degrees of freedom of the space $\mathcal{L}_{k+1}^1 \cap H_0^1(\Omega)$ (if $k = 0$, this is the number of the internal nodes).

4.2. Nodal elements. Let us now consider a nodal element approximation of problem (7). We set

$$\Sigma_h = \left(\mathcal{L}_k^1\right)^3 \cap H_0(\mathrm{curl}; \Omega), \tag{45}$$

that is, the space of continuous piecewise polynomial vector-valued functions $\underline{u}$ of degree $\le k$, with $\underline{u} \times \underline{n} = 0$ at the boundary. Theorem 2.1 states that, for this

choice of Σ_h, all the nonvanishing eigenvalues of problem (8) are the eigenvalues of the following problem:

$$\text{find } \omega_h \text{ s.t. } \exists \underline{0} \not\equiv (\underline{\alpha}_h, \underline{p}_h) \in \Sigma_h \times W_h : \tag{46}$$
$$\begin{cases} (\varepsilon \underline{\alpha}_h, \underline{\tau}_h) - (\mu^{-1/2} \operatorname{curl} \underline{\tau}_h, \underline{p}_h) = 0 & \forall \underline{\tau}_h \in \Sigma_h, \\ (\mu^{-1/2} \operatorname{curl} \underline{\alpha}_h, \underline{q}_h) = \omega_h^2 (\underline{p}_h, \underline{q}_h) & \forall \underline{q}_h \in W_h, \end{cases}$$

where W_h is the image of the "$\mu^{-1/2}\operatorname{curl}$" operator acting on Σ_h.

Assumptions H1 and H2 are consequences of the definition of W_h, while H3 is false on general meshes (the constant C may depend on h).

In section 6, several two-dimensional test cases are shown, in which piecewise linear elements on different meshes are used. For one of them (the so-called criss-cross mesh), it has been proved in [7] that H3 holds true as well, but the results are not completely satisfactory. In fact, for the criss-cross mesh the uniform convergence (24) is not fulfilled (see [7]).

5. Extension to more general cases. In this section we study how the results stated in sections 2 and 4 read in more general situations including mixed boundary conditions, inhomogeneity and anisotropy of the materials, and more complex geometries of the domain. To this aim, we refer to [21], in which suitable functional spaces are introduced and useful compactness results are proved in order to characterize the range of the curl operator in the case of general topologies and mixed boundary conditions.

First of all, we shall also prove that in this case the analysis of Maxwell's eigenproblem can be performed by means of an equivalent mixed formulation. The criteria stated in section 3 can then be applied to this formulation in order to obtain convergence results. The proof of these criteria, under very general assumptions, is still lacking. Hypotheses H3 and H4 have been proved in [5] under some regularity results on Σ_0 which will be made precise in this section. Moreover, they are shown to hold only when lowest-order edge elements are involved. In general, the question of the optimal regularity for the eigensolutions associated with Maxwell's system is still an open problem.

Let us recall the hypotheses and the definitions we shall use in this section. The cavity shape is described by an open, bounded, and connected Lipschitz polyhedral domain $\Omega \subset \mathbb{R}^3$. Its boundary $\Gamma = \partial\Omega$ is then the union of polygonal faces, edges, and vertices. We assume that Γ is split into two subsets Γ_E and Γ_H, where different boundary conditions are given. In the former, boundary conditions are imposed in order to simulate the conducting boundary and the possible "electric mirror" symmetry planes; in the latter, possible "magnetic mirror" symmetry planes are modeled. Hence, Γ_E is never empty, while Γ_H may be.

The two subsets Γ_E and Γ_H satisfy the following assumptions. $\overline{\Gamma}_E$ and $\overline{\Gamma}_H$ are compact submanifolds of Γ satisfying $\overline{\Gamma}_E \cup \overline{\Gamma}_H = \Gamma$. Denoting by $\partial\Gamma_E$ and $\partial\Gamma_H$ the (possibly empty) boundaries of Γ_E and Γ_H, respectively, one has that $\overline{\Gamma}_E \cap \overline{\Gamma}_H = \partial\Gamma_E = \partial\Gamma_H$ is the union of continuous piecewise straight lines whose pieces do not necessarily coincide with the edges of the polyhedral domain Ω.

We assume also that a set of mutually disjoint globally two-sided Lipschitz continuous cuts Λ_j, $j = 1, \ldots, N$, never meeting tangentially Γ, can be chosen in such a way that the set $\dot{\Omega} = \Omega \setminus \Lambda$, with $\Lambda = \cup_{j=1}^{N} \Lambda_j$, is simply connected. Moreover, we assume that both $\Gamma_E \setminus \partial\Lambda$ and $\Gamma_H \setminus \partial\Lambda$ have a finite number of connected components and that the closure of each of them is a Lipschitz two-dimensional submanifold of Γ.

For a deeper discussion of these topological assumptions see [21].

In order to simplify the presentation, we assume (see Proposition 5.6 of [21]) that the cuts Λ_j, $j = 1, \dots, N$, can be chosen such that, for any connected component $\Gamma_{E,i}$, $i = 1, \dots, n$, of Γ_E, the set $\Gamma_{E,i} \setminus \partial\Lambda$ is connected and that, for any j, the boundary $\partial\Lambda_j$ is either included in Γ_H or intersects Γ_E. The cuts are numbered in such a way that, for $j = 1, \dots, N'$, $\partial\Lambda_j \subset \Gamma_H$. Finally, we assume also that another set of cuts Λ^E can be chosen, satisfying the same assumptions with the roles of Γ_E and Γ_H exchanged.

Material properties are described by two real tensors $\varepsilon(\underline{x})$ (electric permittivity) and $\mu(\underline{x})$ (magnetic permeability) satisfying the following symmetry, boundedness, and ellipticity conditions:

$$
\begin{aligned}
&\varepsilon_{ij} = \varepsilon_{ji} \in L^\infty(\Omega), \qquad \mu_{ij} = \mu_{ji} \in L^\infty(\Omega) \qquad \text{for} \quad i,j = 1,2,3,\\
&\sum_{i,j=1}^{3} \varepsilon_{ij}\xi_i\xi_j \geq \varepsilon_* |\xi|^2, \qquad \sum_{i,j=1}^{3} \mu_{ij}\xi_i\xi_j \geq \mu_* |\xi|^2 \qquad \text{a.e. in} \quad \Omega \ \forall \xi \in \mathbb{R}^3
\end{aligned} \tag{47}
$$

for some positive constants ε_* and μ_*. Finally, in order to treat different materials inside the cavity, we assume that Ω can be split into open polyhedral subdomains Ω_k, $k = 1, \dots, r$, and that ε and μ are Lipschitz continuous in each of them, but not necessarily continuous in Ω.

Let us write Maxwell's system in this more general case. With respect to the model we have introduced in section 2, we have to add the boundary conditions along Γ_H. Again, let $\underline{E}$, $\underline{H}$, and ω be the electric field phasor, the magnetic field phasor, and the angular frequency, respectively. The problem is to find nontrivial solutions $(\omega \neq 0, \underline{E}, \underline{H})$ of the Maxwell's system for vector phasors, which in this case reads

$$
\begin{cases}
\text{(i)} & \operatorname{curl} \underline{E} = -j\omega\mu\underline{H} & \text{in} & \Omega,\\
\text{(ii)} & \operatorname{div}(\varepsilon\underline{E}) = 0 & \text{in} & \Omega,\\
\text{(iii)} & \operatorname{curl} \underline{H} = j\omega\varepsilon\underline{E} & \text{in} & \Omega,\\
\text{(iv)} & \operatorname{div}(\mu\underline{H}) = 0 & \text{in} & \Omega,\\
\text{(v)} & \underline{E} \times \underline{n} = 0 & \text{on} & \Gamma_E,\\
\text{(vi)} & (\mu\underline{H}) \cdot \underline{n} = 0 & \text{on} & \Gamma_E,\\
\text{(vii)} & \underline{H} \times \underline{n} = 0 & \text{on} & \Gamma_H,\\
\text{(viii)} & (\varepsilon\underline{E}) \cdot \underline{n} = 0 & \text{on} & \Gamma_H.
\end{cases} \tag{48}
$$

Since the static solutions (i.e., those with $\omega = 0$) are of no interest in this context, they are eliminated by the explicit constraint $\omega \neq 0$. Problem (48) behaves differently from problem (2): the vanishing frequency cannot be directly excluded here because the domain may not be simply connected and Γ_E may not be connected as well.

Since μ is nonsingular, we can eliminate the magnetic field to obtain the equivalent eigenvalue problem for the electric field (now denoted by $\underline{u}$) so that the problem reads

find $(\omega,\ \underline{u} \not\equiv 0)$ such that

$$
\begin{cases}
\text{(i)} & \operatorname{curl}(\mu^{-1} \operatorname{curl} \underline{u}) = \omega^2 \varepsilon \underline{u} & \text{in} & \Omega,\\
\text{(ii)} & \operatorname{div}(\varepsilon\underline{u}) = 0 & \text{in} & \Omega,\\
\text{(iii)} & \underline{u} \times \underline{n} = 0 & \text{on} & \Gamma_E,\\
\text{(iv)} & (\mu^{-1} \operatorname{curl} \underline{u}) \times \underline{n} = 0 & \text{on} & \Gamma_H,\\
\text{(v)} & (\varepsilon\underline{u}) \cdot \underline{n} = 0 & \text{on} & \Gamma_H,\\
\text{(vi)} & \int_{\Gamma_{E,i}} (\varepsilon\underline{u}) \cdot \underline{n} = 0 & i = 1, \dots, n,\\
\text{(vii)} & \int_{\Lambda_j} (\varepsilon\underline{u}) \cdot \underline{n} = 0 & j = 1, \dots, N'.
\end{cases} \tag{49}
$$

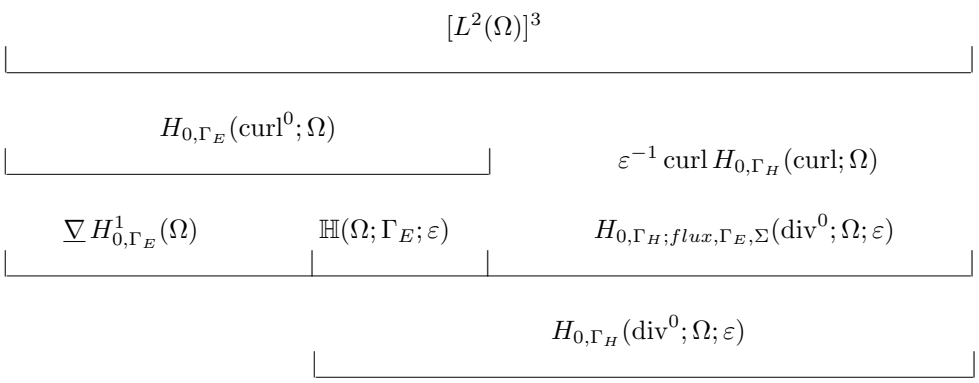

FIG. 1. *ε-orthogonal decompositions of* $[L^2(\Omega)]^3$.

The last two conditions imply that if $\omega = 0$, then $\underline{u} \equiv 0$. Consequently, the explicit constraint $\omega \neq 0$ can be dropped, as it is sufficient to require $\underline{u} \not\equiv 0$.

Let us introduce the functional spaces useful to deal with problem (49).

$$
\begin{aligned}
&H^1_{0,\Gamma_E}(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_E} = 0\};\\
&H_{0,\Gamma_E}(\mathrm{curl};\Omega) = \{\underline{v} \in H(\mathrm{curl};\Omega) : \underline{v} \times \underline{n}|_{\Gamma_E} = 0\};\\
&H_{0,\Gamma_E}(\mathrm{curl}^0;\Omega) = \{\underline{v} \in H_{0,\Gamma_E}(\mathrm{curl};\Omega) : \mathrm{curl}\,\underline{v} = 0\};\\
&H_{0,\Gamma_H}(\mathrm{curl};\Omega;\mu^{-1/2}) = \{\underline{v} \in [L^2(\Omega)]^3 : \mathrm{curl}(\mu^{-1/2}\underline{v}) \in [L^2(\Omega)]^3,\\
&\qquad\qquad (\mu^{-1/2}\underline{v}) \times \underline{n}|_{\Gamma_H} = 0\};\\
&H_{0,\Gamma_H}(\mathrm{div}^0;\Omega;\varepsilon) = \{\underline{v} \in [L^2(\Omega)]^3 : \mathrm{div}(\varepsilon\underline{v}) = 0,\ (\varepsilon\underline{v}) \cdot \underline{n}|_{\Gamma_H} = 0\};\\
&\mathbb{H}(\Omega;\Gamma_E;\varepsilon) = H_{0,\Gamma_E}(\mathrm{curl}^0;\Omega) \cap H_{0,\Gamma_H}(\mathrm{div}^0;\Omega;\varepsilon);\\
&H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\mathrm{div}^0;\Omega;\varepsilon) = \{\underline{v} \in H_{0,\Gamma_H}(\mathrm{div}^0;\Omega;\varepsilon) :\\
&\qquad\qquad \textstyle\int_{\Gamma_{E,i}} (\varepsilon\underline{v}) \cdot \underline{n} = 0\ i = 1,\dots n,\\
&\qquad\qquad \textstyle\int_{\Lambda_j} (\varepsilon\underline{v}) \cdot \underline{n} = 0\ j = 1,\dots,N'\}.
\end{aligned}
\tag{50}
$$

Moreover, the space $H_{0,\Gamma_E;flux,\Gamma_H,\Sigma}(\mathrm{div}^0;\Omega;\mu^{1/2})$ can be defined, using the set of cuts Λ^E.

The diagram in Figure 1 recalls the decomposition of $[L^2(\Omega)]^3$ as the direct sum of closed ε-orthogonal (i.e., orthogonal with respect to the scalar product $(\varepsilon\underline{u}, \underline{v})$) subspaces introduced in [21].

The variational form of (49) reads as follows:

$$
\begin{aligned}
&\text{find } \omega \in \mathbb{R} \text{ s.t. } \exists\ 0 \not\equiv \underline{u} \in H_{0,\Gamma_E}(\mathrm{curl};\Omega) \cap H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\mathrm{div}^0;\Omega;\varepsilon) :\\
&\qquad (\mu^{-1}\,\mathrm{curl}\,\underline{u}, \mathrm{curl}\,\underline{v}) \quad = \quad \omega^2(\varepsilon\underline{u}, \underline{v})\\
&\qquad \forall \underline{v} \in H_{0,\Gamma_E}(\mathrm{curl};\Omega) \quad \cap \quad H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\mathrm{div}^0;\Omega;\varepsilon).
\end{aligned}
\tag{51}
$$

Propositions 7.3 and 7.4, together with Remark 7.5 of [21], state that the space $H_{0,\Gamma_E}(\mathrm{curl};\Omega) \cap H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\mathrm{div}^0;\Omega;\varepsilon)$ is compactly embedded in $[L^2(\Omega)]^3$ and that the bilinear, continuous, and symmetric form $(\mu^{-1}\,\mathrm{curl}\,\underline{u}, \mathrm{curl}\,\underline{v})$ is coercive on the space $H_{0,\Gamma_E}(\mathrm{curl};\Omega) \cap H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\mathrm{div}^0;\Omega;\varepsilon)$. Hence, the self-adjoint operator associated with problem (51) is compact and problem (51) admits a countable set of real and positive eigenvalues. Moreover, each eigenspace is finite-dimensional. Again, the eigenfunctions can always be chosen to be real and the whole analysis can be carried out in real functional spaces.

We point out that, as in the simpler problem (6) of section 2, $\omega^2 = 0$ is not an eigenvalue of (51). As in section 2, let us now consider the related problem

$$\begin{aligned} &\text{find } \omega \in \mathbb{R} \text{ s.t. } \exists\ 0 \not\equiv \underline{u} \in H_{0,\Gamma_E}(\mathrm{curl};\Omega) : \\ &\quad (\mu^{-1}\,\mathrm{curl}\,\underline{u}, \mathrm{curl}\,\underline{v}) = \omega^2(\varepsilon\underline{u},\underline{v}) \qquad \forall \underline{v} \in H_{0,\Gamma_E}(\mathrm{curl};\Omega). \end{aligned} \tag{52}$$

Problem (52) has all the eigensolutions of problem (51) and, possibly, others. If $\underline{u}$ satisfies (52) *with* $\omega \neq 0$, it is easily seen by taking $\underline{v} \in H_{0,\Gamma_E}(\mathrm{curl}^0;\Omega)$ that $\underline{u}$ is ε-orthogonal to $H_{0,\Gamma_E}(\mathrm{curl}^0;\Omega)$ so that $\underline{u} \in H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\mathrm{div}^0;\Omega;\varepsilon)$ and, thus, satisfies (51). On the other hand, $\underline{u}$ is an eigenfunction of (52) *with* $\omega = 0$ if and only if $\underline{u} \in H_{0,\Gamma_E}(\mathrm{curl}^0;\Omega)$. Hence, (51) and (52) are equivalent *for* $\omega \neq 0$. However, $\omega = 0$ is an eigenvalue of (52) with the infinite-dimensional eigenspace $H_{0,\Gamma_E}(\mathrm{curl}^0;\Omega) = \underline{\nabla}\, H^1_{0,\Gamma_E}(\Omega) \oplus \mathbb{H}(\Omega;\Gamma_E;\varepsilon)$. This fact prevents the operator associated with problem (52) from being compact; the eigenfunctions in $\underline{\nabla}\, H^1_{0,\Gamma_E}(\Omega)$ lack any physical meaning, while those in $\mathbb{H}(\Omega;\Gamma_E;\varepsilon)$ are electrostatic solutions of no interest in cavity resonator analysis.

Considering a finite-dimensional subspace Σ_h of $H_{0,\Gamma_E}(\mathrm{curl};\Omega)$, the finite element approximation of (52) reads

$$\begin{aligned} &\text{find } \omega_h \in \mathbb{R} \text{ s.t. } \exists\ 0 \not\equiv \underline{u}_h \in \Sigma_h : \\ &\qquad (\mu^{-1}\,\mathrm{curl}\,\underline{u}_h, \mathrm{curl}\,\underline{v}_h) = \omega_h^2(\varepsilon\underline{u}_h,\underline{v}_h) \qquad \forall \underline{v}_h \in \Sigma_h. \end{aligned} \tag{53}$$

Following the same outline as in section 2, we introduce the mixed eigenproblem

$$\begin{aligned} &\text{find } \omega \in \mathbb{R} \text{ s.t. } \exists \underline{0} \not\equiv (\underline{\alpha},\underline{p}) \in \Sigma \times W : \\ &\qquad \begin{cases} (\varepsilon\underline{\alpha},\underline{\tau}) - (\mu^{-1/2}\,\mathrm{curl}\,\underline{\tau},\underline{p}) = 0 & \forall \underline{\tau} \in \Sigma, \\ (\mu^{-1/2}\,\mathrm{curl}\,\underline{\alpha},\underline{q}) = \omega^2(\underline{p},\underline{q}) & \forall \underline{q} \in W, \end{cases} \end{aligned} \tag{54}$$

where $\Sigma = H_{0,\Gamma_E}(\mathrm{curl};\Omega)$ and $W = H_{0,\Gamma_E;flux,\Gamma_H,\Sigma}(\mathrm{div}^0;\Omega;\mu^{1/2})$. We observe that $W = \mu^{-1/2}\,\mathrm{curl}\,\Sigma$.

We show that the eigensolutions of problem (54) coincide with all the eigensolutions of problem (52) corresponding to $\omega \neq 0$; therefore, they are also the eigensolutions of (51). Taking $\underline{\tau} \in H_{0,\Gamma_E}(\mathrm{curl}^0;\Omega)$ in the first equation of (54) results in $\underline{\alpha}$ being ε-orthogonal to $H_{0,\Gamma_E}(\mathrm{curl}^0;\Omega)$ and, due to the diagram in Figure 1, $\underline{\alpha} \in H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\mathrm{div}^0;\Omega;\varepsilon)$. Then, since the same equation implies $\varepsilon\underline{\alpha} = \mathrm{curl}(\mu^{-1/2}\underline{p})$ in the sense of the distributions, it follows that $\underline{p} \in H_{0,\Gamma_H}(\mathrm{curl};\Omega;\mu^{-1/2})$. Taking into account that $(\underline{\alpha},\underline{p}) \in \Sigma \times W$, we obtain $\underline{\alpha} \in H_{0,\Gamma_E}(\mathrm{curl};\Omega) \cap H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\mathrm{div}^0;\Omega;\varepsilon)$ and $\underline{p} \in H_{0,\Gamma_H}(\mathrm{curl};\Omega;\mu^{-1/2}) \cap H_{0,\Gamma_E;flux,\Gamma_H,\Sigma}(\mathrm{div}^0;\Omega;\mu^{1/2})$.

Since W can be characterized as the image of the $\mu^{-1/2}\,\mathrm{curl}$ operator acting on Σ (see [21]), we can take $\underline{q} = \mu^{-1/2}\,\mathrm{curl}\,\underline{\tau}$ in the second equation of (54), where $\underline{\tau}$ is an element of Σ. Using also the first equation of (54), we have

$$(\mu^{-1/2}\,\mathrm{curl}\,\underline{\alpha}, \mu^{-1/2}\,\mathrm{curl}\,\underline{\tau}) = \omega^2(\underline{p}, \mu^{-1/2}\,\mathrm{curl}\,\underline{\tau}) = \omega^2(\varepsilon\underline{\alpha},\underline{\tau}) \quad \forall \underline{\tau} \in \Sigma. \tag{55}$$

Hence $(\omega,\underline{\alpha})$ is an eigenpair of (51).

The theory of section 3 applies to problem (54). Still using that $W = \mu^{-1/2}\,\mathrm{curl}\,(\Sigma)$, it is easy to check that (12) is verified. Moreover, the compact embedding of the space $H_{0,\Gamma_H}(\mathrm{curl};\Omega;\mu^{-1/2}) \cap H_{0,\Gamma_E;flux,\Gamma_H,\Sigma}(\mathrm{div}^0;\Omega;\mu^{1/2})$ into $[L^2(\Omega)]^3$ (see [21]) guarantees the compactness of the operator associated with (54) (see (14) and (15)).

Setting $W_h = \mu^{-1/2}\,\mathrm{curl}(\Sigma_h)$, the following theorem holds true.

THEOREM 5.1. *If* $(\omega_h, \underline{u}_h) \in \mathbb{R} \times \Sigma_h$ *with* $\omega_h \neq 0$ *is an eigensolution of problem* (53), *then there exists* $\underline{p}_h \in W_h$ *such that* $(\omega_h, \underline{\alpha}_h, \underline{p}_h)$ *with* $\underline{\alpha}_h = \underline{u}_h$ *is an eigensolution of the following problem:*

$$\text{find } \omega_h \in \mathbb{R} \text{ s.t. } \exists \underline{0} \not\equiv (\underline{\alpha}_h, \underline{p}_h) \in \Sigma_h \times W_h\text{:} \qquad (56)$$
$$\begin{cases} (\varepsilon \underline{\alpha}_h, \underline{\tau}_h) - (\mu^{-1/2} \operatorname{curl} \underline{\tau}_h, \underline{p}_h) = 0 & \forall \underline{\tau}_h \in \Sigma_h, \\ (\mu^{-1/2} \operatorname{curl} \underline{\alpha}_h, \underline{q}_h) = \omega_h^2 (\underline{p}_h, \underline{q}_h) & \forall \underline{q}_h \in W_h. \end{cases}$$

Conversely, if $(\omega_h, \underline{\alpha}_h, \underline{p}_h) \in \mathbb{R} \times \Sigma_h \times W_h$ *is an eigensolution of problem* (56), *then* $(\omega_h, \underline{u}_h)$ *is an eigensolution of problem* (53) *with* $\underline{u}_h = \underline{\alpha}_h$.

From now on we limit ourselves to analyzing the case of edge elements. Using the notation already introduced in (31), we set

$$\Sigma_h = \{\underline{v} \in H_{0,\Gamma_E}(\operatorname{curl}; \Omega) : \underline{v}|_K \in \mathcal{R}^k(K) \quad \forall K \in \mathcal{T}_h\}. \qquad (57)$$

The analysis of the convergence properties of the numerical method is based on the two propositions we are going to state (see section 3). In the first one the inf-sup condition is proved, so that the operator T_h is well defined. In the second one we consider the convergence of the eigenvalues and recall the minimal hypotheses under which H3 and H4 are proved in [5]. Notice that the validity of H1 and H2 is a straightforward generalization of the results explained in the previous section.

We assume that there exists s greater than zero such that the continuous embedding holds

$$H_{0,\Gamma_E}(\operatorname{curl}; \Omega) \cap H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\operatorname{div}^0; \Omega; \varepsilon) \subset [H^s(\Omega)]^3. \qquad (58)$$

This assumption is satisfied if some hypotheses on Ω, Γ_E, Γ_H, ε, and μ are made. The individuation of the minimal hypotheses for the validity of (58) is still an open problem (for some results in the three-dimensional case, see [1, 25, 4, 16, 31]), but in all significant applications they seem to hold true.

PROPOSITION 5.2. *For every* $\underline{q}_h \in W_h$ *there exists* $\underline{\alpha}_h \in \Sigma_h$ *such that*

$$\mu^{-1/2} \operatorname{curl} \underline{\alpha}_h = \underline{q}_h \qquad (59)$$

and

$$\|\underline{\alpha}_h\|_{\operatorname{curl}} \leq C \|\underline{q}_h\|_0 \qquad (60)$$

with C *independent of* h.

Proof. This is the same proof as in Proposition 4.1. Notice that it is possible to extend the proof of Theorem 1 of [30] using the regularity assumption (58). ∎

PROPOSITION 5.3. *Let us consider the lowest-order edge approximation, that is,* $k = 0$. *Let us assume, moreover, that* (58) *holds with* $s > 1/2$. *Then there exists* $\rho_2(h)$ *tending to zero as* h *goes to zero such that for every* $\underline{\tau} \in H_{0,\Gamma_E}(\operatorname{curl}; \Omega) \cap H_{0,\Gamma_H;flux,\Gamma_E,\Sigma}(\operatorname{div}^0; \Omega; \varepsilon)$ *there exists* $\Pi_h \underline{\tau} \in \Sigma_h$ *satisfying*

$$\begin{gathered} (\mu^{-1/2} \operatorname{curl}(\underline{\tau} - \Pi_h \underline{\tau}), \underline{q}_h) = 0 \qquad \forall \underline{q}_h \in W_h, \\ \|\Pi_h \underline{\tau}\|_\Sigma \leq C \|\underline{\tau}\|_{\Sigma_0}, \\ \|\underline{\tau} - \Pi_h \underline{\tau}\|_0 \leq \rho_2(h) \|\underline{\tau}\|_{\Sigma_0}. \end{gathered} \qquad (61)$$

Remark 5.1. The minimal assumptions on Ω, μ, and ε in order to get (58) with $s > 1/2$ are still objects of research. For instance, this condition is met if Ω is a

general Lipschitz polyhedron, μ, ε are Lipschitz continuous, and no mixed boundary conditions are involved (see [1] for more details).

Propositions 5.2 and 5.3 constitute the main ingredients for proving the uniform convergence (24).

Hence a generalization of Theorem 4.4 can be stated under the same hypotheses as in Proposition 5.3. Concerning the rate of convergence, it would depend on the regularity of the eigenfunctions and should be checked case by case with the aid of Theorems 11.1 and 11.2 of [2].

The following theorem, analogous to Theorem 4.4, states the exact rate of convergence depending on $\rho_1(h)$ and $\rho_2(h)$. Its proof is similar to that of the proof of Theorem 4.4.

Notice that $\rho_1(h)$ tends to zero thanks to the compact embedding of $W_0 = H_{0,\Gamma_H}(\mathrm{curl};\Omega;\mu^{-1/2}) \cap H_{0,\Gamma_E;flux,\Gamma_H,\Sigma}(\mathrm{div}^0;\Omega;\mu^{1/2})$ in $[L^2(\Omega)]^3$. Moreover, if there exists $r>0$ such that $W_0 \subset [H^r(\Omega)]^3$, then $\rho_1(h) = O(h^r)$. The existence of a function $\rho_2(h)$ tending to zero as h goes to zero is, on the other hand, an open problem in more general situations than those presented in [5]. Moreover, also in the cases considered in [5] it should be interesting to establish the precise convergence rate of $\rho_2(h)$.

THEOREM 5.4. *Let ω_i^2 be an eigenvalue of problem* (54)*, with multiplicity m_i, and denote by E_i the corresponding eigenspace. Assume that hypotheses* H1–H4 *hold true. Then exactly m_i eigenvalues of problem* (56)*, $\omega_{h,i_1}^2,\dots,\omega_{h,i_{m_i}}^2$, converge to ω_i^2. Denoting by $E_{h,i}$ the direct sum of the eigenspaces corresponding to $\omega_{h,i_1}^2,\dots,\omega_{h,i_{m_i}}^2$, we have that there exists h_0 such that for $0<h<h_0$ the following inequalities hold:*

$$
\begin{aligned}
&|\omega_i^2 - \omega_{h,i_j}^2| \le C\,(\rho_1(h)+\rho_2(h))^2 \qquad \forall j=1,\dots,m_i,\\
&\delta(E_i,E_{h,i}) \le C\,(\rho_1(h)+\rho_2(h)),
\end{aligned}
\tag{62}
$$

where C is a constant independent of h.

Moreover, let $\mathcal{K}$ be a real compact subset of the resolvent set of the operator associated with problem (54)*. Then there exists h_0 such that for every $0<h<h_0$ all the nonzero eigenvalues of problem* (56) *lie outside $\mathcal{K}$.*

6. Numerical results. Here we present some numerical experiments concerning edge and nodal elements to confirm the theoretical results of the previous sections. All our examples are two-dimensional and make use of the lowest-order elements: the analysis of the previous sections can be easily applied to this simpler case. The variational formulation used for the numerical tests reads

$$
\begin{aligned}
&\text{find } \omega_h \in \mathbb{R} \text{ s.t. } \exists \underline{u}_h \in \Sigma_h \subset \Sigma \text{ with } \underline{u}_h \not\equiv 0 :\\
&\qquad (\mu^{-1}\,\mathrm{rot}\,\underline{u}_h, \mathrm{rot}\,\underline{v}_h) = \omega_h^2(\varepsilon\underline{u}_h,\underline{v}_h) \qquad \forall \underline{v}_h \in \Sigma_h.
\end{aligned}
\tag{63}
$$

Here $\mathrm{rot}(v_1,v_2) = v_{2,x} - v_{1,y}$, $\Sigma = \{\underline{v} \in [L^2(\Omega)]^2 : \mathrm{rot}\,\underline{v} \in L^2(\Omega),\ \underline{v}\cdot\underline{t} = 0 \text{ on } \partial\Omega\}$, where $\underline{t}$ is the counterclockwise-oriented tangent versor to the boundary.

We observe that, since we solve the problem for the transverse components, our examples have all the significant features of the three-dimensional case (see [20]).

In subsection 6.1, we show that edge elements give reasonable results in all the test problems we have considered. In the first examples, we consider the problem in a square domain with either homogeneous conditions on the tangential component along the whole boundary or mixed boundary conditions; next, discontinuous permeability and permittivity are taken into account; finally, a nonconvex L-shaped domain has been chosen.

Subsection 6.2 deals with the nodal element approximation of the problem in a square domain with homogeneous boundary conditions on the tangential component. We present results obtained with both unstructured and structured meshes. In the former case, the inf-sup condition is not verified and we obtain a poorly approximated spectrum in which it is quite impossible to pick out the physical eigenvalues among the others. In the latter case, the zero frequencies are exactly individuated, but spurious eigenvalues are generated by the numerical scheme. Let us notice that the structured mesh we have used supports C^1 piecewise quadratic polynomials (see [32]), so the conjecture proposed in [36] does not seem to be of general validity.

6.1. Edge elements. In the two-dimensional case, Theorems 4.4 and 5.4 can be made more precise. In fact it is well known that, in this case, the interpolation operator is a Fortin operator so that hypotheses H3 and H4 are always fulfilled thanks to the compact embedding of the solutions space Σ_0 into $[L^2(\Omega)^2]$. Moreover, using the notation of section 3, $\rho_2(h)$ can be estimated to be $O(h^s)$ whenever Σ_0 is contained in $H^s(\Omega)$.

Let us take $\Omega =]0,\pi[\times]0,\pi[$, $\Gamma_E = \Gamma$, $\mu \equiv 1$, and $\varepsilon \equiv 1$. In this case it is easy to find the exact eigenvalues of problem (3), which are given by $0 \neq \omega^2 = n^2 + m^2$, with $n, m = 0, 1, \ldots$. The aim of our first test is to demonstrate the performance of the edge elements. We choose unstructured regular meshes of four different sizes; in Figure 2 the second one is plotted.

TABLE 1
Edge approximation for Dirichlet boundary conditions.

Exact	Computed				Slope
1.00000	0.99754	0.99947	0.99993	0.99999	4.51
1.00000	1.00091	0.99970	0.99998	1.00001	4.01
2.00000	1.98987	1.99980	1.99944	1.99967	2.49
4.00000	3.93252	3.98964	3.99586	3.99891	3.29
4.00000	4.06794	4.00597	4.00086	4.00018	4.90
5.00000	4.89190	4.99541	4.99892	4.99932	4.26
5.00000	4.97885	5.00488	5.00038	4.99961	3.67
8.00000	7.84082	7.99208	8.00064	7.99987	5.94
9.00000	8.73060	8.98023	8.99658	8.99736	3.99
9.00000	8.79263	9.02384	9.00393	8.99886	4.33
d.o.f.	68	252	635	1183	
# zeros	17	73	194	370	

In Table 1 the values of the first 10 nonvanishing eigenvalues corresponding to the four considered grids are compared with the exact ones. The number of computed zeros is indicated. It is interesting to observe that this value coincides with the number of the internal vertices, that is, the dimension of the space containing continuous piecewise linear functions, which vanish at the boundary (see Remark 4.1). The eigenfunctions corresponding to the first four nonvanishing eigenvalues are plotted in Figure 3.

Let us consider again the square $\Omega =]0,\pi[\times]0,\pi[$, and let us split the boundary into $\Gamma_E = \{0,\pi\}\times]0,\pi[$ and $\Gamma_H =]0,\pi[\times\{0,\pi\}$. Hence, the boundary conditions are

$$\begin{aligned} \underline{u}\cdot\underline{t} &= 0 \qquad \text{on } \Gamma_E, \\ \operatorname{rot}\underline{u} &= 0 \qquad \text{on } \Gamma_H. \end{aligned} \tag{64}$$

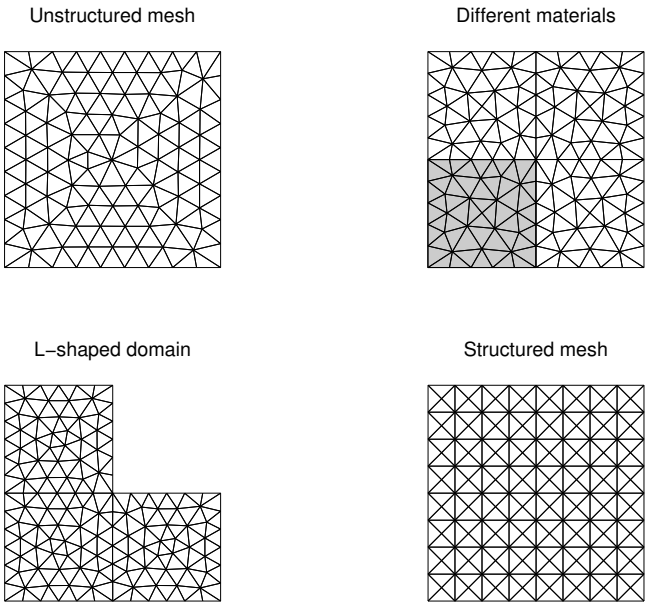

FIG. 2. *Meshes.*

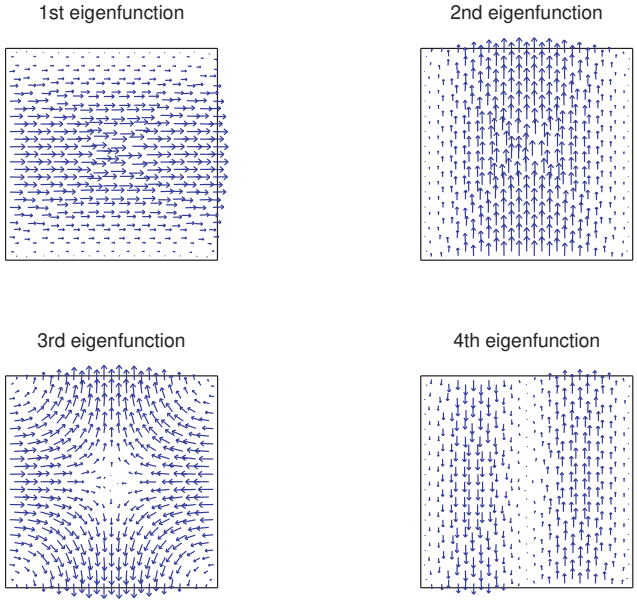

FIG. 3. *Edge approximation for Dirichlet boundary conditions.*

The first condition is contained in the definition of the finite element spaces, while the second is a natural conditions which follows from the variational formulation. Using again the four meshes as in the previous example, we obtain the results listed in Table 2. The eigenfunctions corresponding to the first four nonvanishing eigenvalues are plotted in Figure 4.

The next two tests are relative to discontinuous permeability and permittivity,

TABLE 2
Edge approximation for mixed boundary conditions.

Exact	Computed				Slope
1.00000	0.99735	0.99993	0.99995	0.99997	3.59
2.00000	1.99231	1.99992	2.00035	2.00020	2.59
4.00000	3.92806	3.99530	4.00009	3.99999	7.54
5.00000	4.92700	4.99101	4.99772	4.99930	3.82
5.00000	5.03912	4.99678	4.99887	5.00016	4.34
8.00000	7.95748	7.98688	7.99977	7.99961	4.56
9.00000	8.64988	8.95823	8.99200	8.99888	4.64
10.00000	9.55206	9.95049	9.99040	9.99672	4.09
10.00000	9.87920	10.0078	10.0043	10.0006	4.10
13.00000	12.4482	12.9595	12.9836	12.9951	3.79
d.o.f.	78	270	663	1221	
# zeros	26	90	221	407	

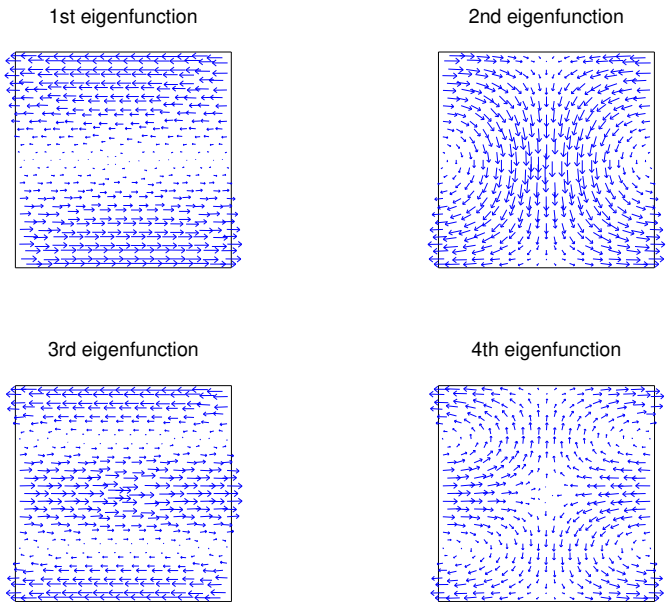

FIG. 4. *Edge approximation for mixed boundary conditions.*

respectively. The square domain Ω has been divided into two subregions (see Figure 2): in the white region we always set $\mu \equiv \varepsilon \equiv 1$; in the grey one we consider $\mu \equiv 0.01$, $\varepsilon \equiv 1$ and $\mu \equiv 1$, $\varepsilon \equiv 100$, respectively. The eigenvalues are reported in Tables 3 and 4, respectively. Similarly, Figures 5 and 6 show the first four significative eigenfunctions.

In the last test with edge elements, the domain is nonconvex and L-shaped, as shown in Figure 2. As usual, we report the first 10 nonzero eigenvalues corresponding to four different meshes in Table 5 and the first four significative eigenfunctions in Figure 7.

Remark 6.1. The last column of each table reports the rates of convergence of the listed eigenvalues. It can be noticed that the computed rates are much higher

TABLE 3
Edge approximation for discontinuous μ.

					Slope
1st	1.03135	1.04050	1.04130	1.04166	3.19
2nd	1.48689	1.48850	1.48871	1.48883	2.69
3rd	4.00423	4.10451	4.10361	4.10594	3.43
4th	4.25724	4.40127	4.40346	4.40656	3.54
5th	4.79431	4.88385	4.88750	4.88975	3.31
6th	6.15395	6.40445	6.42879	6.44024	2.99
7th	8.61611	8.81353	8.87875	8.87229	4.12
8th	8.89628	9.49132	9.52046	9.53619	4.15
9th	9.49640	9.76563	9.77699	9.78137	3.66
10th	12.5910	13.0257	13.1336	13.1373	3.51
d.o.f.	96	292	620	1304	
# zeros	25	85	189	409	

TABLE 4
Edge approximation for discontinuous ε.

					Slope
1st	0.02403	0.02374	0.02372	0.02371	3.14
2nd	0.09039	0.09764	0.09800	0.09820	4.08
3rd	0.10295	0.10420	0.10402	0.10392	2.60
4th	0.16968	0.17877	0.17957	0.17953	6.07
5th	0.21791	0.24621	0.25126	0.25296	3.32
6th	0.28125	0.26275	0.26244	0.26230	4.01
7th	0.31595	0.32875	0.33671	0.33821	3.28
8th	0.33569	0.33692	0.33918	0.33877	2.80
9th	0.42796	0.44265	0.46303	0.46937	2.33
10th	0.47614	0.47485	0.49683	0.49782	4.12
d.o.f.	96	292	620	1304	
# zeros	25	85	189	409	

than the theoretical estimates. This may depend on the symmetry of the domain and consequently of the eigensolutions. This superconvergence phenomenon is not clear at the moment and its explanation is outside the scope of this paper.

6.2. Nodal elements. We apply the nodal elements to the simplest case study of the previous subsection, that is, $\Omega =]0, \pi[\times]0, \pi[$, $\Gamma_E = \Gamma$, $\mu \equiv 1$, and $\varepsilon \equiv 1$. We use an unstructured mesh similar to that in Figure 2, with 440 degrees of freedom. The first 70 eigenvalues are plotted in Figure 8. In this plot one cannot pick out the approximations of the physical eigenvalues from the nonphysical modes. In Figure 9 we present the picture of four eigenfunctions corresponding to eigenvalues close to the value 1. It is apparent that the 49th and 50th eigenfunctions well approximate the correct ones associated with the double eigenvalue 1. The remaining ones are evidently associated with nonphysical eigenvalues. This behavior is due to the fact that on this mesh the inf-sup constant (20) goes to zero with h.

The second test in which nodal elements are used involves structured meshes similar to the last one plotted in Figure 2. Using these meshes, the results are quite

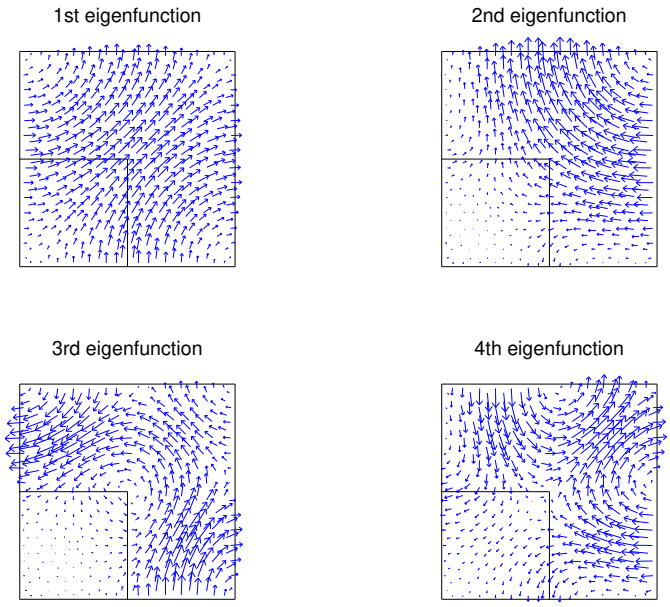

FIG. 5. *Edge approximation for discontinuous* μ.

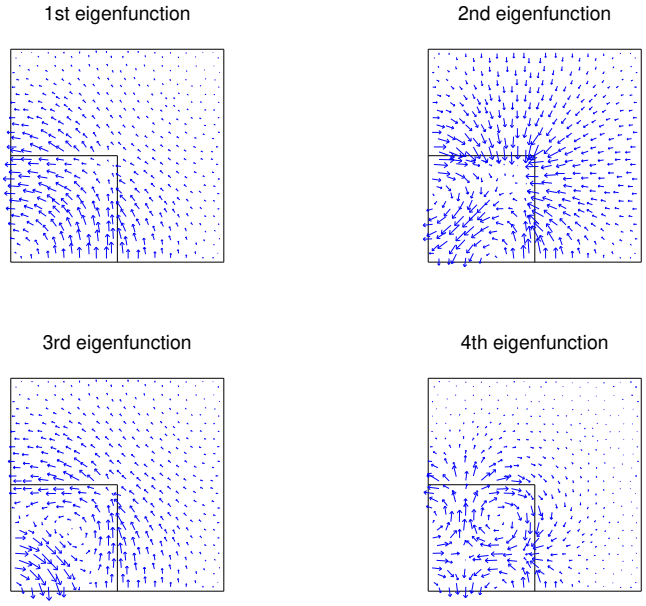

FIG. 6. *Edge approximation for discontinuous* ε.

different. One can see in Table 6 that the zero mode is exactly approximated by a number of zero discrete eigenvalues. Let us notice that this number is equal to the number of gradients which can be represented by a continuous piecewise linear function; i.e., it is equal to the number of C^1 piecewise quadratic functions, which vanish on the boundary (in this case the number of cross-grids minus one; cf. [32]). The first 10 nonvanishing eigenvalues are listed in Table 6. We point out that the

TABLE 5
Edge approximation for L-shaped domain.

					Slope
1st	0.57569	0.58627	0.59012	0.59219	3.76
2nd	1.43441	1.43242	1.43230	1.43228	13.13
3rd	3.97545	3.99209	3.99674	3.99860	6.23
4th	4.01957	3.99805	3.99828	3.99914	6.45
5th	4.62476	4.61321	4.61443	4.61522	5.30
6th	5.03756	5.05317	5.06669	5.07493	2.94
7th	7.94174	8.01297	7.99996	7.99801	9.97
8th	8.33765	8.56963	8.61489	8.63473	5.18
9th	9.39049	9.46310	9.46172	9.46117	11.93
10th	11.2519	11.4449	11.4748	11.4950	4.62
d.o.f.	128	327	607	968	
# zeros	33	94	182	297	

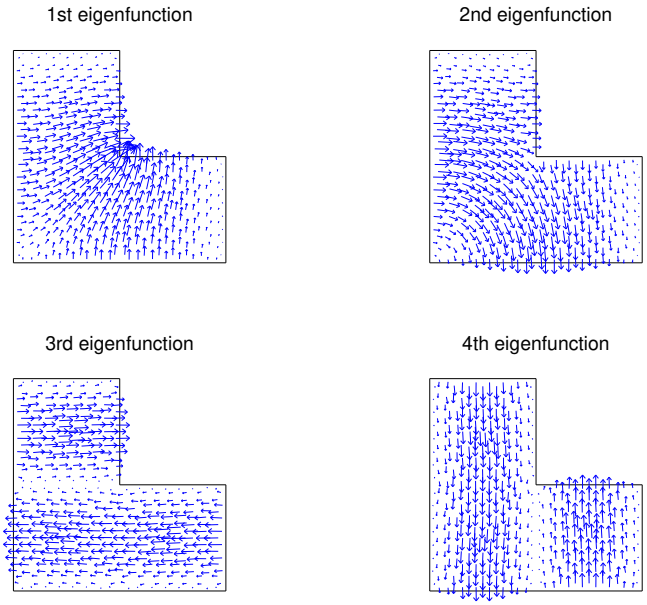

FIG. 7. *Edge approximation for L-shaped domain.*

physical modes are well approximated. However, in the table there is a numerical spurious mode which seems to converge to six.

This fact confutes the conjecture of Wong and Cendes [36] that meshes supporting C^1 piecewise quadratic polynomials give stable approximations.

Figures 10 and 11 refer to the criss-cross 16×16 mesh. In the former the eigenvalues from 240 to 280 are plotted. The marked eigenvalues do not approximate any point in the continuous spectrum and their associated eigenfunctions are shown in the latter.

7. Conclusions. The approximation of the eigensolutions of Maxwell's system has been studied. In particular, sufficient conditions preventing the occurrence of

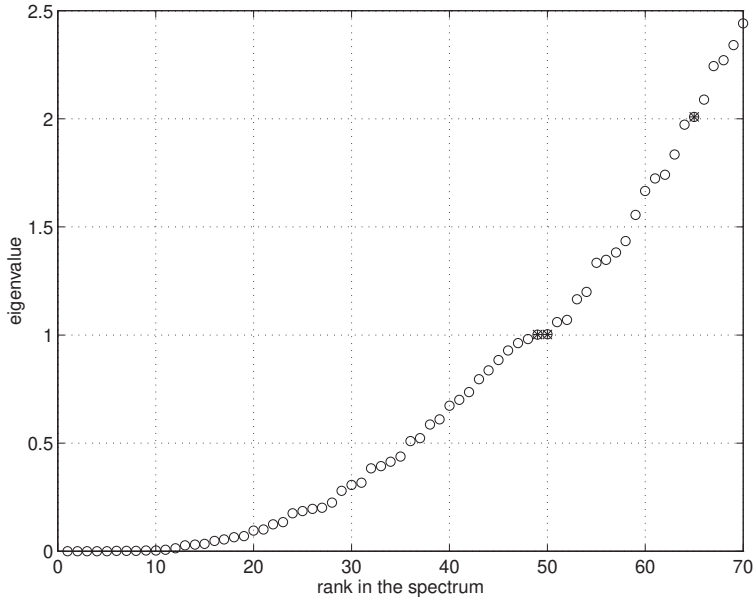

FIG. 8. *Nodal approximation.*

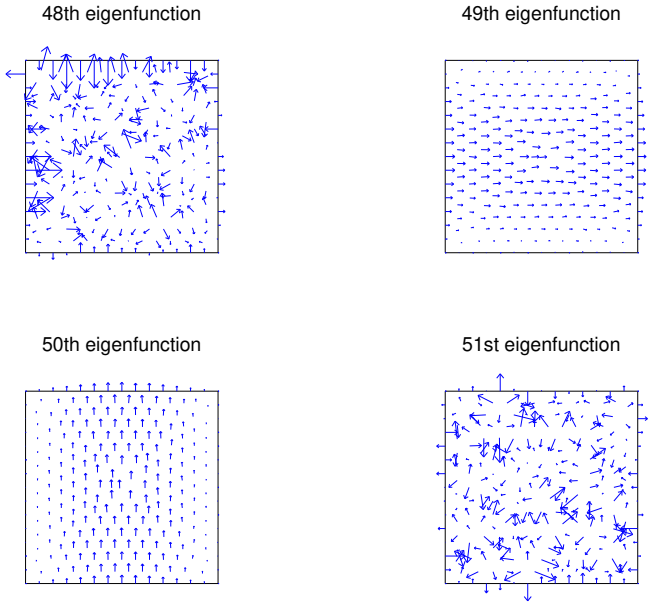

FIG. 9. *Nodal approximation.*

nonphysical modes have been investigated. From the numerical point of view it is convenient to drop the divergence-free constraint. However, this leads to a problem whose associated operator is not compact, since an infinite-dimensional eigenspace corresponds to the zero frequency. Although the zero frequency eigenspace is not physical, it needs to be carefully approximated; otherwise a pollution of the whole spectrum cannot be avoided.

TABLE 6
Nodal approximation on criss-cross mesh.

Exact	Computed			
1.00000	1.00428	1.00190	1.00107	1.00068
1.00000	1.00428	1.00190	1.00107	1.00069
2.00000	2.01711	2.00761	2.00428	2.00274
4.00000	4.06804	4.03037	4.01710	4.01095
4.00000	4.06804	4.03037	4.01710	4.01095
5.00000	5.10634	5.04748	5.02674	5.01712
5.00000	5.10634	5.04748	5.02674	5.01712
	5.92293	5.96578	5.98074	5.98767
8.00000	8.27128	8.12151	8.06845	8.04383
9.00000	9.34085	9.15309	9.08640	9.05537
9.00000	9.34085	9.15309	9.08640	9.05537
d.o.f.	254	574	1022	1598
# zeros	63	143	255	399

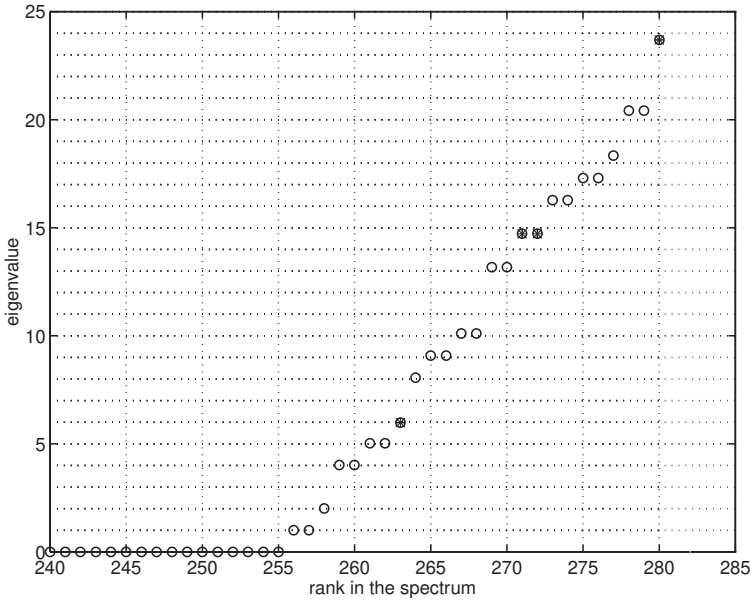

FIG. 10. *Nodal approximation on a criss-cross mesh.*

We analyze the eigenproblem using an equivalent mixed formulation. Following the theory of [6], sufficient conditions have been introduced in order to guarantee the convergence of the eigensolutions and to avoid the spectral pollution. Under suitable hypotheses, the edge elements of lowest degree are proved to satisfy these conditions.

We have also considered the approximation by nodal elements in order to convince the reader that the choice of appropriate elements is crucial. In fact, nodal elements on general meshes give rise to several nonphysical eigenvalues occurring at positive frequencies. On the other hand, the use of structured meshes can produce reasonable results, at least in approximating the zero frequency eigenspace, but subtler troubles

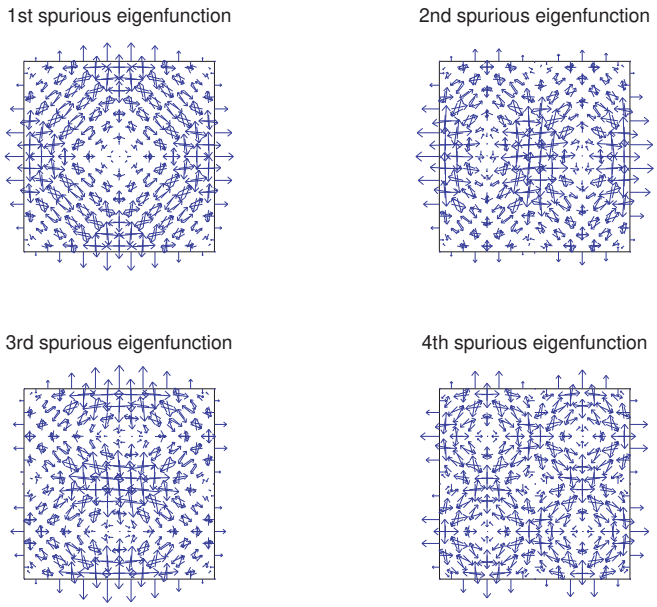

FIG. 11. *Nodal approximation on a criss-cross mesh.*

arise. A deeper investigation into these features can be found in [7].

Some numerical experiences on several case studies were also shown.

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