Convergence to self-similarity for the Boltzmann equation for strongly inelastic Maxwell molecules.

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Abstract. We prove propagation of regularity, uniformly in time, for the scaled solutions of the inelastic Maxwell model for any value of the coefficient of restitution. The result follows from the uniform in time control of the tails of the Fourier transform of the solution, normalized in order to have constant energy. By standard arguments this implies the convergence of the scaled solution towards the stationary state in Sobolev and L^1 norms in the case of regular initial data as well as the convergence of the original solution to the corresponding self-similar cooling state. In the case of weak inelasticity, similar results have been established by Carlen, Carrillo and Carvalho in [CCC] via a precise control of the growth of the Fisher information.

Keywords. Granular gases, kinetic models, Boltzmann equation.

1 Introduction

This paper concerns the regularity properties of solutions of the spatially homogeneous Boltzmann equation for Maxwellian molecules in $\mathbb{R}^3$ with inelastic collisions, introduced in [BCG00]. This equation describes the relaxation towards equilibrium of the distribution function of particles that interact through inelastic binary collisions. Let $f(v,\tau)$ be the probability density for the velocity v of a particle chosen randomly from the collection at time τ. Let $\varphi(v)$ be any bounded and continuous function on $\mathbb{R}^3$. For a dilute gas, with a mean free path of size ε, the equation under investigation is given, in weak form, by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle f,\varphi\rangle = \frac{1}{\varepsilon}\langle Q(f,f),\varphi\rangle \tag{1}$$

where

$$\langle f,\varphi\rangle = \int_{\mathbb{R}^3}\varphi(v)f(v,\tau)\,\mathrm{d}v,$$

and

$$\langle Q(f,f),\varphi\rangle = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v,\tau)f(w,\tau)\Big[\varphi(v')-\varphi(v)\Big]\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}w. \tag{2}$$

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In (2) σ is a unit vector in S^2, $d\sigma$ is the uniform measure on S^2 with total mass 4π and the post collision velocities are given by the collision mechanism written as

$$\begin{aligned} v' &= \frac{1}{2}(v+w) + \frac{1-e}{4}\,q + \frac{1+e}{4}|q|\sigma \\ w' &= \frac{1}{2}(v+w) - \frac{1-e}{4}\,q - \frac{1+e}{4}|q|\sigma \end{aligned} \tag{3}$$

with $q = v - w$. The positive parameter e, with $0 \le e < 1$ is the restitution coefficient. From (3) it follows that the post–collision relative velocity is non increasing, with

$$|v' - w'|^2 = |q'|^2 = |q|^2 - \frac{1-e^2}{2}\left(|q|^2 - |q|q\cdot\sigma\right) \le |q|^2 = |v-w|^2.$$

Note that the dissipation increases with e decreasing. Thus the case $e = 0$ corresponds to the strongest dissipation. Since $e < 1$, the collisions are inelastic, energy is dissipated in each collision, and the collisions are not reversible. This makes a crucial difference with the elastic theory in which there is a complete time reversal symmetry between the pre and post collision velocities. As pointed out in [CCC], it is mainly for this reason that the Boltzmann equation is usually written in the weak form (1), and not because of any difficulty in constructing strong solutions. The post-collision velocities (3) represent one of the possible parameterizations of the inelastic collision mechanism. However, as exhaustively discussed in [CCC] other possible pairs of pre-collision velocities (v_*, w_*) that result in the pair of post collision velocities (v, w) can be constructed [BCG00, GPV04]. It is remarkable that these parameterizations give equivalent collision terms only when the restitution coefficient satisfies $e > 0$ [CCC]. Consequently, any result which is valid for any value of e, including the case $e = 0$, can not be translated, in this limit case, to other collision rules.

Since in each individual collision $(v, w) \to (v', w')$, the total momentum $v + w$ is conserved, the first moment of $f(v, \tau)$ is conserved. In particular, choosing as initial datum a probability density f_0 satisfying

$$\int_{\mathbb{R}^3} f_0(v)\,dv = 1, \quad \int_{\mathbb{R}^3} v_i f_0(v)\,dv = 0,\ i = 1,2,3, \quad \int_{\mathbb{R}^3} |v|^2 f_0(v)\,dv = T_0 < \infty, \tag{4}$$

it follows that while both mass and momentum are preserved in time, the second moment (i.e the temperature or energy) decays according to the law

$$T(\tau) = \int_{\mathbb{R}^3} |v|^2 f(v,\tau)\,dv = T_0 \exp\left\{-\frac{1-e^2}{4\varepsilon}\tau\right\}. \tag{5}$$

This implies that $f(v, \tau)dv$ converges towards a point mass at $v = 0$ as τ tends to infinity. The precise way in which the density collapses into a mass concentrated in zero has been investigated in various previous works [BC03, BCT03, BCT06, BCG08, BC07]. It has been shown that, after a certain relaxation time, each solution converges towards a self-similar solution, known as the *homogeneous cooling state.* The argument to show that this happens, and the respective rate of convergence, is based on an argument which is commonly used in nonlinear diffusion equations [CT00]. If one defines a temperature invariant scaling $h(v, \tau)$ of $f(v, \tau)$ as

$$h(v,\tau) = \left(\frac{T(\tau)}{3}\right)^{3/2} f\left(\left(\frac{T(\tau)}{3}\right)^{1/2} v, \tau\right), \tag{6}$$

so that $\int_{\mathbb{R}^3} |v|^2\, h(v,\tau)\,dv = 3$ for all $\tau \ge 0$, then the scaled density tends to a universal equilibrium state h_∞, and $\int_{\mathbb{R}^3} |v|^2\, h_\infty(v)\,dv = 3$.

The equation for $h(v,\tau)$ now reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle h,\varphi\rangle = \frac{1}{\varepsilon}\langle Q(h,h),\varphi\rangle + \frac{1-e^2}{8\varepsilon}\langle h, v\cdot\nabla\varphi\rangle. \tag{7}$$

The speed at which the second moment converges to zero depends both of the mean free path ε and the coefficient of restitution e. On the other hand, the dependence on the mean free path can be absorbed once and for all by scaling the time. By setting

$$E = \frac{8}{1-e^2}, \tag{8}$$

and scaling the time as

$$t = \frac{1-e^2}{8\varepsilon}\tau, \tag{9}$$

we obtain for

$$g(v,t) = h\left(v, \frac{8\varepsilon}{1-e^2}\,t\right) \tag{10}$$

the equivalent equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle g,\varphi\rangle = E\langle Q(g,g),\varphi\rangle + \langle g, v\cdot\nabla\varphi\rangle. \tag{11}$$

In the rest of the paper, we will study equation (11). We remark that any result on the asymptotic convergence of $g(v,t)$ towards $g_\infty = h_\infty$ can be easily translated into a result on the asymptotic convergence of $h(v,\tau)$. It is worth remarking that thanks to (6) and (10), the initial data g_0 of a scaled solution $g(t)$ is related to the initial data f_0 of the original solution $f(\tau)$ by

$$g_0(v) = \left(\frac{T_0}{3}\right)^{3/2} f_0\left(\left(\frac{T_0}{3}\right)^{1/2} v\right)$$

and so $\int_{\mathbb{R}^3} |v|^2\, g_0(v)\,\mathrm{d}v = 3$ independently of the initial temperature T_0 of f_0.

Both the large-time behavior and the regularity of the solution of equation (11) have been recently studied in [CCC]. In particular, propagation of regularity was found by controlling the growth of the frequencies of the Fourier transform of the solution by means of a precise control of the growth of the Fisher information

$$I(f)(t) = 4\int_{\mathbb{R}^3}\left|\nabla\sqrt{f(v,t)}\right|^2 \mathrm{d}v.$$

Coupling the uniform bound on the regularity of the solution with the time decay of the Fourier metric [BCT06, CT07]

$$d_2(f,g)(t) = \sup_{\xi\neq 0}\frac{|\widehat{f}(\xi,t)-\widehat{g}(\xi,t)|}{|\xi|^2}, \tag{12}$$

convergence in L^1 follows.

The results of [CCC] however, require the assumption of weak inelasticity, namely the coefficient of restitution e sufficiently close to 1.

A different technique which allows to control the growth of the frequencies of the Fourier transform of the solution has been recently applied in [FPTT] to a one-dimensional dissipative model introduced by Ben-Avraham and coworkers [BBLR03]. The results in [FPTT] are independent of the degree of dissipation. By adapting this technique to the three-dimensional case, we will end up with a result which is independent of the coefficient of restitution e. The starting point of this analysis are the results of the paper [CGT99], where the proof of uniform propagation of regularity makes use of the following property for the solution $f(t)$ of the elastic Boltzmann equation

$$\sup_{|\xi|\geq R} |\hat{f}(\xi,t)| \longrightarrow 0, \;\; R \to +\infty \tag{13}$$

uniformly in time. This property is proved by exploiting the pointwise convergence of the Fourier transform of the solution to the Maxwellian equilibrium $\hat{M}(\xi) = e^{-\frac{|\xi|^2}{2}}$ and the decreasing of the Maxwellian itself. In the dissipative case, condition (13) cannot be satisfied by $f(t)$ but continues to hold for the solution to the scaled Boltzmann equation $g(t)$, provided the Fourier transform of the initial data satisfies

$$(1+\kappa|\xi|)^{\mu}|\hat{g}_0(\xi)| \leq K, \quad \xi \in \mathbb{R}^3 \tag{14}$$

for some positive constants κ, K and μ. Indeed, we prove that the solution $g(t)$ satisfies an analogous estimate

$$(1+\kappa|\xi|)^{\mu}|\hat{g}(\xi,t)| \leq K, \quad \xi \in \mathbb{R}^3 \tag{15}$$

uniformly in time, for possibly different κ, K and μ. Condition (15) is difficult to prove directly from the equation satisfied by the scaled density $g(t)$, due to the presence of the drift term in equation (11). The key point of our approach is to consider a semi-implicit discretization of equation (11), where the drift term is absorbed in an integral term which is much easier to handle. The same trick has been successfully exploited in the one-dimensional case in [FPTT] where it has been also pointed out that an equivalent approach could have been to resort to the Duhamel formula for the solution $g(t)$ (see e.g. [CT07]).

Our main results are summarized into the following statements.

Theorem 1
Assume $e \in [0,1)$. Let $g(t)$ be the weak solution of the equation (11), corresponding to the initial probability density g_0 with zero mean velocity, $\int_{\mathbb{R}^3} |v|^2\, g_0(v)\,\mathrm{d}v = 3$ and satisfying $\int_{\mathbb{R}^3} |v|^4\, g_0(v)\,\mathrm{d}v < +\infty$. If in addition

$$|\hat{g}_0(\xi)| \leq \frac{1}{(1+\beta|\xi|)^{\nu}}, \quad |\xi| > R, \tag{16}$$

for some $R > 0$, $\nu > 0$ and $\beta > 0$, then there exist $\rho > 0$, $k > 0$, $\nu' > 0$ such that $g(t)$ satisfies

$$|\hat{g}(\xi,t)| \leq \begin{cases} \dfrac{1}{1+k|\xi|^2}, & |\xi| \leq \rho, \quad t \geq 0 \\[2mm] \dfrac{1}{(1+\beta|\xi|)^{\nu'}}, & |\xi| > \rho, \quad t \geq 0. \end{cases} \tag{17}$$

Theorem 2
Assume $e \in [0,1)$. Let $g(t)$ be the weak solution of the equation (11), corresponding to the initial probability density g_0 with zero mean velocity, $\int_{\mathbb{R}^3} |v|^2\, g_0(v)\,\mathrm{d}v = 3$ and satisfying $\int_{\mathbb{R}^3} |v|^4\, g_0(v)\,\mathrm{d}v < +\infty$. Let us suppose moreover $g_0 \in H^{\eta}(\mathbb{R}^3)$ for some $\eta > 0$ large enough, $\sqrt{g_0} \in \dot{H}^{\nu}(\mathbb{R}^3)$ for some $\nu > 0$. Then $g(t)$ converges strongly in L^1 with an exponential rate towards the stationary solution g_∞, i.e., there exist positive constants C and γ explicitly computable such that

$$\|g(t) - g_\infty\|_{L^1(\mathbb{R}^3)} \leq C e^{-\gamma t}, \quad t \geq 0.$$

Thanks to the scaling invariance of the L^1 norm, Theorem 2 allows to deduce also the strong convergence of the original non scaled solution $f(\tau)$ to the self-similar state

$$\|f(\tau) - f_\infty(\tau)\|_{L^1(\mathbb{R}^3)} \leq C e^{-\gamma \frac{1-e^2}{8\varepsilon}\tau}, \quad \tau \geq 0,$$

where

$$f_\infty(v,\tau) = \left(\frac{3}{T(\tau)}\right)^{3/2} g_\infty\left(\left(\frac{3}{T(\tau)}\right)^{1/2} v\right)$$

for $T(\tau)$ as in (5) and this independently of the initial temperature T_0 of f_0.

2 Preliminary results

Following Bobylev [Bob88] it is convenient to consider equation (1) for Maxwell molecules in the dissipative case in the Fourier variables:

$$\frac{\partial}{\partial\tau}\hat{f}(\xi,\tau) = \frac{1}{4\pi\,\varepsilon}\int_{\sigma\in S^2}\left(\hat{f}(\xi^+,\tau)\hat{f}(\xi^-,\tau) - \hat{f}(\xi,\tau)\hat{f}(\xi,0)\right)\mathrm{d}\sigma \tag{18}$$

where

$$\begin{aligned} \xi^+ &= \frac{3-e}{4}\xi + \frac{1+e}{4}|\xi|\sigma \\ \xi^- &= \frac{1+e}{4}\left(\xi - |\xi|\sigma\right) = \xi - \xi^+. \end{aligned} \tag{19}$$

The existence and uniqueness of a solution of (18) for any initial data f_0 satisfying (4) can be established through the application of Wild sums [CT07].

Theorem 3 (Theorem of existence and uniqueness [Bob88, CT07])
We consider $f_0 \geq 0$ satisfying the normalization conditions (4) and the following Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial\tau}\hat{f}(\xi,\tau) = \dfrac{1}{4\pi\,\varepsilon}\displaystyle\int_{\sigma\in S^2}\left(\hat{f}(\xi^+,\tau)\hat{f}(\xi^-,\tau) - \hat{f}(\xi,\tau)\hat{f}(\xi,0)\right)\mathrm{d}\sigma, & \tau > 0 \\ \hat{f}(\xi,0) = \hat{f}_0(\xi). \end{cases} \tag{20}$$

Then, there exists a unique nonnegative solution $f \in C^1\left([0,+\infty), L^1(\mathbb{R}^3)\right)$ to equation (20). This solution preserves mass and momentum, while the energy decays at the exponential rate given by (5).

Let $g(v,t)$ be defined by (10). One can easily show that $g(v,t)$ preserves the temperature, and moreover

$$\int_{\mathbb{R}^3} |v|^2\, g(v,t)\,\mathrm{d}v = 3, \quad t \geq 0.$$

Moreover if the initial data g_0 has a diagonal pressure tensor which is unitary

$$\int_{\mathbb{R}^3} v_i v_j\, g_0(v)\,\mathrm{d}v = \delta_{i,j}, \quad i,j = 1,2,3, \tag{21}$$

then all the second moments are also preserved. If on the contrary the pressure tensor is not unitary, then its non isotropic part ($\int_{\mathbb{R}^3} v_i v_j\, g(v,t)\,\mathrm{d}v$ for $i \neq j$) vanishes if initially does (cfr. [BCT06]), whereas the isotropic part ($\int_{\mathbb{R}^3} v_i^2\, g(v,t)\,\mathrm{d}v$) is in general not preserved. Nevertheless, the condition $\int_{\mathbb{R}^3} |v|^2\, g_0(v)\,\mathrm{d}v = 3$ implies that the matrix $\left[\int_{\mathbb{R}^3} v_i v_j\, g_0(v)\,\mathrm{d}v\right]_{i,j}$ is real and symmetric. Consequently there exists a suitable orthonormal system in $\mathbb{R}^3$ in which it is diagonal and therefore it remains diagonal for all $t > 0$. Owing to this property, throughout this paper we will assume, without any additional assumption, that g_0 has diagonal pressure tensor, although not unitary.

It is well-known that in Maxwell models the time evolution of moments can be evaluated exactly. In particular, this can be done for the diagonal terms $\int_{\mathbb{R}^3} v_i^2\, g(v,t)\,\mathrm{d}v$. We have

Proposition 4
Assume $e \in [0,1)$. Let $g(t)$ be the weak solution of the equation (11), corresponding to the initial probability density g_0 with zero mean velocity, diagonal pressure tensor and satisfying $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$. Then, we have

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2\, g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2\, g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2\, g(v,t)\,\mathrm{d}v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-\frac{1+e}{1-e}t} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-\frac{1+e}{1-e}t}, \quad t \geq 0 \tag{22}$$

where $C_1 = \int_{\mathbb{R}^3} v_2^2\, g_0(v)\,\mathrm{d}v - 1$ and $C_2 = \int_{\mathbb{R}^3} v_3^2\, g_0(v)\,\mathrm{d}v - 1$.

Proof: Recalling equation (11), the expression (3) of the post collisional variables, the conservations of the mass and the vanishing both of the momentum and of the non isotropic terms $\int_{\mathbb{R}^3} v_i v_j\, g(v,t)\,\mathrm{d}v$ along the solution, we obtain the following linear differential system:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \int_{\mathbb{R}^3} v_1^2\, g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2\, g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2\, g(v,t)\,\mathrm{d}v \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} \\ \frac{1}{3}\frac{1+e}{1-e} & -\frac{2}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} \\ \frac{1}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} & -\frac{2}{3}\frac{1+e}{1-e} \end{bmatrix} \begin{bmatrix} \int_{\mathbb{R}^3} v_1^2\, g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2\, g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2\, g(v,t)\,\mathrm{d}v \end{bmatrix}.$$

Since the matrix of coefficients has eigenvalues 0 simple and $-\frac{1+e}{1-e}$ double with eigenspaces $\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$ and $\left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$ respectively, the general solution of the system is

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2\, g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2\, g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2\, g(v,t)\,\mathrm{d}v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-\frac{1+e}{1-e}t} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-\frac{1+e}{1-e}t}, \quad t \geq 0$$

for C_1, C_2 real constants to be determined by the initial conditions. □

In Fourier variables, the function $g(t)$ satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{g}(\xi,t) - (\xi \cdot \nabla_\xi)\hat{g}(\xi,t) = \dfrac{E}{4\pi} \displaystyle\int_{\sigma \in S^2} \big(\hat{g}(\xi^+,t)\hat{g}(\xi^-,t) - \hat{g}(\xi,t)\hat{g}(\xi,0)\big)\,\mathrm{d}\sigma, \\ \hat{g}(\xi,0) = \hat{g}_0(\xi). \end{cases} \tag{23}$$

We will denote

$$Q_+(\hat{g},\hat{g})(\xi,t) = \frac{1}{4\pi} \int_{\sigma \in S^2} \hat{g}(\xi^+,t)\hat{g}(\xi^-,t)\,\mathrm{d}\sigma.$$

Hence equation (23) can be written

$$\frac{\partial}{\partial t}\hat{g}(\xi,t) - (\xi \cdot \nabla_\xi)\hat{g}(\xi,t) = E\left(Q_+(\hat{g},\hat{g})(\xi,t) - \hat{g}(\xi,t)\right).$$

For $s > 0$ let us define $\mathcal{P}_s(\mathbb{R}^3)$ the set of probability densities satisfying

$$\int_{\mathbb{R}^3} |v|^s\, f(v)\,\mathrm{d}v < +\infty.$$

Consider on $\mathcal{P}_s(\mathbb{R}^3)$ the distance

$$d_s(f,g) = \sup_{\xi \neq 0} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^s}.$$

It is not difficult to show through a Taylor expansion that $d_s(f, g)$ is finite for any pair of probability densities f and g with equal moments $\int_{\mathbb{R}^3} v^\beta f(v)\, \mathrm{d}v$ for any multi-index $\beta \in \mathbb{N}^3$ of length smaller than or equal to $[s]$ (if $s \in \mathbb{N}$, it is enough to suppose equal moments up to $s-1$ order). For $\alpha \in (0, 1]$ and initial data g_1, $g_2 \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ which share the same moments up to the second order and have unitary pressure tensor as in (21), it is possible to estimate the $d_{2+\alpha}$ distance between the two solutions as follows.

Theorem 5 (Strict contraction of $d_{2+\alpha}$ [BCT06])
Assume $e \in [0, 1)$. For $\alpha \in (0, 1]$ there exists an explicit constant $C(\alpha, e) > 0$, $C(\alpha, e) \searrow 0$ as $\alpha \to 0$, such that for any $g_1(t)$ and $g_2(t)$ solutions of (11) corresponding to initial values g_1^0, g_2^0 in $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with unit mass, zero mean velocity and unitary pressure tensor, then

$$d_{2+\alpha}(g_1(t), g_2(t)) \leq d_{2+\alpha}(g_1^0, g_2^0) e^{-C(\alpha,e)t}, \quad t \geq 0.$$

In our framework, the constant $C(\alpha, e)$ has the following expression:

$$C(\alpha, e) = E(1 - A(\alpha, e)) - (2 + \alpha) \tag{24}$$

where

$$A(\alpha, e) = \frac{1}{4\pi} \int_{S^2} \frac{|\xi^+|^{2+\alpha} + |\xi^-|^{2+\alpha}}{|\xi|^{2+\alpha}}\, \mathrm{d}\sigma. \tag{25}$$

A detailed analysis of $C(\alpha, e)$ can be found in [BCT06].

It is worth noticing that we cannot deduce from the previous theorem the uniform boundedness in time neither of $\int_{\mathbb{R}^3} |v|^{2+\alpha} g_1(v, t)\, \mathrm{d}v$ nor of $\int_{\mathbb{R}^3} |v|^{2+\alpha} g_2(v, t)\, \mathrm{d}v$. Nevertheless, if the initial data g_0 belongs to $\mathcal{P}_4(\mathbb{R}^3)$, it is possible to prove that the solution keeps on satisfying the same property uniformly in time.

Theorem 6 (Uniform control of 4th moment [BC07])
If g_0 is a Borel probability measure on $\mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} |v|^4\, g_0(v)\, \mathrm{d}v < \infty,$$

then the solution $g(t)$ to (11) with initial data g_0 satisfies

$$\sup_{t \geq 0} \int_{\mathbb{R}^3} |v|^4\, g(v, t)\, \mathrm{d}v < \infty.$$

Thanks to the uniform boundedness of the fourth moment, it is possible to prove, via a fixed point argument, the existence of a universal stationary state in a suitable subspace of $\mathcal{P}_2(\mathbb{R}^3)$. Moreover, the strict contraction of the $d_{2+\alpha}$ metric allows also to prove the stability of this stationary state in this subspace.

Theorem 7 (Stationary states [BC03, BCT06, CT07])
Let $e \in [0, 1)$ be fixed. Equation (11) has a unique stationary state g_∞ which is a probability density in

$$\mathcal{H} = \Big\{ g \geq 0,\ \int_{\mathbb{R}^3} g(v)\, \mathrm{d}v = 1,\ \int_{\mathbb{R}^3} v_i\, g(v)\, \mathrm{d}v = 0,\ \int_{\mathbb{R}^3} v_i\, v_j\ g(v)\, \mathrm{d}v = \delta_{i,j},\ i, j = 1, 2, 3,$$
$$\int_{\mathbb{R}^3} |v|^{2+\alpha}\, g(v)\, \mathrm{d}v < +\infty,\ \text{for } \alpha \in (0, 1] \Big\}.$$

This stationary state is a radial function and belongs to $\mathcal{P}_4(\mathbb{R}^3)$ *with* $\int_{\mathbb{R}^3} |v|^4 g_\infty(v)\,\mathrm{d}v \leq M_4$*, where* M_4 *depends only on* $e \in [0,1)$*. Moreover, for* $\alpha \in (0,1]$*, given any* $g(t)$ *solution of* (11) *issued from an initial datum* $g_0 \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ *with unit mass, zero mean velocity and unitary pressure tensor, then*

$$d_{2+\alpha}(g(t), g_\infty) \leq d_{2+\alpha}(g_0, g_\infty) e^{-C(\alpha,e)t}, \quad t \geq 0.$$

where $C(\alpha, e)$ *is the constant (24).*

The hypothesis of unitary pressure tensor can in fact be removed. Under the only assumption that the initial datum belongs to $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$ (and same moments up to the first order as the stationary state), the d_2 distance between the solution and the stationary state is proved to be exponentially decreasing.

Theorem 8 (Stability in d_2 without unitary tensor pressure [BCT06, CT07])
Let $e \in [0,1)$ *be fixed. For any* $\alpha \in (0,1]$ *and any initial datum* $g_0 \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ *with unit mass, zero mean velocity and* $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$*, there exist positive constants* K_1 *and* K_2 *depending on* e, α, g_0 *such that given* $g(t)$ *the solution of* (23) *issued from* g_0 *and* g_∞ *the stationary state, then*

$$d_2(g(t), g_\infty) \leq K_1 e^{-K_2 t}, \quad t \geq 0.$$

As a consequence, the uniqueness of the stationary state holds in a larger space where the second moments are not prescribed.

Corollary 9 (Uniqueness of stationary states [BC03, CT07])
Uniqueness of the stationary state g_∞ *found in Theorem 7 holds true in*

$$\tilde{\mathcal{H}} = \Big\{ g \geq 0,\ \int_{\mathbb{R}^3} g(v)\,\mathrm{d}v = 1,\ \int_{\mathbb{R}^3} v_i\, g(v)\,\mathrm{d}v = 0,\ i = 1,2,3,\ \int_{\mathbb{R}^3} |v|^2\, g(v)\,\mathrm{d}v = 3,$$
$$\int_{\mathbb{R}^3} |v|^{2+\alpha}\, g(v)\,\mathrm{d}v < +\infty,\ \text{for } \alpha \in (0,1] \Big\}.$$

As far as the smoothness of the stationary state is concerned, it is known that $g_\infty \in H^s(\mathbb{R}^3)$, for all $s \geq 0$. This is as a consequence of the following result.
è vero per $e = 0$??

Theorem 10 (Smoothness of the stationary state [BC03], Theorem 5.3)
For $e \in [0,1)$*, the stationary state* g_∞ *satisfies the bounds*

$$e^{-\frac{|\xi|^2}{2}} \leq |\hat{g}_\infty(\xi)| \leq (1 + |\xi|) e^{-|\xi|}, \quad \xi \in \mathbb{R}^3.$$

3 The iteration process

The goal of this section is to build up a sequence of functions $\{g^N(\xi,t)\}$ which approximates uniformly the solution $\hat{g}(\xi,t)$. In order to do this, for any fixed $T > 0$ we consider firstly a semi-implicit discretization in time of equation (23) by partitioning the interval $[0,T]$ into N subintervals and we define thus the approximate solution at any time $t = j\frac{T}{N}$ for $j = 0, \ldots, N$. Secondly, we define $g^N(\xi,t)$ on the whole interval $[0,T]$ by interpolation and lastly we show the convergence of the approximation to the solution. Other details and the missing proofs of this section can be found in [FPTT].

We begin by introducing a semi-implicit discretization in time of equation (23) as follows. Let $T > 0$ and $\Delta t = \frac{T}{N}$ for $N \in \mathbb{N}$, $N > T$. Let $\hat{\varphi}_j^N(\xi)$, $j = 0, \ldots, N$ be the sequence:

$$\begin{cases} \hat{\varphi}_0^N(\xi) = \hat{g}_0(\xi) \\ \dfrac{\hat{\varphi}_{j+1}^N(\xi) - \hat{\varphi}_j^N(\xi)}{\Delta t} = \xi \cdot \nabla_\xi \, \hat{\varphi}_{j+1}^N + E\left(Q_+(\hat{\varphi}_j^N, \hat{\varphi}_j^N)(\xi) - \hat{\varphi}_j^N(\xi)\right), \quad j = 0, \ldots, N-1. \end{cases} \tag{26}$$

Proposition 11
Assume $e \in [0,1)$. If g_0 is a probability density with zero mean velocity and satisfying $\int_{\mathbb{R}^3} |v|^2 \, g_0(v) \, \mathrm{d}v = 3$, then there exists a unique sequence of bounded functions $\hat{\varphi}_j^N$ for $j = 1, \ldots, N$ satisfying (26). This sequence is defined as follows

$$\begin{cases} \hat{\varphi}_0^N(\xi) = \hat{g}_0(\xi) \\ \hat{\varphi}_{j+1}^N(\xi) = \dfrac{1}{\Delta t} \displaystyle\int_1^{+\infty} \left(E\Delta t \, Q_+(\hat{\varphi}_j^N, \hat{\varphi}_j^N)(\eta\xi) + (1 - E\Delta t)\hat{\varphi}_j^N(\eta\xi)\right) \frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad j = 0, \ldots, N-1. \end{cases} \tag{27}$$

Proof: Let us begin by proving that $\hat{\varphi}_1^N$ is well defined. We proceed in the same way as in Proposition 7 in [FPTT]. For $\xi \neq 0$ we multiply equation (26) by $\left(-\frac{1}{\Delta t}\right) \, |\xi|^{-\frac{1}{\Delta t}-1}$ and obtain

$$\left(-\frac{1}{\Delta t}\right) \, |\xi|^{-\frac{1}{\Delta t}-1}\hat{\varphi}_1^N(\xi) + |\xi|^{-\frac{1}{\Delta t}} \frac{\xi}{|\xi|} \cdot \nabla_\xi \hat{\varphi}_1^N(\xi) =$$
$$\left(-\frac{1}{\Delta t}\right) \, |\xi|^{-\frac{1}{\Delta t}-1} \left(E\Delta t Q_+(\hat{g}_0, \hat{g}_0)(\xi) + (1 - E\Delta t)\hat{g}_0(\xi)\right)$$

which is

$$\frac{\partial}{\partial |\xi|} \left(|\xi|^{-\frac{1}{\Delta t}} \hat{\varphi}_1^N\right)(\xi) = \left(-\frac{1}{\Delta t}\right) \, |\xi|^{-\frac{1}{\Delta t}-1} \left(E\Delta t Q_+(\hat{g}_0, \hat{g}_0)(\xi) + (1 - E\Delta t)\hat{g}_0(\xi)\right).$$

Now, we can write $\xi = (|\xi| \cos\theta \sin\psi, |\xi| \sin\theta \sin\psi, |\xi| \cos\psi)$, $\theta \in [0, 2\pi)$, $\psi \in [0, \pi]$ and so we integrate on $[|\xi|, +\infty)$ and since $\hat{g}_0(\xi)$ is bounded we get:

$$|\xi|^{-\frac{1}{\Delta t}} \, \hat{\varphi}_1^N(\xi) = \frac{1}{\Delta t} \int_{|\xi|}^{+\infty} \left(E\Delta t \; Q_+(\hat{g}_0, \hat{g}_0)(s, \theta, \psi) + (1 - E\Delta t)\hat{g}_0(s, \theta, \psi)\right) s^{-\frac{1}{\Delta t}-1} \mathrm{d}s.$$

Finally by the change of variables $\eta = \frac{s}{|\xi|}$ and since

$$(\eta|\xi| \cos\theta \sin\psi, \eta|\xi| \sin\theta \sin\psi, \eta|\xi| \cos\psi) = \eta\xi,$$

we are led to:

$$\hat{\varphi}_1^N(\xi) = \frac{1}{\Delta t} \int_1^{+\infty} \left(E\Delta t \; Q_+(\hat{g}_0, \hat{g}_0)(\eta\xi) + (1 - \Delta t) \; \hat{g}_0(\eta\xi)\right) \frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}. \tag{28}$$

Since g_0 has unit mass, zero mean velocity and bounded temperature, then $\hat{g}_0$ belongs to $\mathcal{C}^1(\mathbb{R}^3)$ and there exists $C > 0$ such that

$$\hat{g}_0(0) = 1 \quad |\partial_k \hat{g}_0(\xi)| \leq C \quad \partial_k \hat{g}_0(0) = 0, \quad k = 1, 2, 3. \tag{29}$$

Therefore the function $\hat{\varphi}_1^N$ can be defined by continuity in $\xi = 0$ as $\hat{\varphi}_1^N(0) = 1$ and it is the unique, bounded and $\mathcal{C}^1(\mathbb{R}^3)$ solution of (26).

By an iteration argument the same conclusion holds for any $\hat{\varphi}_j^N$ obtaining for $j = 0, \dots, N-1$

$$\hat{\varphi}_{j+1}^N(\xi) = \frac{1}{\Delta t} \int_1^{+\infty} \left(E\Delta t\, Q_+(\hat{\varphi}_j^N, \hat{\varphi}_j^N)(\eta\xi) + (1 - E\Delta t)\hat{\varphi}_j^N(\eta\xi) \right) \frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}.$$

□

Remark 12

Applying Fubini's theorem we can remark that any $\hat{\varphi}_{j+1}^N(\xi)$ is the Fourier transform of $\varphi_{j+1}^N(v)$ where for $j = 0, \dots, N-1$ and $v \in \mathbb{R}^3$

$$\begin{cases} \varphi_0^N(v) = g_0(v) \\ \varphi_{j+1}^N(v) = \dfrac{1}{\Delta t} \displaystyle\int_1^{+\infty} \left(E\Delta t \frac{1}{\eta^3} Q_+(\varphi_j^N, \varphi_j^N) \left(\frac{v}{\eta}\right) + (1 - E\Delta t) \frac{1}{\eta^3} \varphi_j^N \left(\frac{v}{\eta}\right) \right) \dfrac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}. \end{cases} \tag{30}$$

We gather in the following proposition the results on the moments of the approximation φ_j^N which will be useful for proving Theorem 1. We will prove the proposition in the Appendix.

Proposition 13

Assume $e \in [0, 1)$ and let g_0 be a probability density with zero mean velocity and satisfying

$$\int_{\mathbb{R}^3} v_i\, v_k\, g_0(v)\,\mathrm{d}v = 0, \ i \neq k, \ \int_{\mathbb{R}^3} |v|^2\, g_0(v)\,\mathrm{d}v = 3, \ \int_{\mathbb{R}^3} |v|^4\, g_0(v)\,\mathrm{d}v < +\infty. \tag{31}$$

Let φ_j^N be the approximation defined in (30). Then, there exists $C > 0$ such that for N large enough, for $j = 0, \dots, N$ we get

$$\varphi_j^N(v) \geq 0, \ \int_{\mathbb{R}^3} \varphi_j^N(v)\,\mathrm{d}v = 1, \ \int_{\mathbb{R}^3} v_k\, \varphi_j^N(v)\,\mathrm{d}v = 0, \ k = 1, 2, 3, \tag{32}$$

$$\int_{\mathbb{R}^3} v_i\, v_k\, \varphi_j^N(v)\,\mathrm{d}v = 0, \ i \neq k, \tag{33}$$

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2\, \varphi_j^N(v)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2\, \varphi_j^N(v)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2\, \varphi_j^N(v)\,\mathrm{d}v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \left(\frac{1 - \Delta t\, \frac{3-e}{1-e}}{1 - 2\Delta t} \right)^j + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \left(\frac{1 - \Delta t\, \frac{3-e}{1-e}}{1 - 2\Delta t} \right)^j, \tag{34}$$

$$\int_{\mathbb{R}^3} |v|^4\, \varphi_j^N(v)\,\mathrm{d}v \leq C. \tag{35}$$

where $C_1 = \int_{\mathbb{R}^3} v_2^2\, g_0(v)\,\mathrm{d}v - 1$ and $C_2 = \int_{\mathbb{R}^3} v_3^2\, g_0(v)\,\mathrm{d}v - 1$.

We define the approximate solution $g^N(\xi, t)$ at any time $t = j\frac{T}{N}$ as $g^N(\xi, j\frac{T}{N}) = \hat{\varphi}_j^N(\xi)$ for $j = 0, \dots, N$. We extend afterwards the definition on the whole interval by interpolation. More precisely, let us define:

$$g^N(\xi, t) = \begin{cases} \hat{g}_0(\xi) & t = 0 \\ \alpha(t)\hat{\varphi}_{K_N-1}^N(\xi) + (1 - \alpha(t))\hat{\varphi}_{K_N}^N(\xi) & 0 < t \leq T \end{cases}$$

where for $0 < t \leq T$ we have $(K_N - 1)\frac{T}{N} < t \leq K_N \frac{T}{N}$ for $K_N \in \{1, \dots, N\}$ and more specifically there is a function $0 \leq \alpha(t) < 1$ such that $t = \alpha(t)(K_N - 1)\frac{T}{N} + (1 - \alpha(t))K_N \frac{T}{N}$. Any $g^N(\xi, t)$ is continuous on $\mathbb{R}^3 \times [0, T]$ and for any $t \in [0, T]$ it belongs to $\mathcal{C}^2(\mathbb{R}^3)$.

The result of convergence is therefore as follows.

Proposition 14
There is a subsequence $\left\{g^{N_l}(\xi,t)\right\}_l$ *of* $\left\{g^N(\xi,t)\right\}_N$ *which converges uniformly on any compact set of* $\mathbb{R}^3\times[0,T]$ *to the solution* $\hat{g}(\xi,t)$*.*

4 Control of tails of the Fourier transform of the solution

In this section we prove Theorem 1. Thanks to the uniform convergence of a subsequence of the approximate solutions $g^N(\xi,t)$ to the solution $\hat{g}(\xi,t)$ and to the definition of $g^N(\xi,t)$, it is enough to prove the bounds (17) for any $\hat{\varphi}_j^N(\xi)$ uniformly for $N\in\mathbb{N}$ large enough and $j=0,\dots,N$. The control of low frequencies is a direct consequence of the properties (4) and $\int_{\mathbb{R}^3}|v|^4\,g_0(v)\,\mathrm{d}v<+\infty$ on the initial data.

Lemma 15
Assume $e\in[0,1)$*. Let* $g(t)$ *be the weak solution of the equation (11), corresponding to the initial probability density* g_0 *with zero mean velocity,* $\int_{\mathbb{R}^3}|v|^2\,g_0(v)\,\mathrm{d}v=3$ *and satisfying* $\int_{\mathbb{R}^3}|v|^4\,g_0(v)\,\mathrm{d}v<+\infty$*. Let* $\hat{\varphi}_j^N$ *be the approximation defined in* (27)*. There exist* $k>0$ *and* $\rho>0$ *such that for any fixed* $T>0$ *and any* $N\in\mathbb{N}$ *large enough we get*

$$|\hat{g}(\xi,t)|\le\frac{1}{1+k|\xi|^2},\quad |\xi|\le\rho,\quad t\ge0, \tag{36}$$

$$|\hat{\varphi}_j^N(\xi)|\le\frac{1}{1+k|\xi|^2},\quad |\xi|\le\rho,\quad j=0,\dots,N. \tag{37}$$

Proof: Both estimates will be performed through a Mac Laurin expansion in which we will be able to bound uniformly the terms thanks to Proposition 4, Theorem 6 and Proposition 13. In what follows, constants may vary for one line to another. Let us begin by proving (36). From the hypotheses on g_0 and Proposition 4, we get the Mac Laurin expansion in Fourier variables

$$\hat{g}(\xi,t)=1-\frac{1}{2}\sum_{k=1}^{3}\left(\int_{\mathbb{R}^3}v_k^2\,g(v,t)\,\mathrm{d}v\right)\xi_k^2+\int_0^1 D^3\hat{g}(s\xi,t)(\xi,\xi,\xi)\,\mathrm{d}s,\quad \xi\in\mathbb{R}^3,\ t\ge0.$$

Since for all $i,j,k=1,2,3$ and all $t\ge0$

$$\int_{\mathbb{R}^3}|v_iv_jv_k|\,g(v,t)\,\mathrm{d}v\le C<+\infty,$$

due to the conservation of the mass and the uniform boundedness of the fourth moment (Theorem 6) and since for all $i,j,k=1,2,3$ and all $t\ge0$

$$\left|\partial^3_{ijk}\hat{g}(\xi,t)\right|\le C\int_{\mathbb{R}^3}|v_iv_jv_k|\,g(v,t)\,\mathrm{d}v,$$

we can deduce the uniform upper bound

$$\left|\int_0^1 D^3\hat{g}(s\xi,t)(\xi,\xi,\xi)\,\mathrm{d}s\right|\le C|\xi|^3,\quad \xi\in\mathbb{R}^3,\ t\ge0.$$

On the other hand, the evolution of the second moments of $g(t)$ recalled in Proposition 4 allows to give a strictly positive uniform lower bound on $\int_{\mathbb{R}^3}v_k^2\,g(v,t)\,\mathrm{d}v$ for $k=1,2,3$ and $t\ge0$. We get therefore for $\xi\in\mathbb{R}^3$ and $t\ge0$

$$|\hat{g}(\xi,t)|\le1-C|\xi|^2+D|\xi|^3$$

with C and D suitably chosen and so there exist $\rho > 0$ and $k > 0$ such that

$$|\hat{g}(\xi,t)| \leq \frac{1}{1+k\xi^2}, \quad |\xi| \leq \rho, \ t \geq 0.$$

As for (37), we will proceed exactely in the same way, thanks to the uniform estimates collected in Proposition 13.

□

We are now in position to prove Theorem 1.

Theorem 1
Assume $e \in [0,1)$. Let $g(t)$ be the weak solution of the equation (11), corresponding to the initial probability density g_0 with zero mean velocity, $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$ and satisfying $\int_{\mathbb{R}^3} |v|^4 g_0(v)\,\mathrm{d}v < +\infty$. If in addition

$$|\hat{g}_0(\xi)| \leq \frac{1}{(1+\beta|\xi|)^{\nu}}, \quad |\xi| > R, \tag{38}$$

for some $R > 0$, $\nu > 0$ and $\beta > 0$, then there exist $\rho > 0$, $k > 0$, $\nu' > 0$ such that $g(t)$ satisfies

$$|\hat{g}(\xi,t)| \leq \begin{cases} \dfrac{1}{1+k|\xi|^2}, & |\xi| \leq \rho, \quad t \geq 0 \\ \dfrac{1}{(1+\beta|\xi|)^{\nu'}}, & |\xi| > \rho, \quad t \geq 0. \end{cases} \tag{39}$$

Proof: The bound on the low frequencies $|\xi| \leq \rho$ has been established in Lemma 15. Moreover, as a consequence of Proposition 3.3 in [DFT09], we can suppose that condition (38) holds for any $|\xi| > \rho$ with a possibly smaller exponent ν'. We will prove that for any $N \in \mathbb{N}$ and $j = 0, \ldots, N$ we get

$$|\hat{\varphi}_j^N(\xi)| \leq \frac{1}{(1+\beta|\xi|)^{\nu'}}, \quad |\xi| > \rho,$$

for positive ν' small enough. By induction we have only to check the bound on

$$\hat{\varphi}_1^N(\xi) = \frac{1}{\Delta t}\int_{\eta=1}^{+\infty} \left(E\Delta t\, Q_+(\hat{g}_0,\hat{g}_0)(\eta\xi) + (1-E\Delta t)\hat{g}_0(\eta\xi)\right)\frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}.$$

Let $|\xi| > \rho$ and $\eta > 1$. We would like to bound the term

$$Q_+(\hat{g}_0,\hat{g}_0)(\eta\xi) = \frac{1}{4\pi}\int_{S^2} \hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\mathrm{d}\sigma,$$

where we have denoted $\tilde{\xi} = \eta\xi$. In particular $|\tilde{\xi}| = \eta|\xi| > |\xi| > \rho$. We recall that for $\sigma \in S^2$

$$\tilde{\xi}^+ = \frac{3-e}{4}\tilde{\xi} + \frac{1+e}{4}|\tilde{\xi}|\sigma$$
$$\tilde{\xi}^- = \frac{1+e}{4}\left(\tilde{\xi} - |\tilde{\xi}|\sigma\right),$$

which imply $|\tilde{\xi}^+ + \tilde{\xi}^-| = |\tilde{\xi}|$. Therefore

$$|\tilde{\xi}^+| + |\tilde{\xi}^-| \geq |\tilde{\xi}|, \tag{40}$$

and so either $|\tilde{\xi}^+| \geq \frac{|\tilde{\xi}|}{2}$ or $|\tilde{\xi}^-| \geq \frac{|\tilde{\xi}|}{2}$. Moreover, for any $\sigma \in S^2$ we get

$$|\tilde{\xi}^+|^2 \geq |\tilde{\xi}|^2\left(\frac{1-e}{2}\right)^2 \geq |\xi|^2\left(\frac{1-e}{2}\right)^2. \tag{41}$$

We can split S^2 into six non overlapping pieces

$$S^2 = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$$

where

$$\begin{aligned}
A_0 &= \left\{ \sigma \in \mathcal{S}^2 : |\tilde{\xi^+}| > \rho, |\tilde{\xi^-}| > \rho \right\}, \\
A_1 &= \left\{ \sigma \in \mathcal{S}^2 : |\tilde{\xi^+}| \leq \rho,\ |\tilde{\xi^-}| > \rho \right\}, \\
A_2 &= \left\{ \sigma \in \mathcal{S}^2 : |\tilde{\xi^-}| \leq \rho,\ |\tilde{\xi^-}| \leq \frac{|\tilde{\xi}|}{2},\ |\tilde{\xi^+}| > \frac{|\tilde{\xi}|}{2} \right\}, \\
A_3 &= \left\{ \sigma \in \mathcal{S}^2 : |\tilde{\xi^-}| \leq \rho,\ |\tilde{\xi^-}| > \frac{|\tilde{\xi}|}{2},\ |\tilde{\xi^+}| > \frac{|\tilde{\xi}|}{2},\ |\tilde{\xi^+}| \leq \rho \right\}, \\
A_4 &= \left\{ \sigma \in \mathcal{S}^2 : |\tilde{\xi^-}| \leq \rho,\ |\tilde{\xi^-}| > \frac{|\tilde{\xi}|}{2},\ |\tilde{\xi^+}| > \frac{|\tilde{\xi}|}{2},\ |\tilde{\xi^+}| > \rho \right\}, \\
A_5 &= \left\{ \sigma \in \mathcal{S}^2 : |\tilde{\xi^-}| \leq \rho, |\tilde{\xi^+}| \leq \frac{|\tilde{\xi}|}{2} \right\}.
\end{aligned}$$

Let us estimate now

$$|Q_+(\hat{g}_0, \hat{g}_0)(\eta\xi)| \leq \frac{1}{4\pi} \sum_{i=0}^{5} \int_{A_i} |\hat{g}_0(\tilde{\xi^+})\hat{g}_0(\tilde{\xi^-})| \mathrm{d}\sigma$$

by estimating each integral separately. For $\sigma \in A_0$, thanks to (40) we have:

$$\begin{aligned}
|\hat{g}_0(\tilde{\xi^+})\hat{g}_0(\tilde{\xi^-})| &\leq \frac{1}{\left(1 + \beta|\tilde{\xi^+}|\right)^{\nu'}} \frac{1}{\left(1 + \beta|\tilde{\xi^-}|\right)^{\nu'}} \leq \frac{1}{\left(1 + \beta(|\tilde{\xi^+}| + |\tilde{\xi^-}|)\right)^{\nu'}} \\
&\leq \frac{1}{(1 + \beta|\xi|)^{\nu'}}.
\end{aligned}$$

Hence

$$\int_{A_0} |\hat{g}_0(\tilde{\xi^+})\hat{g}_0(\tilde{\xi^-})| \mathrm{d}\sigma \leq \frac{1}{(1 + \beta|\xi|)^{\nu'}} |A_0| \tag{42}$$

where we have denoted $|A_0|$ the Lebesgue measure of A_0. Now, let us suppose $\sigma \in A_1$. Thanks to (41) and to $|\tilde{\xi^-}| \geq \frac{|\tilde{\xi}|}{2}$ (which holds true since $|\tilde{\xi^-}| \geq |\tilde{\xi^+}|$), we get

$$\begin{aligned}
|\hat{g}_0(\tilde{\xi^+})\hat{g}_0(\tilde{\xi^-})| &\leq \frac{1}{1 + k|\tilde{\xi^+}|^2} \ \frac{1}{\left(1 + \beta|\tilde{\xi^-}|\right)^{\nu'}} \leq \frac{1}{1 + k\left(\frac{1-e}{2}\right)^2 |\xi|^2} \ \frac{1}{\left(1 + \frac{\beta}{2}|\tilde{\xi}|\right)^{\nu'}} \\
&\leq \frac{1}{1 + k\left(\frac{1-e}{2}\right)^2 |\xi|^2} \ \frac{1}{\left(1 + \frac{\beta}{2}|\xi|\right)^{\nu'}}.
\end{aligned}$$

By choosing $\nu' > 0$ small enough we can show that

$$\frac{1}{1 + k\left(\frac{1-e}{2}\right)^2 |\xi|^2} \ \frac{1}{\left(1 + \frac{\beta}{2}|\xi|\right)^{\nu'}} \leq \frac{1}{(1 + \beta|\xi|)^{\nu'}}.$$

Indeed, since $\frac{1+x}{1+\frac{1}{2}x} \leq 2$ for any $x \geq 0$ we obtain

$$\left(\frac{1 + \beta|\xi|}{1 + \frac{\beta}{2}|\xi|}\right)^{\nu'} \ \frac{1}{1 + k\left(\frac{1-e}{2}\right)^2 |\xi|^2} \leq 2^{\nu'} \frac{1}{1 + k\left(\frac{1-e}{2}\right)^2 \rho^2}.$$

Finally the last term is smaller than 1 for $\nu' \le \log_2(1 + k\left(\frac{1-e}{2}\right)^2 \rho^2)$. So

$$\int_{A_1} |\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)| \mathrm{d}\sigma \le \frac{1}{(1+\beta|\xi|)^{\nu'}} |A_1|. \tag{43}$$

Let us come to $\sigma \in A_2$. We get

$$\int_{A_2} |\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)| \mathrm{d}\sigma \le \frac{1}{\left(1+\beta|\tilde{\xi}^+|\right)^{\nu'}} \int_{A_2 \cap \{|\tilde{\xi}^+| > \rho\}} |\hat{g}_0(\tilde{\xi}^-)| \, \mathrm{d}\sigma$$
$$+ \frac{1}{1+k|\tilde{\xi}^+|^2} \int_{A_2 \cap \{|\tilde{\xi}^+| \le \rho\}} |\hat{g}_0(\tilde{\xi}^-)| \, \mathrm{d}\sigma. \tag{44}$$

Since $|\tilde{\xi}^+|^2 > \rho^2 \left(\frac{1-e}{2}\right)^2$ as recalled in (41), we have for ν' small enough

$$\frac{1}{1+k|\tilde{\xi}^+|^2} \le \frac{1}{\left(1+\beta|\tilde{\xi}^+|\right)^{\nu'}}$$

and so we can write

$$\int_{A_2} |\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)| \mathrm{d}\sigma \le \frac{1}{\left(1+\beta|\tilde{\xi}^+|\right)^{\nu'}} \int_{A_2} |\hat{g}_0(\tilde{\xi}^-)| \, \mathrm{d}\sigma \le \frac{1}{\left(1+\frac{\beta}{2}|\xi|\right)^{\nu'}} \int_{A_2} \frac{1}{1+k|\tilde{\xi}^-|^2} \mathrm{d}\sigma.$$

We parameterize, as it is usually done, the sphere S^2 centered at the collision center between two particles with z-axis defined by ξ (or $\tilde{\xi}$, which is just a multiple) and by denoting $\vartheta \in [0,\pi]$ the longitude and $\varphi \in [0, 2\pi)$ the latitude of the point on the sphere corresponding to the vector σ. We have

$$|\tilde{\xi}^+|^2 = \frac{10+2e^2-4e}{16} |\tilde{\xi}|^2 + \frac{(3-e)(1+e)}{8} |\tilde{\xi}|^2 \cos\vartheta$$
$$|\tilde{\xi}^-|^2 = 2\left(\frac{1+e}{4}\right)^2 |\tilde{\xi}|^2 (1-\cos\vartheta).$$

We can therefore write

$$A_2 = \left\{ \varphi \in [0,2\pi),\ \vartheta \in [0,\pi] \ : \ (1-\cos\vartheta) \le \min\left(\frac{\rho^2}{2\left(\frac{1+e}{4}\right)^2 |\tilde{\xi}|^2}, \ \frac{6}{(3-e)(1+e)}, \frac{1}{8\left(\frac{1+e}{4}\right)^2} \right) \right\}.$$

But $\frac{6}{(3-e)(1+e)} \ge \frac{1}{8\left(\frac{1+e}{4}\right)^2}$ for all $e > 0$ (since $|\tilde{\xi}^-| \le \frac{|\tilde{\xi}|}{2}$ implies indeed $|\tilde{\xi}^+| > \frac{|\tilde{\xi}|}{2}$), so in fact

$$A_2 = \left\{ \varphi \in [0,2\pi),\ \vartheta \in [0,\pi] \ : \ (1-\cos\vartheta) \le \min\left(\frac{\rho^2}{2\left(\frac{1+e}{4}\right)^2 |\tilde{\xi}|^2}, \ \frac{1}{8\left(\frac{1+e}{4}\right)^2} \right) \right\}.$$

Let us suppose first $\tilde{\xi}$ such that

$$|\tilde{\xi}|^2 \ge 4\rho^2.$$

In this case, we have

$$\int_{A_2} \frac{1}{1+k|\tilde{\xi}^-|^2} \mathrm{d}\sigma = \int_{\varphi\in[0,2\pi)} \int_{(1-\cos\vartheta) \le \frac{\rho^2}{2\left(\frac{1+e}{4}\right)^2 |\tilde{\xi}|^2}} \frac{1}{1+2k\left(\frac{1+e}{4}\right)^2 |\tilde{\xi}|^2 (1-\cos\vartheta)} \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi$$
$$= \frac{2\pi}{2\left(\frac{1+e}{4}\right)^2 |\tilde{\xi}|^2} \int_0^{\rho^2} \frac{\mathrm{d}x}{1+kx} = \frac{2\pi\rho^2}{2\left(\frac{1+e}{4}\right)^2 |\tilde{\xi}|^2} \frac{\log\left(1+k\rho^2\right)}{k\rho^2} = \frac{\log\left(1+k\rho^2\right)}{k\rho^2} |A_2|.$$

If on the contrary

$$|\tilde{\xi}|^2 < 4\rho^2$$

then, we have

$$\int_{A_2} \frac{1}{1+k|\tilde{\xi}^-|^2}\mathrm{d}\sigma = \int_{\varphi\in[0,2\pi)} \int_{(1-\cos\vartheta)\le \frac{1}{8\left(\frac{1+e}{4}\right)^2}} \frac{1}{1+2k\left(\frac{1+e}{4}\right)^2|\tilde{\xi}|^2(1-\cos\vartheta)} \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi$$

$$= \frac{2\pi}{2\left(\frac{1+e}{4}\right)^2|\tilde{\xi}|^2}\int_0^{\frac{|\tilde{\xi}|^2}{4}} \frac{\mathrm{d}x}{1+kx} = \frac{2\pi}{2\left(\frac{1+e}{4}\right)^2|\tilde{\xi}|^2}\frac{\log\left(1+k\frac{|\tilde{\xi}|^2}{4}\right)}{k}$$

$$= \frac{2\pi}{8\left(\frac{1+e}{4}\right)^2}\frac{\log\left(1+k\frac{|\tilde{\xi}|^2}{4}\right)}{k\frac{|\tilde{\xi}|^2}{4}} = \frac{\log\left(1+k\frac{|\tilde{\xi}|^2}{4}\right)}{k\frac{|\tilde{\xi}|^2}{4}}|A_2|.$$

In both cases we get

$$\int_{A_2} \frac{1}{1+k|\tilde{\xi}^-|^2}\mathrm{d}\sigma \le \sup_{t\ge\frac{\rho^2}{4}} \frac{\log(1+kt)}{kt}|A_2|$$

and so

$$\int_{A_2} |\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)|\mathrm{d}\sigma \le \frac{1}{\left(1+\frac{\beta}{2}|\xi|\right)^{\nu'}} \sup_{t\ge\frac{\rho^2}{4}} \frac{\log(1+kt)}{kt}|A_2|.$$

By choosing $\nu' > 0$ small enough we can show that

$$\frac{1}{\left(1+\frac{\beta}{2}|\xi|\right)^{\nu'}} \sup_{t\ge\frac{\rho^2}{4}} \frac{\log(1+kt)}{kt} \le \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

Indeed,

$$\left(\frac{1+\beta|\xi|}{1+\frac{\beta}{2}|\xi|}\right)^{\nu'} \sup_{t\ge\frac{\rho^2}{4}} \frac{\log(1+kt)}{kt} \le 2^{\nu'} \sup_{t\ge\frac{\rho^2}{4}} \frac{\log(1+kt)}{kt} \le 1$$

as soon as

$$\nu' \le \log_2\left(\inf_{t\ge\frac{\rho^2}{4}} \frac{kt}{\log(1+kt)}\right).$$

Therefore

$$\int_{A_2} |\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)|\mathrm{d}\sigma \le \frac{1}{(1+\beta|\xi|)^{\nu'}}\ |A_2|. \tag{45}$$

Let us suppose now $\sigma \in A_3$. We have

$$|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)| \le \frac{1}{1+k|\tilde{\xi}^+|^2}\ \frac{1}{1+k|\tilde{\xi}^-|^2} \le \frac{1}{1+k\left(|\tilde{\xi}^+|^2+|\tilde{\xi}^-|^2\right)}$$

$$\le \frac{1}{1+k\frac{|\xi|^2}{2}} \le \frac{1}{(1+\beta|\xi|)^{\nu'}}$$

for $\nu' > 0$ small enough. So

$$\int_{A_3} |\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)|\mathrm{d}\sigma \le \frac{1}{(1+\beta|\xi|)^{\nu'}}\ |A_3|. \tag{46}$$

For $\sigma \in A_4$ we have

$$|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)| \le \frac{1}{\left(1+\beta|\tilde{\xi}^+|\right)^{\nu'}}\ \frac{1}{1+k|\tilde{\xi}^-|^2} \le \frac{1}{\left(1+\frac{\beta}{2}|\xi|\right)^{\nu'}}\ \frac{1}{1+k\frac{|\xi|^2}{4}} \le \frac{1}{(1+\beta|\xi|)^{\nu'}}$$

for $\nu' > 0$ small enough. So

$$\int_{A_4} |\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)| \mathrm{d}\sigma \le \frac{1}{(1+\beta|\xi|)^{\nu'}} \; |A_4|. \tag{47}$$

Let us come to $\sigma \in A_5$. It is worth noticing that

$$A_5 \subset \left\{ |\tilde{\xi}^-| > \frac{|\tilde{\xi}|}{2}, \; |\tilde{\xi}^+| \le \rho \right\}$$

and so

$$|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)| \le \frac{1}{1+k|\tilde{\xi}^+|^2} \; \frac{1}{1+k|\tilde{\xi}^-|^2} \le \frac{1}{1+k\frac{|\xi|^2}{4}} \le \frac{1}{(1+\beta|\xi|)^{\nu'}}$$

for $\nu' > 0$ small enough. So

$$\int_{A_5} |\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)| \mathrm{d}\sigma \le \frac{1}{(1+\beta|\xi|)^{\nu'}} \; |A_5|. \tag{48}$$

Grouping (42), (43), (45), (46), (47) and (48) we get for ν' small enough (depending on ρ and k fixed once and for all in the first part of this proof): for $|\xi| > \rho$ and $\eta > 1$

$$|Q_+(\hat{g}_0, \hat{g}_0)(\eta\xi)| \le \frac{1}{4\pi}|S^2| \frac{1}{(1+\beta|\xi|)^{\nu'}} = \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

Coming back to $\hat{\varphi}_1^N(\xi)$ we obtain for N large enough and $|\xi| > \rho$:

$$|\hat{\varphi}_1^N(\xi)| \le \frac{1}{\Delta t} \int_{\eta=1}^{+\infty} \left(E\Delta t \, \frac{1}{(1+\beta|\xi|)^{\nu'}} + (1 - E\Delta t) \frac{1}{(1+\beta|\xi|)^{\nu'}} \right) \frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}} = \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

□

Remark 16

The bounds in (39) *are also equivalent to*

$$|\hat{g}(\xi, t)| \le \frac{C}{(1+\kappa|\xi|)^{\mu}}, \quad \xi \in \mathbb{R}^3, \quad t \ge 0.$$

5 Propagation of regularity and strong convergence

In this section, we are going to prove Theorem 2 on the strong L^1 convergence of the scaled solution $g(t)$ to the stationary state g_∞.

Theorem 2

Assume $e \in [0, 1)$. Let $g(t)$ be the weak solution of the equation (11)*, corresponding to the initial probability density g_0 with zero mean velocity, $\int_{\mathbb{R}^3} |v|^2 \, g_0(v) \, \mathrm{d}v = 3$ and satisfying $\int_{\mathbb{R}^3} |v|^4 \, g_0(v) \, \mathrm{d}v < +\infty$. Let us suppose moreover $g_0 \in H^\eta(\mathbb{R}^3)$ for some $\eta > 0$ large enough, $\sqrt{g_0} \in \dot{H}^\nu(\mathbb{R}^3)$ for some $\nu > 0$. Then $g(t)$ converges strongly in L^1 with an exponential rate towards the stationary solution g_∞, i.e., there exist positive constants C and γ explicitly computable such that*

$$\|g(t) - g_\infty\|_{L^1(\mathbb{R}^3)} \le Ce^{-\gamma t}, \quad t \ge 0.$$

Let us begin by the following lemma (cfr. [LT95]), which makes a link between the hypothesis $\sqrt{g_0} \in \dot{H}^\nu(\mathbb{R}^3)$ and the Fourier decreasing appearing in Theorem 1. For a proof see also [FPTT].

Lemma 17
If g_0 is a probability density such that $\sqrt{g_0} \in \dot{H}^\nu(\mathbb{R}^3)$ for some $\nu > 0$, then g_0 satisfies

$$|\hat{g}_0(\xi)| \leq \frac{C}{(1+\beta|\xi|)^\nu}, \quad \xi \in \mathbb{R}^3$$

for positive constants C and β.

In order to prove Theorem 2, we have to convert the weak convergence in the Fourier distance d_2 of the solution $g(t)$ to the stationary state g_∞ (Theorem 7) into a L^1 convergence exploiting the propagation of regularity established in Theorem 1. The main ingredient at this point is the uniform boundedness of the solution in Sobolev norms. We only reproduce the statement, since its proof is exactely the same as in [CCC], Lemma 3.4 and Theorem 3.6. In that paper though, the decreasing of the solution (39) was proved only for small inelasticity $e \simeq 1$ exploiting a control of the growth of the Fisher information (cfr. [CCC], Theorem 1.2).

Theorem 18
Assume $e \in [0,1)$. Let $g(t)$ be the weak solution of the equation (11), corresponding to the initial probability density g_0 with zero mean velocity, $\int_{\mathbb{R}^3} |v|^2\, g_0(v)\, \mathrm{d}v = 3$ and satisfying $\int_{\mathbb{R}^3} |v|^4\, g_0(v)\, \mathrm{d}v < +\infty$. Let us suppose moreover $g_0 \in H^\eta(\mathbb{R}^3)$ for some $\eta > 0$, $\sqrt{g_0} \in \dot{H}^\nu(\mathbb{R}^3)$ for some $\nu > 0$. Then $g(t)$ is uniformly bounded in $\dot{H}^\eta(\mathbb{R}^3)$.

Proof of Theorem 2. The proof of Theorem 2 is completed as soon as one recalls the following interpolation bounds (see Theorems 4.1 and 4.2 in [CGT99]):

- there exists a positive constant C such that

$$\|h\|_{L^1} \leq C \||v|^2 h\|_{L^1}^{\frac{3}{7}} \|h\|_{L^2}^{\frac{4}{7}};$$

- for any $s \geq 0$ there exist positive constants M, N, β and γ such that

$$\|h\|_{H^s} \leq C \left(\sup_{\xi \neq 0} \frac{|\hat{h}(\xi)|}{|\xi|^2} \right)^\beta \left(\|h\|_{H^M} + \|h\|_{H^N} \right)^\gamma .$$

So, letting $h = g(t) - g_\infty$, $s = 0$ and keeping in mind the regularity of the stationary state as recalled in Theorem 10, we get

$$\|g(t) - g_\infty\|_{L^1} \leq C \left(\||v|^2 g(t)\|_{L^1} + \||v|^2 g_\infty\|_{L^1} \right)^{\tilde{\alpha}} \times \\ d_2(g(t), g_\infty)^{\tilde{\beta}} \left(\|g(t)\|_{H^M} + \|g(t)\|_{H^N} + \|g_\infty\|_{H^M} + \|g_\infty\|_{H^N} \right)^{\tilde{\gamma}}$$

for suitable exponents $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$.

□

6 Appendix

Proof of Proposition 13: The estimates on the moments up to the first order (32) are easily obtained by a recursive procedure through expression (30) as well as the vanishing of the non isotropic second moments (33).

Let us come to the diagonal second moments (34). After some calculations (as in Proposition 22) we get the following first order linear difference system

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2\, \varphi_{j+1}^N \,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2\, \varphi_{j+1}^N \,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2\, \varphi_{j+1}^N \,\mathrm{d}v \end{bmatrix} = \frac{1}{1-2\Delta t}\ \left(\mathrm{Id} + \Delta t\ A\right) \begin{bmatrix} \int_{\mathbb{R}^3} v_1^2\, \varphi_{j}^N \,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2\, \varphi_{j}^N \,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2\, \varphi_{j}^N \,\mathrm{d}v \end{bmatrix}$$

where

$$A = \begin{bmatrix} -\frac{4}{3}\frac{2-e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} \\ \frac{1}{3}\frac{1+e}{1-e} & -\frac{4}{3}\frac{2-e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} \\ \frac{1}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} & -\frac{4}{3}\frac{2-e}{1-e} \end{bmatrix}.$$

Since the eigenvalues of the matrix A are -2 simple, with eigenspace $\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$ and $-\frac{3-e}{1-e}$ double with eigenspace $\left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$, the matrix $\frac{1}{1-2\Delta t}\ (\mathrm{Id} + \Delta t\ A)$ has eigenvalues 1 simple and $\frac{1}{1-2\Delta t}\left(1 - \Delta t\,\frac{3-e}{1-e}\right)$ double, with the same corresponding eigenspaces. Since for N large enough depending on $e \in [0,1)$ we have $0 < \frac{1}{1-2\Delta t}\left(1 - \Delta t\,\frac{3-e}{1-e}\right) < 1$, we get

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2\, \varphi_{j}^N(v) \,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2\, \varphi_{j}^N(v) \,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2\, \varphi_{j}^N(v) \,\mathrm{d}v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \left(\frac{1 - \Delta t\,\frac{3-e}{1-e}}{1-2\Delta t} \right)^j + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \left(\frac{1 - \Delta t\,\frac{3-e}{1-e}}{1-2\Delta t} \right)^j$$

for C_1, C_2 real constants to be determined by the initial conditions. As a consequence of (34) we get for all N, for all $j = 0, \ldots, N$

$$\int_{\mathbb{R}^3} |v|^2 \varphi_j^N(v)\,\mathrm{d}v = 3.$$

Let us come to the fourth moment (35). Again through expression (30) we get

$$\begin{aligned} &\int_{\mathbb{R}^3} |v|^4\, \varphi_{j+1}^N(v)\,\mathrm{d}v \\ &= \frac{1}{\Delta t} \int_1^{+\infty} \left(E\Delta t \int_{\mathbb{R}^3} \frac{|v|^4}{\eta^3} Q_+(\varphi_j^N, \varphi_j^N) \left(\frac{v}{\eta}\right) \mathrm{d}v + (1 - E\Delta t) \int_{\mathbb{R}^3} \frac{|v|^4}{\eta^3} \varphi_j^N \left(\frac{v}{\eta}\right) \mathrm{d}v \right) \frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}} \\ &= \frac{1}{\Delta t} \frac{1}{\frac{1}{\Delta t} - 4} \left(E\Delta t \int_{\mathbb{R}^3} |v|^4\, Q_+(\varphi_j^N, \varphi_j^N)(v)\,\mathrm{d}v + (1 - E\Delta t) \int_{\mathbb{R}^3} |v|^4\, \varphi_j^N(v)\,\mathrm{d}v \right). \end{aligned}$$

We denote

$$m_{4,j} = \int_{\mathbb{R}^3} |v|^4\, \varphi_j^N(v)\,\mathrm{d}v.$$

Exploiting Lemma 5 in [BC07], we get that there exist λ, μ_1 and μ_2 positive constants depending only on e such that

$$\begin{aligned} \int_{\mathbb{R}^3} |v|^4\, Q(\varphi_j^N, \varphi_j^N)(v)\,\mathrm{d}v &= -\lambda m_{4,j} + 9\mu_1 + \mu_2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v \cdot w)^2 \varphi_j^N(v) \varphi_j^N(w)\,\mathrm{d}v\,\mathrm{d}w \\ &\le -\lambda m_{4,j} + C \end{aligned}$$

for a suitable positive constant C depending on g_0. Remembering that

$$\int_{\mathbb{R}^3} |v|^4\, Q_+(\varphi_j^N, \varphi_j^N)(v)\,\mathrm{d}v = \int_{\mathbb{R}^3} |v|^4\, Q(\varphi_j^N, \varphi_j^N)(v)\,\mathrm{d}v + \int_{\mathbb{R}^3} |v|^4\, \varphi_j^N(v)\,\mathrm{d}v,$$

we end up with the following recursive relation

$$\begin{aligned} m_{4,j+1} &\leq \frac{1}{1-4\,\Delta t}\left(E\Delta t((1-\lambda)m_{4,j}+C)+(1-E\Delta t)m_{4,j}\right) \\ &= \frac{1}{1-4\,\Delta t}\left((1-\lambda E\Delta t)m_{4,j}+CE\Delta t\right). \end{aligned}$$

By a Taylor expansion we get for $\Delta t \to 0$:

$$\begin{aligned} m_{4,j+1} &\leq (1-\lambda E\Delta t)\,(1+4\,\Delta t+o(\Delta t))\,m_{4,j}+CE\Delta t\,(1+4\,\Delta t+o(\Delta t)) \\ &= (1-(\lambda E-4)\,\Delta t+o(\Delta t))\,m_{4,j}+CE\Delta t+o(\Delta t). \end{aligned}$$

Since $\lambda > \frac{4}{E}$ for $e \in [0,1)$ ([BC07]), we get for N large enough

$$m_{4,j+1} \leq \left(1-\frac{\lambda E-4}{2}\Delta t\right)m_{4,j}+2CE\Delta t.$$

Thus

$$\begin{aligned} m_{4,j+1} &\leq \left(1-\frac{\lambda E-4}{2}\Delta t\right)^{j+1} m_{4,0}+2CE\Delta t\sum_{k=0}^{j}\left(1-\frac{\lambda E-4}{2}\Delta t\right)^{k} \\ &\leq m_{4,0}+\frac{2CE\Delta t}{\frac{\lambda E-4}{2}\Delta t} = m_{4,0}+\frac{4CE}{\lambda E-4} \end{aligned}$$

and so $m_{4,j}$ is uniformly bounded (for all N large enough and for all $j = 0, \ldots, N$) if initially bounded.

□

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