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LAGRANGIAN NUMERICAL APPROXIMATIONS TO ONE-DIMENSIONAL CONVOLUTION-DIFFUSION EQUATIONS*

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Abstract. This work focuses on the numerical analysis of 1D nonlinear diffusion equations involving a convolution product. First, homogeneous friction equations are considered. Algorithms follow recent ideas on mass transportation methods and lead to simple schemes which can be proved to be stable, to decrease entropy and to converge toward the unique solution of the continuous problem. In particular, for the first time, homogeneous cooling states are displayed numerically. Further, we present results on the more delicate fourth-order thin-films equation for which a nonnegativity-preserving scheme is derived. Dead core phenomenon is presented for the Hele–Shaw cell.

Key words. granular flows, friction equations, Lagrangian approximation, Wasserstein metric, homogeneous cooling states, Hele–Shaw cell

AMS subject classifications. Primary, 65M06, 65M12; Secondary, 76T25, 76X05, 76D08

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1. Introduction.

One objective for the present paper is to study some constructive approximations for one-dimensional dissipative Boltzmann equations modelling granular flows; they read

$$\partial_t f + v\partial_x f = Q(f,f) + R, \qquad t,x,v \in \mathbb{R}_*^+ \times \mathbb{R} \times \mathbb{R},$$

for kinetic densities $f(t,x,v) \geq 0$. Here, $Q(f,f)$ stands for the term modeling the inelastic collisions [1, 2, 27, 30] whereas R renders the effects of e.g. external heating processes, like Brownian agitation which leads to a diffusion term; see [11] for a large survey on granular materials. First, it is customary now to decouple the transport and collision steps relying on a time-splitting technique, see e.g. [28, 26, 18]; thus it makes sense to restrict ourselves to homogeneous densities $f(t,v)$. We shall therefore concentrate on the following Cauchy problem for $t,v \in \mathbb{R}_*^+ \times \mathbb{R}$,

$$\partial_t f(t,v) = \partial_v \left\{ f(t,v) \int_{\mathbb{R}} \mathcal{I}'(v-\omega) f(t,\omega).d\omega \right\} + \mu\partial_v(vf(t,v)) + \sigma\partial_{vv} f(t,v), \tag{1.1}$$

where $\mathcal{I}$ stands for an **even convex** interaction potential. Generally, it is taken as $\mathcal{I}(\xi) = |\xi|^{\gamma+2}/\gamma + 2$, $\gamma \geq 0$. Equation (1.1) rewrites as a convolution Fokker-Planck:

$$\partial_t f = \partial_v \left\{ f\partial_v \Big(f *_v \mathcal{I}(v) \Big) + \sigma\partial_v f \right\} + \mu\partial_v(vf).$$

Several particular cases of (1.1) are of interest; the first one corresponds to setting $\sigma = \mu = 0$ and models dissipative flows of granular media. This model was proposed by McNamara and Young [27] and then studied from a mathematical viewpoint in [1, 2, 7, 13, 14]. Roughly speaking, it describes a huge number of particles that undergo

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inelastic collisions, see [30, 26], which lead to a monotonic decay of the temperature. Steady-states display a very elementary structure, namely a Dirac mass concentrated onto the null velocity; however, a rescaling process allows to derive a particular class of intermediate asymptotics called *homogeneous cooling states*, [2, 24, 30]. These ones have been singled out mainly because they are minimizers of a "free energy functional", (which is to be intensively used in section 3.1)

$$\mathcal{J}(f) := \int_{\mathbb{R}^2} f(v)f(w)\mathcal{I}(v-w).dv.dw - \int_{\mathbb{R}} f(v)v^2.dv, \tag{1.2}$$

among probability measures on $\mathbb{R}$. However, despite stability could be hoped for, it has recently been shown that these states are too poor to play a role of similar importance than Barenblatt-Pattle source solutions for porous media equations; detailed proofs are available in [10]. It will be one of our objectives to display numerically such homogeneous cooling states in section 3.2; to the authors' knowledge, this hasn't been performed before.

Another objective is to develop on the numerical analysis of the so–called lubrication approximation of the Navier–Stokes equations for the movement of thin films driven by surface tension, see [25]; such a derivation leads to a fourth-order Cauchy problem,

$$\partial_t u + \partial_x(a(u)\partial_{xxx}u) = 0, \qquad u(t=0,.) = u_0 \geq 0, \quad t,x \in \mathbb{R}_*^+ \times \mathbb{R}, \tag{1.3}$$

The theory for this equation is still sparse, see [3, 4, 5, 6, 16]; especially, little is known about uniqueness of strong, i.e. $H^2(\mathbb{R})$ nonnegative solutions. A special and somewhat more tractable case of (1.3) corresponds to $a(u) = u$ and is generally referred to as the Hele–Shaw cell: we shall focus on it within section 5 where certain ideas proposed in [29] will be expanded. Let us first explain why this problem can fall inside the class of "convolutive diffusion equations". Our main trick is to consider, for a small number $0 < \varepsilon \ll 1$, a smoothered equation,

$$\partial_t u^\varepsilon + \partial_x\Big(a(u^\varepsilon)\partial_{xxx}u^\varepsilon *_x \Gamma_\varepsilon(x)\Big) = \partial_t u^\varepsilon + \partial_x\Big(a(u^\varepsilon)u^\varepsilon *_x \Gamma_\varepsilon'''(x)\Big) = 0,$$

where Γ_ε is a standard Friedrichs' mollifier. At this point, one can derive an approximate problem for (1.3) written in the convenient form for $t,x \in \mathbb{R}_*^+ \times \mathbb{R}$,

$$\partial_t u^\varepsilon + \partial_x\Big(a(u^\varepsilon)\partial_x(u^\varepsilon *_x \Gamma_\varepsilon''(x))\Big) = 0, \quad u(t=0,.) = u_0 \geq 0. \tag{1.4}$$

However, even if sharing some features with the aforementioned models of granular flows (especially for $a(u) = u$), a major difficulty arises from the fact that $-\Gamma_\varepsilon''$ will never be a convex function; it is nonetheless possible to obtain a numerical process able to preserve nonnegativity of solutions, a property commonly considered important in practice, [34, 21].

Going a bit deeper into the contents of this work, we aim at showing that modern Wasserstein techniques can be successful at producing simple numerical schemes for (1.1) which are positivity preserving, stable, and consistent with the continuous problem. Indeed, it has been shown in [15], and we shall briefly recall in section 2.1 how one can pass by means of a Lagrangian-type change of variables from (1.1) to the much more tractable equation holding for $t, \varrho \in \mathbb{R}_*^+ \times (0,1)$,

$$\partial_t V(t,\varrho) + \sigma\partial_\varrho\left(\frac{1}{\partial_\varrho V(t,\varrho)}\right) + \mu V(t,\varrho) + \int_0^1 \mathcal{I}'\Big(V(t,\varrho) - V(t,\varrho')\Big).d\varrho' = 0, \tag{1.5}$$

which reduces, in case $\sigma = \mu = 0$ to some sort of integro-differential equation. This trick is reminiscent of a previous work, [19], and allows to circumvent the issues which are connected to numerical approximation of probabillity measures with finite-differences, see [9, 17, 26] for more in this direction. One feature of paramount importance for (1.5) is the easiness in proving contractivity in any p-Wasserstein metric for various numerical approximations, just relying on the standard Jensen's inequality. It is not difficult also to convert such an estimate into a BV-bound thus showing compactness for a sequence of approximations for the V variable. Such a program is to be realized in sections 2.2–3. The BV framework (functions of bounded variation) corresponds exactly to measure solutions for (1.1) hence provides a convenient approach to the problem. section 4 is concerned by the modifications brought by the immersion of such granular media inside a thermal fluid. Following [12], we are most of all interested in classical solutions even if we shall keep on using numerical schemes based on the "reciprocal equation" (1.5). Convergence is shown in section 4.1 whereas the "tails" of the solutions for large $|v|$ (see [2, 7, 8, 9, 13]) are to be studied numerically in section 4.2. Section 5 is entirely devoted to the study of the Hele–Shaw cell's equation for which we can propose a nonnegativity-preserving scheme based on similar ideas. An energy estimate is given in section 5.2 ad some numerical results are shown in section 5.3; an extension to mobilities $a(u) \neq u$ is briefly discussed in Remark 2 and in more detail within section 6. At last, concluding remarks are given in section 7 and the Appendices A and B contain the proofs of two important contraction estimates. Appendix C deals with some special features of the Fourier scheme (6.2) for equation (1.3) with $a(u) = u$.

2. Lagrangian approximation of 1D friction equations; $\sigma = \mu = 0$. The object under the scope is now:

$$\partial_t f(t,v) = \partial_v \left\{ f(t,v) \int_{\mathbb{R}} \mathcal{I}'(v-\omega) f(t,\omega).d\omega \right\}; \quad t, v \in \mathbb{R}_*^+ \times \mathbb{R}, \tag{2.1}$$

as proposed in [27] from the Boltzmann equation in a so–called quasi-elastic limit in case $\mathcal{I}(\xi) = |\xi|^3/3$ i.e. $\gamma = 1$. We use the notation $\mathbb{R}_*^+ = (0, +\infty)$ throughout the paper. Since (2.1) is known to admit Dirac-type similarity solutions, it is convenient to introduce the following functional framework for $p \geq 1$,

$$\mathcal{M}_p(\mathbb{R}) = \left\{ v \mapsto f(.,v) \text{ probability measure such that } \int_{\mathbb{R}} |v|^p f(.,v).dv < +\infty \right\},$$

equipped with the topology of weak convergence of measures. A distinguished class of metrics on $\mathcal{M}_p(\mathbb{R})$ is given by the so-called Wasserstein (or Monge–Kantorovitch) distance, [19, 24, 32], which can be written as the usual L^p distance of the *reciprocal mappings* (to be defined in section 2.1 below),

$$d_p(f,g) := \left(\int_0^1 |V(.,\varrho) - W(.,\varrho)|^p .d\varrho \right)^{1/p}, \qquad 1 \leq p < +\infty.$$

By a weak solution to the Cauchy problem for (2.1), we mean any $f \in C^1(\mathbb{R}_*^+; \mathcal{M}_p(\mathbb{R}))$ satisfying for all $\varphi \in C^1(\mathbb{R})$ and $t > 0$,

$$\begin{array}{rl} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v) f(t,v).dv & = -\int_{\mathbb{R}^2} \mathcal{I}'(v-w) \varphi'(v) f(t,v) f(t,w).dv.dw \\ & = \frac{1}{2} \int_{\mathbb{R}^2} \mathcal{I}'(v-w)(\varphi'(w) - \varphi'(v)) f(t,v) f(t,w).dv.dw, \end{array} \tag{2.2}$$

and $f(t,.) \to f(t=0,.) = f_0 \in \mathcal{M}_p(\mathbb{R})$ weakly as a measure as $t \to 0$. The second equality holds because $\mathcal{I}'$ is an odd function. Obviously, choosing $\varphi(v) = 1, v$ in (2.2) yields conservation of mass and momentum; thus it makes sense to restrict ourselves to centered probability measures. Further, inserting $\varphi(v) = v^2$ leads to,

$$\frac{d}{dt}\theta(t) := \frac{d}{dt}\int_{\mathbb{R}^2} v^2 f(t,v).dv = -\int_{\mathbb{R}^2} \mathcal{I}'(v-w)(v-w)f(t,v)f(t,w).dv.dw \leq 0.$$

Thus the temperature is meant to decrease monotonically in time (at most at an exponential rate, see [30] and Figure 3.1) because $\mathcal{I}'$ is monotone nondecreasing. We also deduce from [24] the following uniqueness theorem:

THEOREM 2.1. *Let $f_0 \in \mathcal{M}_p(\mathbb{R})$ for $p = 2$ and $\mathcal{I}$ be a smooth even convex function; then the weak solution to* (2.1) *is unique.*

The proof is forwarded to Appendix A.

2.1. The reciprocal mapping. From [10], the decay towards stable similarity solutions (homogeneous cooling states) can be expected to be slow and because of their singular structure, one cannot use high-order discretizations. This clearly constitutes a numerical difficulty we propose to overcome by means of the change of variable described as follows:

- Let us introduce the distribution function associated to the initial probability measure f_0,

$$\varrho_0(v) = \int_{-\infty}^{v} f_0(\omega).d\omega \in (0,1), \qquad \varrho_0 \in BV(\mathbb{R}),$$

which is obviously nondecreasing in the v variable. We can thus define a (nondecreasing) pseudo-inverse:

$$v_0 : \begin{array}{ccc} (0,1) & \to & \mathbb{R} \\ \bar{\varrho} & \mapsto & v_0(\bar{\varrho}) := \inf\{\omega \in \mathbb{R} \text{ such that } \varrho_0(\omega) = \bar{\varrho}\}. \end{array} \tag{2.3}$$

- For any given value $\bar{\varrho} \in (0,1)$, we can define the **reciprocal mapping**,

$$V : \begin{array}{ccc} \mathbb{R}^+ & \to & \mathbb{R} \\ t & \mapsto & V(t,\bar{\varrho}), \end{array}$$

by means of the implicit function theorem in case $\partial_v \varrho \neq 0$, such that

$$V(t=0,\bar{\varrho}) = v_0(\bar{\varrho}), \qquad \varrho(t, V(t,\bar{\varrho})) = \bar{\varrho}. \tag{2.4}$$

Integrating (2.1) in v,

$$\partial_t \varrho(t,v) = \partial_v \varrho(t,v) \int_{\mathbb{R}} \mathcal{I}'(v-\omega)\partial_v \varrho(t,\omega).d\omega, \qquad \partial_v \varrho(t,v) = f(t,v),$$

and differentiating the second equality of (2.4) with respect to time, one deduces easily the time evolution of $V(.,\varrho)$ (we drop the $\bar{.}$ for ease of reading), see also [15, 24]:

$$\partial_t V(t,\varrho) + \int_0^1 \mathcal{I}'\Big(V(t,\varrho) - V(t,\varrho')\Big).d\varrho' = 0, \qquad t > 0, \tag{2.5}$$

and $V(t=0,\varrho) = v_0(\varrho)$ for any $\varrho \in (0,1)$.

We define a Cartesian computational grid by means of the two positive parameters $\Delta t, \Delta v$. From now on, let v_k and t^n stand for $k\Delta v$ and $n\Delta t$ respectively. Therefore our numerical approach to (2.1) with convenient (unit mass, centered) initial data is built on computing the pseudo-inverse of ϱ_0, $V(t=0,.)$, evolving it in time by means of an *explicit* marching scheme for (2.5) in order to deduce the values of $\varrho(t, X(t,.)) \in [0,1]$ thanks to (2.4). Working on this pseudo-inverse $V(t,.)$ allows to bypass the usual problems in approximating Dirac measures, [9, 17]. This trick can be used mainly because equation (2.1) is *mass–preserving*. It will prove useful to introduce the **equivalent** definition of weak solution to (2.1) (see [24] p. 412):

DEFINITION 1. *Let $f_0 \in \mathcal{M}_p(\mathbb{R})$; $f(t,v) \in C^1(\mathbb{R}_*^+; \mathcal{M}_p(\mathbb{R}))$ is a weak solution to* (2.1), (2.5) *if it satisfies for all $\varphi \in C^1(\mathbb{R})$ and $t>0$,*

$$\begin{aligned}\tfrac{d}{dt}\int_0^1 \varphi(V(t,\varrho)).d\varrho &= -\int_{(0,1)^2} \mathcal{I}'(V(t,\varrho)-V(t,\varrho'))\varphi'(V(t,\varrho)).d\varrho.d\varrho' \\ &= -\int_{(0,1)^2} \mathcal{I}'(V(t,\varrho)-V(t,\varrho'))\tfrac{\varphi'(V(t,\varrho)-\varphi'(V(t,\varrho')}{2}.d\varrho.d\varrho'.\end{aligned}$$

And $f(t,.) \to f_0$ as $t \to 0$ weakly as a measure.

We now discretize the ϱ and t axes and define:

$$V_k^n \simeq V(t^n, \varrho_k); \qquad \varrho_k := \varrho_0(v_k),\ t^n = n\Delta t \text{ for } k \in \mathcal{K} \subset \mathbb{N},\ n \in \mathbb{N}. \tag{2.6}$$

A numerical scheme for (2.5) reads:

$$V_k^{n+1} = V_k^n - \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| \mathcal{I}'(V_k^n - V_{k'}^n), \tag{2.7}$$

where $|C_k| = \varrho_{k+\frac12} - \varrho_{k-\frac12}$ stands for the width of the control cell centered on ϱ_k with $\varrho_{k+\frac12} = \varrho_0(x_{k+\frac12})$. As ϱ_0 is at least of bounded variation (hence it makes sense to speak about left/right values at a given point), a convenient choice is given by linear interpolation, $\varrho_{k+\frac12} = \frac12(\varrho_k + \varrho_{k+1})$, which yields $|C_k| = \frac12\big(\varrho_{k+1} - \varrho_{k-1}\big)$. Since no divided differences show up in (2.7), there is no need for boundary conditions. The scheme (2.7) is conservative *by construction*. This furthermore yields, since $\mathcal{I}$ is an even function:

$$\forall n \in \mathbb{N}, \qquad \sum_k |C_k| V_k^n = \sum_k |C_k| V_k^0 \simeq \int_0^1 V(t=0,\varrho).d\varrho = \int_{\mathbb{R}} v.f_0(x).dv.$$

We stress that the ϱ_k's do **not** depend on time. In order to reconstruct $\tilde{\varrho}(t^n,.)$, an approximation of $\varrho(t,.)$ at a given time $t \simeq t^n$, one has to interpolate the family of numerical values ϱ_k, X_k^n, t^n since

$$\tilde{\varrho}(t^n, V_k^n) \stackrel{def}{=} \varrho_k \simeq \varrho(t^n, V_k^n),$$

$\partial_v \varrho$ being the weak solution to (2.1), up to the numerical truncation errors on V_k^n coming from the discretization (2.7). Then one deduces $f(t^n,.)$ by e.g. centered divided differences.

2.2. Wasserstein contraction estimate. The fundamental lemma reads:

LEMMA 2.2. *Let f_0, g_0 be two nonnegative initial data in $\mathcal{M}_p(\mathbb{R})$ for* (2.1) *and V, W their reciprocal mappings. Under the CFL restriction,*

$$\Delta t \sup_{V_k^0, W_{k'}^0} \Big(\mathcal{I}''\Big) \le 1, \qquad k, k' \in \mathcal{K}^2, \tag{2.8}$$

the scheme (2.7) *is contractive in any Wasserstein metric* d_p*; more precisely,*

$$\forall n \in \mathbb{N}, \qquad \sum_k |C_k||V_k^{n+1} - W_k^{n+1}|^p \leq \sum_k |C_k||V_k^n - W_k^n|^p, \qquad p \geq 1. \tag{2.9}$$

In practice, for (2.8), the supremum can be taken on the convex hull of the given set of data.

Proof. We readily substract the schemes (2.7) on each variable to derive

$$\forall n \in \mathbb{N}, \qquad V_k^{n+1} - W_k^{n+1} = V_k^n - W_k^n - \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| \Big(\mathcal{I}'(V_k^n - V_{k'}^n) - \mathcal{I}'(W_k^n - W_{k'}^n) \Big).$$

We apply the mean-value theorem for each k, k' to obtain,

$$\mathcal{I}'(V_k^n - V_{k'}^n) - \mathcal{I}'(W_k^n - W_{k'}^n) = (\mathcal{I}'')_{k,k'}^n \Big((V_k^n - W_k^n) - (V_{k'}^n - W_{k'}^n) \Big),$$

where convexity manifests itself through

$$0 \leq (\mathcal{I}'')_{k,k'}^n := \mathcal{I}'' \Big(\lambda (V_k^n - V_{k'}^n) + (1-\lambda)(W_k^n - W_{k'}^n) \Big),$$

for some $\lambda \in [0,1]$. Hence rearranging terms, one finds out that $V_k^{n+1} - W_k^{n+1}$ is a convex combination of each $V_{k'}^n - W_{k'}^n$, $k' \in \mathcal{K}$, under the CFL condition (2.8) since $\mathcal{I}$ is convex:

$$V_k^{n+1} - W_k^{n+1} = (V_k^n - W_k^n) \Big(1 - \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| (\mathcal{I}'')_{k,k'}^n \Big) + \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| (\mathcal{I}'')_{k,k'}^n (V_{k'}^n - W_{k'}^n)$$

The result (2.9) follows from summing on $k \in \mathcal{K}$ and invoking Jensen's inequality. ∎

This result is but a numerical equivalent of Theorem 1 in [24]. As an immediate consequence of Lemma 2.2, one obtains the L^p-stability of the method, just taking $W_k^0 \equiv 0$ (which implies $W_k^n \equiv 0$ since $\mathcal{I}'(0) = 0$):

Proposition 2.3. *Under the hypotheses of Lemma* 2.2*, there holds:*

$$\forall n \in \mathbb{N}, \qquad \sum_k |C_k| \Big(|V_k^{n+1}|^p - |V_k^n|^p \Big) \leq 0, \qquad p \geq 1.$$

This somehow highlights the solution's support decay proved in [24]. It shows in particular the constant decrease of the numerical solution's temperature since $\theta(t^n) \simeq \sum_k |C_k||V_k^n|^2$. Clearly more involved treatments of the integral term of (2.5) would yield an estimate of the same flavour. It would also be possible to consider higher-order Runge-Kutta methods in time, see e.g. [20].

Corollary 2.4. *Under the hypotheses of Lemma* 2.2*, if* f_0 *has compact support, Lipschitz time-regularity holds:*

$$\forall k, n \in \mathcal{K} \times \mathbb{N}, \qquad |V_k^{n+1} - V_k^n| \leq \Delta t \sup_{V_k^0} |\mathcal{I}'|, \tag{2.10}$$

Proof. From Proposition 2.3, the sequence V_k^n is bounded in $L^\infty(0,1)$. The estimate (2.10) follows from $\sum_{k \in \mathcal{K}} |C_k| = \|f_0\|_{L^1(\mathbb{R})} = 1$. ∎

2.3. Compactness and entropy consistency of the scheme. By analogy with (1.2), we can define an "entropy" functional for (2.1),

$$\forall t \geq 0, \qquad \mathcal{E}(t,f) = \int_{\mathbb{R}^2} \mathcal{I}(v-v')f(t,v)f(t,v').dv.dv',$$

which is known to decrease as times grow at least for certain potentials. It can be easily checked that (2.7) keeps this property; this is the purpose of the next result.

PROPOSITION 2.5. *Let* $\mathcal{E}^n = \sum_{k,k'} |C_k||C_{k'}|\mathcal{I}(V_k^n - V_{k'}^n)|$ *for* $\mathcal{I}(\xi) = |\xi|^{2+\gamma}/\gamma + 2$. *Under the hypotheses of Lemma* 2.2, *there holds:*

$$\forall n \in \mathbb{N}, \qquad \mathcal{E}^n \leq \Big(1 - \Delta t \inf_{V_k^0}(\mathcal{I}'')\Big)^{n(\gamma+2)} \mathcal{E}^0. \tag{2.11}$$

Proof. The proof proceeds as Lemma 2.2's; substracting and linearizing (2.7) with $W_k^n = V_\ell^n$ for some $\ell \in \mathcal{K}$ gives:

$$\begin{aligned} |V_k^{n+1} - V_\ell^{n+1}| &= |V_k^n - V_\ell^n - \Delta t \textstyle\sum_{k' \in \mathcal{K}} |C_{k'}| \Big(\mathcal{I}'(V_k^n - V_{k'}^n) - \mathcal{I}'(V_\ell^n - V_{k'}^n) \Big)| \\ &= |V_k^n - V_\ell^n - \Delta t \textstyle\sum_{k' \in \mathcal{K}} |C_{k'}| (\mathcal{I}'')_{k,\ell,k'}^n \Big((V_k^n - V_\ell^n) - (V_{k'}^n - V_{k'}^n) \Big)| \\ &= |V_k^n - V_\ell^n| \Big(1 - \Delta t \textstyle\sum_{k' \in \mathcal{K}} |C_{k'}| (\mathcal{I}'')_{k,\ell,k'}^n \Big), \end{aligned}$$

where $(\mathcal{I}'')_{k,\ell,k'}^n = \mathcal{I}''(\lambda V_k^n + (1-\lambda)V_\ell^n - V_{k'}^n)$ and from which we deduce that $|V_k^{n+1} - V_\ell^{n+1}| \leq |V_k^n - V_\ell^n| \big(1 - \Delta t \inf_{V_k^n}(\mathcal{I}'')\big)$ under the CFL condition (2.8). Summing leads to (2.11). Thus we are done. ∎

Of course, if one lets $\Delta t \to 0$, (2.11) provides the exponential decay of the entropy, see [24]. Lemma 2.2 implies also a bound on the total variation of the scheme as we show now.

COROLLARY 2.6. *Under the hypotheses of Lemma* 2.2, *there holds:*

$$\forall n \in \mathbb{N}, \qquad \sum_{k \geq 1} |V_k^n - V_{k-1}^n| \leq \Big(1 - \Delta t \inf_{V_k^0}(\mathcal{I}'')\Big)^n \sum_{k \geq 1} |V_k^0 - V_{k-1}^0|. \tag{2.12}$$

Moreover, the scheme (2.7) *is monotonicity-preserving.*

Proof. The proof is essentially the same than Proposition 2.5's; substracting and linearizing (2.7) with $\ell = k-1$ gives again $|V_k^{n+1} - V_{k-1}^{n+1}| \leq |V_k^n - V_{k-1}^n| \big(1 - \Delta t \inf_{V_k^n}(\mathcal{I}'')\big)$ using the CFL condition (2.8). This also implies monotonicity preservation. By summing for $k, k' \geq 1$, one obtains the decrease of the total variation which is initially bounded if f_0 has compact support. ∎

Remark 1. One clearly sees that the CFL (2.8) condition is indeed very mild in this case, since the values of Δt and $|C_k|$ are completely decorralated. In particular, no implicit time integrator is needed to ensure stability, in sharp contrast with [9].

The decrease of the total variation was to be expected; one can imagine considering $V(t=0,\varrho)$ as an initial value and $W(t=0,\varrho) := V(t=0,\varrho - h)$, $0 < h \ll 1$. Thus the Wasserstein contraction (2.9) ensures that the corresponding numerical approximations (2.7) will go closer to each other as times grow. For $p = 1$, this is exactly what is meant by (2.12). Eventually, and at least for $\mathcal{I}(\xi) = |\xi|^{2+\gamma}/\gamma + 2$, both V and W will get stuck onto a self-similar profile which depends only on the value of $\gamma \geq 0$. We shall return to this in section 3.

At this level, it is convenient to define a sequence of approximate "Lagrangian solutions" $V^{\Delta t} \in BV_{loc}(\mathbb{R}_*^+ \times (0,1))$ as for all $t, \varrho \in \mathbb{R}_*^+ \times (0,1)$,

$$V^{\Delta t}(t,\varrho) = V_k^n, \text{ if } |t-t^n| \leq \frac{\Delta t}{2} \text{ and } |\varrho - \varrho_k| \leq \frac{|C_k|}{2}. \tag{2.13}$$

Obviously, this implies the existence of another sequence $\varrho^{\Delta t}(t,v)$ defined as follows: for all $t, v \in \mathbb{R}_*^+ \times \mathbb{R}$,

$$\varrho^{\Delta t}(t,v) = \varrho_k, \text{ if } |t-t^n| \leq \frac{\Delta t}{2} \text{ and } \frac{V_{k-1}^n - V_k^n}{2} \leq v - V_k^n \leq \frac{V_{k+1}^n - V_k^n}{2}, \tag{2.14}$$

from which one can recover an approximation $f^{\Delta t}(t,v)$ to (2.1) by divided differences. As a consequence of the monotonicity of the sequence $(V_k^n)_{k\in\mathcal{K}}$ for all $n \in \mathbb{N}$, $f^{\Delta t} \geq 0$ as soon as $f_0 \geq 0$. We note in passing that the total variation of $\varrho^{\Delta t}$ in the v variable is constant by construction.

Thus it just remains to extract a subsequence still denoted $V^{\Delta t}$ converging strongly in $L^1_{loc}(\mathbb{R}_*^+ \times (0,1))$ by Helly's compactness principle. From (2.7), we can compute $\varphi(V_k^{n+1})$ for ant smooth function φ; then applying the mean-value theorem leads to:

$$\forall k,n, \qquad \frac{1}{\Delta t}\Big(\varphi(V_k^{n+1}) - \varphi(V_k^n)\Big) = -\sum_{k'\in\mathcal{K}} |C_{k'}| \mathcal{I}'(V_k^n - V_{k'}^n)\varphi'(V_k^n) + o(1).$$

It is now easy to deduce that $V^{\Delta t}$, (2.13) will satisfy the requirements of Definition 1 as $\Delta t, \Delta v \to 0$. Hence we deduce:

THEOREM 2.7. *Let $f_0 \in \mathcal{M}_p(\mathbb{R})$ with compact support and $\mathcal{I}$ be a convex smooth even function; under the CFL condition* (2.8)*, the sequence $V^{\Delta t}$ defined by* (2.13) *and the scheme* (2.7) *converges strongly in $L^1_{loc}(\mathbb{R}_*^+ \times (0,1))$ as $\Delta t, \Delta v \to 0$ towards the unique solution of* (2.1) *in the sense of Definition* 1*. Moreover, it decreases the entropy $\mathcal{E}$ in case $\mathcal{I}(\xi) = |\xi|^{2+\gamma}/(\gamma+2)$*

3. Study of the solution's large-time behaviour. From [10], we know that the so–called homogeneous cooling states cannot be hoped to play the role of paramount importance as it is the case for Barenblatt-Pattle similarity solutions for nonlinear diffusion equations. However, we are about to show that they still can be evidenced numerically by means of our approach.

3.1. Entropy dissipation properties and similarity variable. In this section, we must restrict ourselves to usual interaction potentials $\mathcal{I}(\xi) = |\xi|^{\gamma+2}/\gamma + 2$, $\gamma \geq 0$, because we shall use homogeneity properties. The plan is to change variables in order to make potential self-dimilar profiles stationary; this opens the way to the study of long-time asymptotics for (2.1). We proceed by considering the associated problem with an anti-drift term: (it corresponds to (1.1) with $\mu = -1$ and $\sigma = 0$)

$$\partial_t g(t,v) = \partial_v \left\{ g(t,v) \int_{\mathbb{R}} \mathcal{I}'(v-\omega) g(t,\omega).d\omega - v g(t,v) \right\}; \quad t, v \in \mathbb{R}_*^+ \times \mathbb{R}, \tag{3.1}$$

Indeed, if $g_\infty(v)$ is a steady-state solution for (3.1), then we can deduce a similarity solution of (2.1) quite easily, (see [31])

$$f_s(t,v) = \frac{1}{\alpha(t)} g_\infty\left(\frac{v}{\alpha(t)}\right); \qquad \alpha'(t) + \alpha^{1+\gamma}(t) = 0, \ \alpha(0) = 1.$$

Let us now consider $g(\tau, v)$ solution of (3.1) for $\tau(t) = -\log(\alpha(t))$ in order to state one result from [1, 10, 24].

THEOREM 3.1. *Let $f_0 \in \mathcal{M}_p(\mathbb{R})$ for $p = 2$, and let f be the unique weak solution to (2.1). Let $g(\tau, v)$ be defined by (3.1), then*

$$g \overset{\tau \to +\infty}{\longrightarrow} \frac{1}{2}\Big(\delta(v - 1/2) + \delta(v + 1/2)\Big),$$

weakly as a measure. (δ stands for the Dirac mass in zero)

Notice that $\alpha(0) = 1$ implies that $\theta_s(0) = \int_{\mathbb{R}} f_s(t = 0, v)v^2.dv = \int_{\mathbb{R}} v^2 g_\infty(v).dv = 1/4$. Following [1, 24], we recall the free energy functional (1.2) defined in the introduction. However, we rewrite it in terms of the reciprocal mapping,

$$\mathcal{J}(g, \tau) = \frac{1}{2+\gamma} \int_{(0,1)^2} |\tilde{V}(\tau, \varrho) - \tilde{V}(\tau, \varrho')|^{2+\gamma}.d\varrho.d\varrho' - \int_{\mathbb{R}} \tilde{V}(\tau, \varrho)^2.d\varrho,$$

where $\tilde{V}(\tau, \varrho)$ is the reciprocal mapping of $g(\tau, v)$. We can give at once the value of $\mathcal{J}(g_\infty)$:

$$\begin{aligned}
\mathcal{J}(g_\infty) &= \tfrac{1}{2+\gamma} \textstyle\int_{(0,1)^2} |\tilde{V}_\infty(\varrho) - \tilde{V}_\infty(\varrho')|^{\gamma+2}.d\varrho.d\varrho' - \tfrac{1}{4} \\
&= \tfrac{2}{2+\gamma} \textstyle\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\tilde{V}_\infty(\varrho) - \tilde{V}_\infty(\varrho')|^{2+\gamma}.d\varrho.d\varrho' - \tfrac{1}{4} \\
&= \tfrac{1}{2(2+\gamma)} - \tfrac{1}{4} = -\tfrac{\gamma}{4(2+\gamma)} \leq 0
\end{aligned}$$

In particular, there holds:

$$\tilde{V}(\tau, \varrho) = \frac{V(\tau, \varrho)}{\alpha(t)},$$

where V still stands for the reciprocal mapping of $f(t, v)$ solution of (2.1). Hence $\mathcal{J}(g, \tau)$ rewrites (at this level we use that $\mathcal{I}(V/\alpha) = \alpha^{-(2+\gamma)}\mathcal{I}(V)$),

$$\mathcal{J}(f, t) = \frac{1}{2+\gamma} . \frac{1}{\alpha(t)^{(2+\gamma)}} \int_{(0,1)^2} |V(t, \varrho) - V(t, \varrho')|^{2+\gamma}.d\varrho.d\varrho' - \frac{\theta(t)}{\alpha(t)^2},$$

with the temperature $\theta(t)$ defined in section 2, $\theta(t) = \int_0^1 V(t, \varrho)^2.d\varrho$ can be shown to satisfy the following equation:

$$\begin{aligned}
\tfrac{d\theta(t)}{dt} &= \textstyle\int_0^1 \partial_t V(t, \varrho) V(t, \varrho).d\varrho \\
&= -2 \textstyle\int_{(0,1)^2} V(t, \varrho)\mathcal{I}'(V(t, \varrho) - V(t, \varrho')).d\varrho.d\varrho' \\
&= -\textstyle\int_{(0,1)^2} \mathcal{I}(V(t, \varrho) - V(t, \varrho')).d\varrho.d\varrho' \leq 0.
\end{aligned}$$

In the last step we use explicitly that $\mathcal{I}(\xi)$ is a power function. It is now necessary to make an **assumption** concerning the speed of decay for $\mathcal{J}$, namely we assume that,

$$\exists z \text{ such that } 0 \leq \mathcal{J}(g, \tau) - \mathcal{J}(g_\infty) \leq \underbrace{[\mathcal{J}(g, \tau = 0) - \mathcal{J}(g_\infty)]z(\tau)}_{Z(\tau(t))}, \tag{3.2}$$

upon which relies the following result.

PROPOSITION 3.2. *Let $V(t = 0, \varrho)$ be the reciprocal mapping associated to the initial datum $f_0(v) \in \mathcal{M}_p(\mathbb{R})$; assumption (3.2) implies:*

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t Z(\tau(s)).ds = 0 \Rightarrow \lim_{t \to +\infty} \frac{\theta(t)}{\alpha(t)^2} = \frac{1}{4}.$$

As an immediate consequence, this statement allows to use the temperature as a way of rescaling the solution, as already done for nonlinear diffusion equations in [19] (exacept that here, the task is to zoom out a Dirac measure); this might be of interest if considering more general interaction potentials $\mathcal{I}(\xi)$, in the spirit of [31], for which no similarity variable α is available yet.

Proof. From the evolution of θ, we deduce that:

$$\mathcal{J}(g,\tau) = -\frac{\alpha^{-(2+\gamma)}(t)}{\gamma(2+\gamma)}\frac{d\theta(t)}{dt} - \frac{\theta(t)}{\alpha^2(t)}.$$

Hence we compute:

$$\begin{array}{rl} \frac{d}{dt}\left(\theta(t)\alpha^{-(2+\gamma)}(t)\right) & = \frac{d\theta(t)}{dt}\alpha^{-(2+\gamma)}(t) - (2+\gamma)\alpha^{-(3+\gamma)}(t)\alpha'(t)\theta(t) \\ & = \frac{d\theta(t)}{dt}\alpha^{-(2+\gamma)}(t) + (2+\gamma)\alpha^{-2}(t)\theta(t) \\ & = -(2+\gamma)\mathcal{J}(g,\tau). \end{array}$$

If (3.2) holds, then

$$-\mathcal{J}(g_\infty) \geq \frac{1}{2+\gamma}\frac{d\theta(t)}{dt}\left(\theta(t)\alpha^{-(2+\gamma)}(t)\right) \geq -\mathcal{J}(g_\infty) - Z(\tau(t)).$$

Thus we integrate in time and multiply by $\alpha^\gamma(t)$ to derive:

$$-\mathcal{J}(g_\infty).t.\alpha^\gamma(t) \geq \frac{1}{2+\gamma}\frac{\theta(t)}{\alpha^2(t)} - \theta(0)\alpha^\gamma(t) \geq \alpha^\gamma(t)\left(-\mathcal{J}(g_\infty).t - \int_0^t Z(s).ds\right).$$

The final step is to notice that $\alpha^\gamma(t) = (1+\gamma t)^{-1}$ and that $-\mathcal{J}(g_\infty)(2+\gamma)/\gamma = \frac{1}{4}$. □

3.2. Numerical validation: Homogeneous cooling states. Proposition 3.2 provides us with a simple and systematic way to visualize approximate similarity solutions of (2.1) just computing

$$g_\infty^{\Delta t}(v) = f^{\Delta t}\left(t, 2v\sqrt{\theta(t)}\right)2\sqrt{\theta(t)}, \tag{3.3}$$

which can be easily obtained from (2.7). Indeed, Proposition 3.2 (relying on assumption (3.2)) ensures that (3.3) will behave asymptotically well. This is illustrated in Figure 3.1 for various values of $\gamma \geq 0$. To the authors' knowledge, this is the first time such intermediate asymptotics are displayed numerically. Clearly, as γ decreases, θ gets closer to the zero-machine and the accuracy in computing (3.3) is reduced. However, the correct shape is kept even with $\theta \simeq 10^{-10}$. The parameters we used were $\Delta v = 0.05$ and the time step is chosen adaptively. The initial data is a Gaussian distribution, $f_0(v) = \frac{1}{2\pi}\exp(-x^2/2)$.

4. Inclusion in a "thermal bath": the 1D friction–diffusion equation $\sigma > 0$. We now deal with the Cauchy problem for the **granular diffusion** equation:

$$\partial_t f(t,v) = \partial_v\left\{f(t,v)\int_{\mathbb{R}}\mathcal{I}'(v-\omega)f(t,\omega).d\omega\right\} + \sigma\partial_{vv}f; \quad t,v \in \mathbb{R}_*^+ \times \mathbb{R}. \tag{4.1}$$

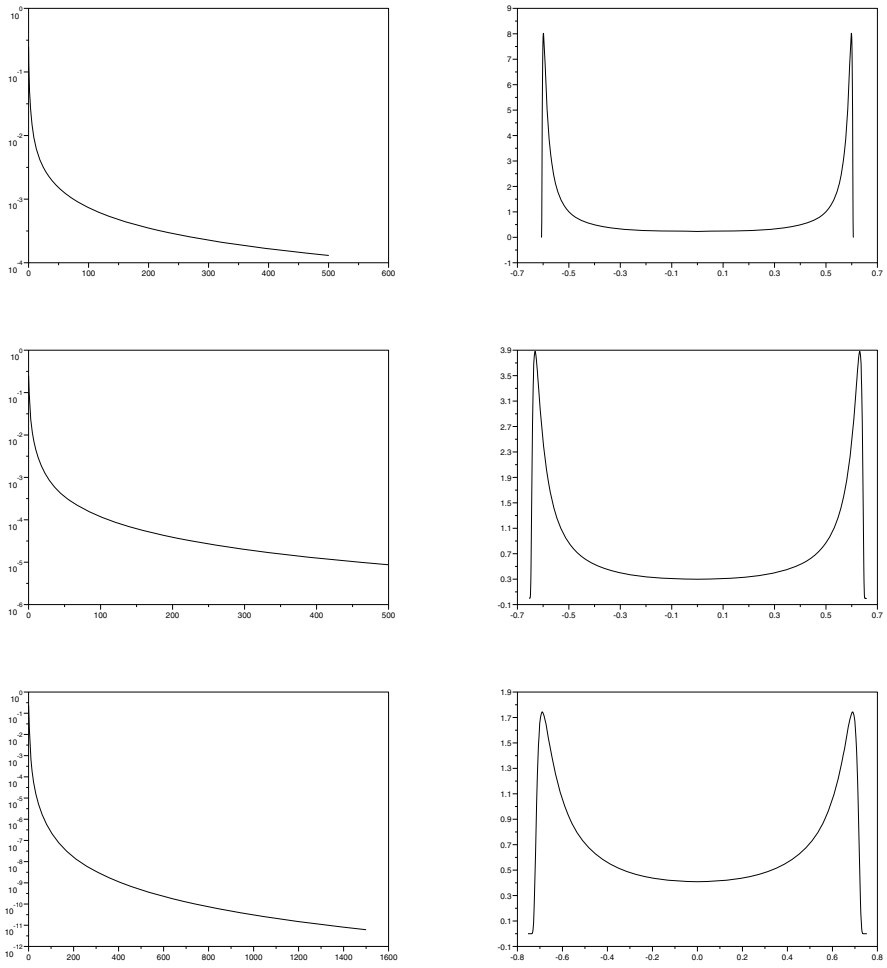

FIG. 3.1. *Decay of the temparatures $t \mapsto \theta(t)$ (left) and numerical similarity solutions (right) for $\gamma = 1.8, 1.2, 0.5$ (top to bottom). Notice the duration of stabilizing processes as γ is decreased.*

Existence and uniqueness have been shown in [12]. As in the preceding sections, we recall the weak solution to (4.1) as a distribution $f(t,v) \in C^1(\mathbb{R}_*^+; \mathcal{M}_p(\mathbb{R}))$ satisfying,

$$\begin{array}{rcl} \frac{d}{dt}\int_{\mathbb{R}} \varphi(v) f(t,v).dv & = & -\int_{\mathbb{R}^2} \mathcal{I}'(v-w)\varphi'(v) f(t,v) f(t,w).dv.dw \\ & & +\sigma \int_{\mathbb{R}} \varphi''(v) f(t,v).dv \\ & = & \frac{1}{2}\int_{\mathbb{R}^2} \mathcal{I}'(v-w)(\varphi'(w)-\varphi'(v)) f(t,v) f(t,w).dv.dw \\ & & +\sigma \int_{\mathbb{R}} \varphi''(v) f(t,v).dv, \end{array} \tag{4.2}$$

and $f(t,.) \to f(t=0,.) = f_0 \in \mathcal{M}_p(\mathbb{R})$ weakly as a measure as $t \to 0$. Clearly from this definition, one deduces again the conservation of mass and momentum, which corresponds to $\varphi(v) = 1, v$. However, the temperature isn't meant to vanish as for the case $\sigma = 0$.

Uniqueness still holds for these measure solutions:

THEOREM 4.1. *Let $f_0 \in \mathcal{M}_p(\mathbb{R})$ for $p = 2$ and $\mathcal{I}$ is a smooth even convex*

function; then the weak solution to (4.1) *is unique.*

The proof is given in Appendix B. The solution of (4.1) has been shown to converge weakly to the one of (2.1) as $\sigma \to 0$ in [26] However, the results of [12] are more interesting at a computational level because they deal with smoother solutions. Let us recall:

THEOREM 4.2. *Let $f_0 \in \mathcal{M}_p(\mathbb{R})$ be such that $\int_{\mathbb{R}}(1+v^4)f_0(v).dv$ is finite and $\mathcal{I}$ is a smooth even convex function; then the solution to* (4.1) *is unique and satisfies $(1+v^4)f(t,v) \in C^0(\mathbb{R}_*^+; L^1(\mathbb{R}))$. Moreover, if $f_0 \in C^2(\mathbb{R})$, the solution f is classical.*

From this result, one sees immediately that the behaviour of solutions is very different because of the diffusive term: global smooth solutions do exist.

4.1. Numerical scheme and Wasserstein contraction estimate. We don't recall the whole derivation of the equation satisfied by the reciprocal mapping (which can be easily derived from the considerations in section 2.1 and [19]): $\forall t, \varrho \in \mathbb{R}_*^+ \times (0,1)$,

$$\partial_t V(t,\varrho) + \sigma \partial_\varrho \left(\frac{1}{\partial_\varrho V(t,\varrho)} \right) + \int_0^1 \mathcal{I}'\Big(V(t,\varrho) - V(t,\varrho')\Big).d\varrho' = 0. \tag{4.3}$$

We shall also rely on the same type of discretization, hence we give at once the expression of the resulting scheme:

(4.4)
$$V_k^{n+1} = V_k^n - \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| \mathcal{I}'(V_k^n - V_{k'}^n) - \frac{\sigma \Delta t}{|C_k|} \left\{ \left(\frac{\varrho_{k+1} - \varrho_k}{V_{k+1}^n - V_k^n} \right) - \left(\frac{\varrho_k - \varrho_{k-1}}{V_k^n - V_{k-1}^n} \right) \right\}.$$

This scheme can be easily proven to be conservative and preserves momentum. It is also clearly Asymptotic-Preserving (AP) in the sense of Jin [22]. In contrast with section 2.1, (4.4) has to be completed by boundary conditions in $\varrho = 0, 1$; we observe that,

$$\frac{1}{\partial_\varrho V(t,\varrho)} = f(t, V(t,\varrho)) = 0 \text{ for } \varrho = 0, 1,$$

at least for smooth functions vanishing at infinity. Hence in $\varrho = 0$, we have:

$$V_0^{n+1} = V_0^n - \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| \mathcal{I}'(V_0^n - V_{k'}^n) - \frac{\Delta t \sigma}{|C_k|} \left(\frac{\varrho_1 - \varrho_0}{V_1^n - V_0^n} \right).$$

And a similar experssion in $\varrho = 1$. In the spirit of Lemma 2.2, we can prove:

LEMMA 4.3. *Let f_0, g_0 be two nonnegative initial data in $\mathcal{M}_p(\mathbb{R})$ for* (4.1) *and V, W their reciprocal mappings. Under the CFL restriction,*

$$\Delta t \sup_{V_k^n, W_{k'}^n} \left\{ \mathcal{I}'' + \sigma \Big(\frac{\varrho_{k+1} - \varrho_k}{(V_{k+1}^n - V_k^n)^2} + \frac{\varrho_{k+1} - \varrho_k}{(W_{k'+1}^n - W_{k'}^n)^2} \Big) \right\} \leq 1, \quad k, k' \in \mathcal{K}^2, \tag{4.5}$$

the scheme (4.4) *is contractive in any Wasserstein metric d_p; more precisely,*

$$\forall n \in \mathbb{N}, \qquad \sum_k |C_k| |V_k^{n+1} - W_k^{n+1}|^p \leq \sum_k |C_k| |V_k^n - W_k^n|^p, \qquad p \geq 1. \tag{4.6}$$

Proof. We substract the schemes (4.4) and rearrange the terms corresponding to each variable. The integral term is linearized the same way and with the notation of Lemma 2.2; the diffusive part gives

$$\begin{array}{rl} V_k^{n+1} - W_k^{n+1} = (V_k^n - W_k^n)\Big(1 - \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| (\mathcal{I}'')_{k,k'}^n + \frac{\Delta t}{|C_k|}(\Phi'_{k+\frac{1}{2}} + \Phi'_{k-\frac{1}{2}})\Big) \\ \quad + \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| (\mathcal{I}'')_{k,k'}^n (V_{k'}^n - W_{k'}^n) \\ \quad - \frac{\Delta t}{|C_k|} \Phi'_{k+\frac{1}{2}} (V_{k+1}^n - W_{k+1}^n) - \frac{\Delta t}{|C_k|} \Phi'_{k-\frac{1}{2}} (V_{k-1}^n - W_{k-1}^n), \end{array}$$

with the following function:

$$\Phi_{k+\frac{1}{2}}(V) = \frac{\sigma(\varrho_{k+1} - \varrho_k)}{V}, \qquad \Phi'_{k+\frac{1}{2}}(V) = -\frac{\sigma(\varrho_{k+1} - \varrho_k)}{V^2} \leq 0.$$

Then $\Phi'_{k+\frac{1}{2}}$ stands for a mid-point value obtained through the mean-value theorem. From the CFL condition (4.5), the combination is convex thus Jensen's inequality yields the announced result. □

We stress that Lemma 4.3 doesn't imply any L^p decay no more simply because initializing with $W_k^0 \equiv C^t \in (0,1)$ is forbidden by the CFL condition (4.5). Nevertheless, the order-preserving property still holds.

However, as said before, this result allows one to restrict the attention to smooth C^2 initial data (which give rise to classical solutions as well) in order to take advantage of classical techniques in numerical analysis. Indeed, even in case the initial data is singular, one can argue that the error between the solutions emerging from its regularization and the genuine one decay with time in any Wasserstein metric d_p.

So, going down this track, we first oberve (by means of Taylor expansions) that the scheme (4.4) is clearly a first-order discretization of (4.1); hence its **local truncation error** is $e_k^n = O(\Delta t + |C_k|)$. Hence in order to establish convergence, we must study the evolution in time of its **global error** $e^{\Delta t}(t,\varrho) \overset{def}{=} V^{\Delta t}(t,\varrho) - V(t,\varrho)$ for all $\varrho \in (0,1)$. It satisfies,

$$\begin{array}{c} \partial_t e^{\Delta t}(t,\varrho) + \sigma \partial_\varrho \left(\frac{1}{\partial_\varrho V^{\Delta t}(t,\varrho)} - \frac{1}{\partial_\varrho V(t,\varrho)} \right) \\ + \int_0^1 \mathcal{I}'\Big(V^{\Delta t}(t,\varrho) - V^{\Delta t}(t,p)\Big) - \mathcal{I}'\Big(V(t,\varrho) - V(t,p)\Big).dp = e_k^n, \end{array}$$

for $t = t^n$ and $\varrho = \varrho_k$. At this point, one multiplies by $e^{\Delta t}(t,\varrho)$ and integrates in $\varrho \in (0,1)$. The integral term is positive by the uniqueness proof of Theorem 2.1 given in Appendix A. The diffusive term is handled thanks to a monotonicity argument, as in the proof of Theorem 4.1 given in Appendix B; indeed, an integration by parts yields

$$-\sigma \int_0^1 (\partial_\varrho V^{\Delta t} - \partial_\varrho V) \left(\frac{1}{\partial_\varrho V^{\Delta t}} - \frac{1}{\partial_\varrho V} \right)(t,\varrho).d\varrho \geq 0, \qquad \sigma \geq 0.$$

Thus remains only

$$\frac{1}{2}\partial_t \int_0^1 e^{\Delta t}(t,\varrho)^2.d\varrho \leq O(\Delta t + \sup_k |C_k|) \int_0^1 |e^{\Delta t}(t,\varrho)|.d\varrho,$$

and since $(0,1)$ is bounded, the L^2 norm dominates the L^1. Hence the error grows at most exponentially in time depending on the local truncation error of (4.4). We deduce:

THEOREM 4.4. *Let $f_0 \in C^2(\mathbb{R})$ with compact support and $\mathcal{I}$ be a convex smooth even function; under the CFL condition* (4.5)*, the sequence $V^{\Delta t}$ defined by* (2.13) *and the scheme* (4.4) *converges pointwise as $\Delta t, \Delta v \to 0$ towards the unique classical solution of* (4.3)*, or equivalently of* (4.1).

The equivalence of the two notions of solution to (4.1) has been used previously in [26]. It is easy to obtain a formal notion of consistency for the scheme (4.4) following the ideas presented in [19], namely, one writes down the basic equality coming directly from (2.4),

$$\forall k, n \in \mathcal{K} \times \mathbb{N}, \qquad \varrho^{\Delta t}(t^n, V_k^n) = \varrho^{\Delta t}(t^{n+1}, V_k^{n+1}),$$

and then plugs (4.4). A partly implicit finite-volume discretization of (4.1) is derived by means of Taylor expansions assuming that $\varrho^{\Delta t}$ is at least C^1 in the v variable. We close this section mentioning that (4.4) isn't the unique choice leading to the contraction property (4.6); a time-splitting strategy would behave equally well. In this last case, the proof would be the concatenation of Lemma 2.2's and [19] (the CFL would be slightly lighter though). The last situation emanating from (2.1) which hasn't been treated yet is $\sigma > 0$ and $\mu \neq 0$ and corresponds to a so–called **granular Fokker–Planck** equation, as studied in [12]. However, when expressed by means of the reciprocal mapping, the confining term rewrites simply $\mu V(t, \varrho)$ thus leading to a simple exponential term which can be integrated explicitly. Indeed, upon multiplying (1.5) by $\exp(\mu t)$, one derives the following equation:

$$\partial_t \tilde{V}(t,\varrho) + \exp(\mu t)\left\{ \sigma \partial_\varrho \left(\frac{1}{\exp(-\mu t)\partial_\varrho \tilde{V}(t,\varrho)} \right) + \int_0^1 \mathcal{I}'\Big(\exp(-\mu t)(\tilde{V}(t,\varrho) - \tilde{V}(t,\varrho')) \Big) . d\varrho' \right\} = 0, \tag{4.7}$$

where $\tilde{V}(t,.) = \exp(\mu t) V(t,.)$. The aforementioned techniques allow to cover this case with minor changes.

4.2. Numerical results; study of the solution's "tails". Our first validation for (4.4) has been to look at the behaviour of rescaled numerical solutions (3.3) as σ is decreased of several orders of magnitude. This has to do with checking the so–called Asymptotic-Preserving property in the sense of [22]. In this case, the decay of the approximate solution's temperature is interesting to look at since it should strengthen as $\sigma \to 0$. This is indeed the case as seen on Figure 4.1; parameters used were $\Delta v = 0.27$ and $\gamma = 1.2$. The time-step Δt is chosen adaptively according to (4.5).

It is a well-known fact that for great values of $|v|$, the solution of (4.1) with $\gamma = 1$ behaves like $\exp(-|v|^3)$, see [7, 8, 9]. We tried to check it out by iterating (4.4) with $\sigma = 1$ for quite a long time starting from a Gaussian initial datum. Results are shown in Figure 4.2. However, as the values of $f^{\Delta t}$ become very small (less than 10^{-50}), results could be slightly corrupted by spurious noise; it is nevertheless seeable that the behaviour is qualitatively correct.

5. Numerical approximations of the Hele–Shaw cell.

5.1. Derivation of the numerical process. As announced in the introduction, we are about to concentrate mainly on a particular case of (1.4) corresponding to $a(u) = u$ (we drop the superscripts ε as there is no ambiguity),

$$\partial_t u + \partial_x \Big(u \partial_x (u *_x \Gamma_\varepsilon''(x)) \Big) = 0, \qquad 0 \leq u_0 \in L^1 \cap H^1(\mathbb{R}).$$

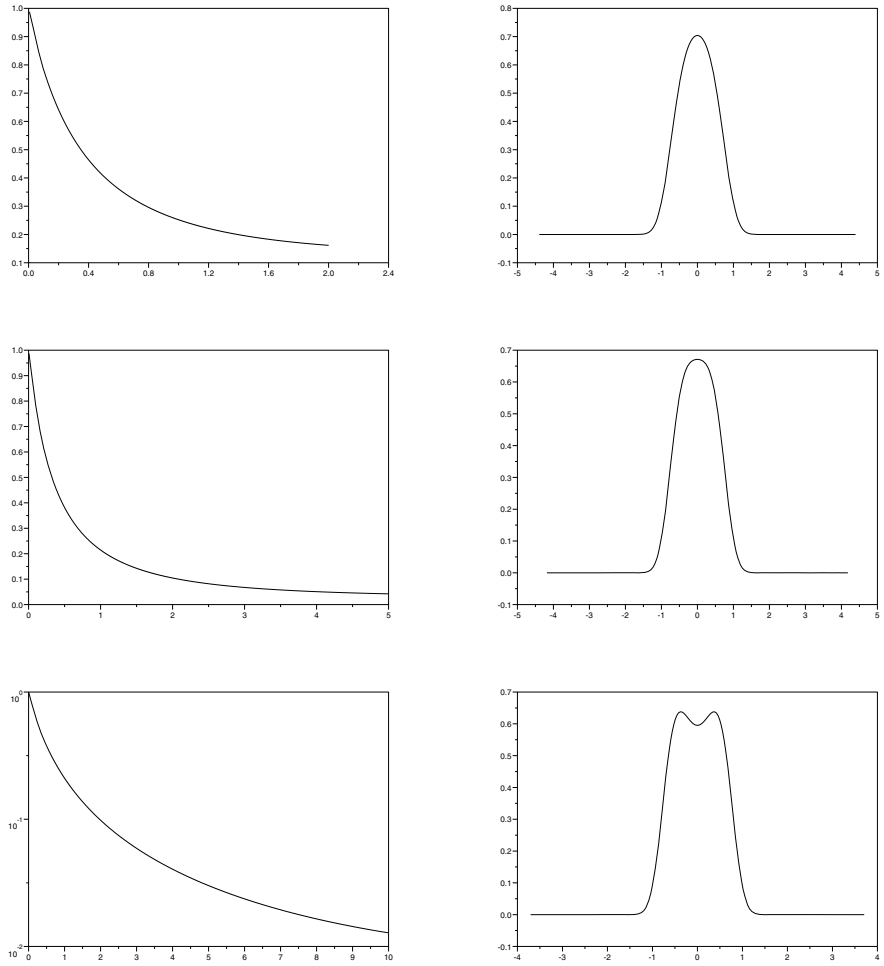

FIG. 4.1. *Decay of the temparatures* $t \mapsto \theta(t)$ *(left) and rescaled numerical solutions* (3.3) *(right) for* $\sigma = 0.1, 0.01, 0.001$ *(top to bottom).*

The general case with mobility $a(u) \neq u$ is to be tackled in section **??** by means of quite different techniques. It is well-known that equation (1.3) is mass-preserving, hence there is no loss of generality in considering only nonnegative initial data satisfying the additional requirement $\int_{\mathbb{R}} u_0(x).dx = 1$. One easily shows that it also dissipates energy, i.e.

$$\frac{d}{dt}\int_{\mathbb{R}} |\partial_x u(t,x)|^2.dx \leq 0, \qquad t \in \mathbb{R}^+.$$

Nonnegativity preservation has been proved to hold as a consequence of entropy dissipation, [4, 5, 6]; however, in one space dimension, positive solutions to (1.3) are classical for $a(u) = u^p$, p big enough. As the solution vanishes, it typically suffers a loss of regularity and propagates at finite speed (it admits in particular source-type solutions, [16]). This is strongly reminiscent of second-order degenerate parabolic

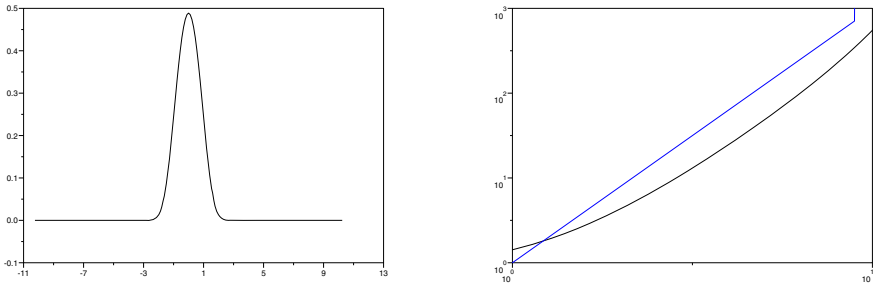

FIG. 4.2. *Large-time behaviour for* (4.1) *at time* $t \simeq 33$ *(left) and comparison of its solution's tails with* $v \mapsto \exp(-|v|^3)$ *(right, we plotted* $-\ln(f^{\Delta t})$ *and* $|v^3|$ *in logscale).*

equations already treated numerically in [19]. In this context, we first recall that approximations of the type (1.4) have been shown to be consistent with the original problem (1.3) for the special choice of Γ_ε being a Gaussian kernel,

$$\Gamma_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{x^2}{2\varepsilon}\right),$$

see Theorem 3.1 in [29]. Relying on this result, we shall pursue with this choice even if (many) other choices could be admissible too, as for instance the compactly supported Landau kernel given by ($m \in \mathbb{N}$)

$$\Gamma_{\varepsilon,m}(x) = \begin{cases} \frac{(\varepsilon^2-x^2)^m}{\int_{-\varepsilon}^{\varepsilon}(\varepsilon^2-y^2)^m.dy}, & |x| \leq \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

For our purpose, one would get:

$$\Gamma_{\varepsilon,4}(x) = \frac{315(\varepsilon^2-x^2)^4}{128\varepsilon^9}, \qquad \Gamma'''_{\varepsilon,4}(x) = \frac{105}{2\varepsilon^6} x(9\varepsilon^2 - 21x^2)\Gamma_{\varepsilon,1}(x).$$

Other kernels can be found in e.g. [33]. Anyway, following ideas from the preceding sections, one can derive a numerical scheme for (1.4): by integration,

$$\partial_t \varrho + (\partial_x \varrho)(\partial_x \varrho *_x \Gamma'''_\varepsilon) = 0, \qquad \varrho(t,x) = \int_{-\infty}^{x} u(t,y).dy,$$

and introducing $X(t,\varrho)$ as the reciprocal mapping of $u(t,x)$, ($\|u_0\|_{L^1(\mathbb{R})} = 1$)

$$\partial_t X(t,\varrho) - \int_0^1 \Gamma'''_\varepsilon(X(t,\varrho) - X(t,\varrho')).d\varrho' = 0, \qquad (t,\varrho) \in \mathbb{R}^+ \times (0,1). \tag{5.1}$$

For Γ_ε being Gaussian, the scheme is completely determined with:

$$\Gamma_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{x^2}{2\varepsilon}\right), \qquad \Gamma'''_\varepsilon(x) = \left(\frac{3x}{\varepsilon^2} - \frac{x^3}{\varepsilon^3}\right)\Gamma_\varepsilon(x).$$

Hence, defining a cartesian grid by means of two positive parameters $\Delta t, \Delta x$ and denoting $t^n = n\Delta t$, $x_k = k\Delta x$, $\varrho_k = \varrho_0(x_k)$, one obtains finally:

$$X_k^{n+1} = X_k^n + \Delta t \sum_{k' \in \mathcal{K}} |C_{k'}| \Gamma'''_\varepsilon(X_k^n - X_{k'}^n). \tag{5.2}$$

Now, the novelty lies in the fact that Γ_ε''' has no sign; this somehow reflects the high-order space derivative in (1.3). As in section 2, no boundary conditions are needed for (5.2).

5.2. Stability estimates for $\varepsilon > 0$. As in former papers, [34, 21], we aim at establishing a nonnegativity-preserving property; however, our scheme isn't contractive in any Wassertein distance. Only remains the monotonicity-preserving property:

PROPOSITION 5.1. *Let $u_0 \geq 0$ and $X(t,\varrho)$ stand for the reciprocal mapping satisfying* (5.1) *with $X(t=0,.)$ stands for the pseudo-inverse of $\varrho(t=0,.)$. Under the CFL condition,*

$$\Delta t \leq \frac{1}{6}\varepsilon^2\sqrt{2\pi\varepsilon\exp(3)}, \tag{5.3}$$

the scheme (5.2) *is monotonicity-preserving.*

Proof. The proof proceeds by substracting and linearizing (5.2); one gets

$$\begin{aligned} X_k^{n+1} - X_{k-1}^{n+1} &= X_k^n - X_{k-1}^n + \Delta t \textstyle\sum_{k'\in\mathcal{K}} |C_{k'}| \Big(\Gamma_\varepsilon'''(X_k^n - X_{k'}^n) - \Gamma_\varepsilon'''(X_{k-1}^n - X_{k'}^n) \Big) \\ &= (X_k^n - X_{k-1}^n)\Big(1 + \Delta t \textstyle\sum_{k'\in\mathcal{K}} |C_{k'}| (\Gamma_\varepsilon'''')_{k,k-1,k'}^n \Big), \end{aligned}$$

where $(\Gamma_\varepsilon'''')_{k,k-1,k'}^n = \Gamma_\varepsilon''''(\lambda X_k^n + (1-\lambda)X_{k-1}^n - X_{k'}^n)$. Now, for Γ_ε being Gaussian, its fourth derivative reads:

$$\Gamma_\varepsilon''''(x) = \frac{\Gamma_\varepsilon(x)}{\varepsilon^2}\Big(y^2 - 6y + 3\Big), \qquad y = \frac{x^2}{\varepsilon}.$$

By elementary calculus, this polynomial has a minimum for $y = 3$ and the corresponding value is -6. It remains to observe that (5.3) ensures that $1 + \Delta t \inf \Gamma_\varepsilon'''' \geq 0$. □

As an immediate consequence, nonnegativity for the u variable holds under the same condition (5.3). At this point, we stress that our scheme (5.2) is endowed with a very light CFL restriction $\Delta t = O(\varepsilon^{5/2})$; indeed the parameters Δt and Δx are completely decorralated as in section 2.2. Clearly, the smaller ε, the more precise the numerical approximation, but the higher the computational cost. The choice of the Landau kernel would impose $\Delta t = O(\varepsilon^5)$.

We now move on to an energy estimate for the scheme (5.2); it is convenient to recall some short-hand notation from [19], namely,

$$\delta X_{k-\frac12}^n := X_k^n - X_{k-1}^n, \qquad \delta\varrho_{k-\frac12} := \varrho_k - \varrho_{k-1}.$$

PROPOSITION 5.2. *Let us denote $\alpha = \frac{6}{\varepsilon^2\sqrt{2\pi\varepsilon\exp(3)}}$; under the hypotheses of Proposition* 5.1, *the following holds for* (5.2):

$$\forall n \in \mathbb{N}, \quad \left| \frac{\delta\varrho_{k+\frac12}}{\delta X_{k+\frac12}^n} - \frac{\delta\varrho_{k-\frac12}}{\delta X_{k-\frac12}^n} \right| \leq \exp(\alpha t^n) \left\{ \left| \frac{\delta\varrho_{k+\frac12}}{\delta X_{k+\frac12}^0} - \frac{\delta\varrho_{k-\frac12}}{\delta X_{k-\frac12}^0} \right| + O(\varepsilon^2) \right\}. \tag{5.4}$$

Proof. We observe that from the proof of Prop. 5.1, the divided difference satisfies,

$$\left| \frac{\delta\varrho_{k-\frac12}}{\delta X_{k-\frac12}^n} \right| \leq \left| \frac{\delta\varrho_{k-\frac12}}{\delta X_{k-\frac12}^{n-1}} \right| \frac{1}{1 + \Delta t \inf \Gamma_\varepsilon''''} \leq \left| \frac{\delta\varrho_{k-\frac12}}{\delta X_{k-\frac12}^0} \right| \exp(-t^n \inf(\Gamma_\varepsilon'''')),$$

since the CFL condition (5.3) ensures monotonicity preservation and $\inf(\Gamma''''_\varepsilon) \leq 0$. For any choice of a, b, c, d in $\mathbb{R}^+$, there holds $\left|\frac{a}{b} - \frac{c}{d}\right| \leq \frac{|a-c|}{b} + c\left|\frac{1}{b} - \frac{1}{d}\right|$. Hence we get that:

$$\left|\frac{\delta\varrho_{k+\frac{1}{2}}}{\delta X^{n+1}_{k+\frac{1}{2}}} - \frac{\delta\varrho_{k-\frac{1}{2}}}{\delta X^{n+1}_{k-\frac{1}{2}}}\right| \leq \frac{\left|\frac{\delta\varrho_{k+\frac{1}{2}}}{\delta X^{n}_{k+\frac{1}{2}}} - \frac{\delta\varrho_{k-\frac{1}{2}}}{\delta X^{n}_{k-\frac{1}{2}}}\right|}{1+\Delta t\inf(\Gamma''''_\varepsilon)} + O(\Delta t)\left|\frac{\delta\varrho_{k-\frac{1}{2}}}{\delta X^{0}_{k-\frac{1}{2}}}\right| \exp(-t^n \inf(\Gamma''''_\varepsilon)).$$

By induction, this yields:

$$\left|\frac{\delta\varrho_{k+\frac{1}{2}}}{\delta X^{n+1}_{k+\frac{1}{2}}} - \frac{\delta\varrho_{k-\frac{1}{2}}}{\delta X^{n+1}_{k-\frac{1}{2}}}\right| \leq \exp(-(n+1)\Delta t\inf(\Gamma''''_\varepsilon)) \times \left\{\left|\frac{\delta\varrho_{k+\frac{1}{2}}}{\delta X^{0}_{k+\frac{1}{2}}} - \frac{\delta\varrho_{k-\frac{1}{2}}}{\delta X^{0}_{k-\frac{1}{2}}}\right| + O(\Delta t)\left|\frac{\delta\varrho_{k-\frac{1}{2}}}{\delta X^{0}_{k-\frac{1}{2}}}\right| \frac{1+\Delta t\inf(\Gamma''''_\varepsilon)}{-\Delta t\inf(\Gamma''''_\varepsilon)}\right\}.$$

It remains to plug the value of $\inf(\Gamma''''_\varepsilon) \leq 0$ and to observe that the last term is at least $O(\varepsilon^2)$. Thus (5.4) follows. □

Unfortunately, it doesn't seem easy to establish the equivalent of the energy decay which holds for the continuous problem (1.3) as a consequence of the nonnegativity of $a(u)$. However, (5.4) can be considered as a "cheap" estimate which ensures that approximate solutions generated by (5.2) won't blow up in finite time.

5.3. Numerical results: Dead core phenomenon and similarity solution. We finally aim at illustrating our approach's outcome on two test-cases from the literature, each describing a typical phenomenon for equation (1.3): the first one is the appearance of the so–called "dead core", which corresponds to film rupture (see section 9 in [34] and [21]), the second is the spreading of the source-type solution already studied in [16, 29].

We first consider the following initial data, taken from [6, 34]:

$$u_0(x) = 0.8 - \cos(\pi x) + 0.25\cos(2\pi x), \qquad x \in (-2, 2). \tag{5.5}$$

Despite the fact the authors of [34] consider $a(u) = \sqrt{u}$, we obtained dead core evidence around time $t \simeq 0.003$ as shown in Figure 5.1; initial data are plotted for comparison. Parameters used for this run were $\Delta x = 2^{-7}$, $\varepsilon = 0.002$ and Δt has been computed from (5.3). Observe in particular the loss of regularity when passing from $t = 0.002$ to $t = 0.0035$. There is no clear touchdown onto the x axis as our scheme is endowed with an error in $\varepsilon > 0$ from the very beginning, left apart the ones coming from the discretisation in $\Delta t, \Delta x$.

The second case of interest is the simulation of a droplet's spreading, rendered through a similarity solution which is only $C^1(\mathbb{R})$; the following formula is (partially) taken from [29],

$$S(t,x) = \frac{1}{24(5t+\tau)^{\frac{1}{5}}}\max\left[0, \left(\omega^2 - \frac{x^2}{(5t+\tau)^{\frac{2}{5}}}\right)^2\right], \quad u_0(x) = \frac{S(0,x)}{\|S(0,.)\|_{L^1(\mathbb{R})}}. \tag{5.6}$$

Parameters used here were $\Delta x = 2^{-8}$, $\varepsilon = 0.03$, $\omega = 2$, $\tau = 0.5$ and we iterated up to time $t = 0.11$, see Figure 5.2. The main difference with the preceding case is that the rendering of our scheme (5.2) shows up oscillations (which decay with

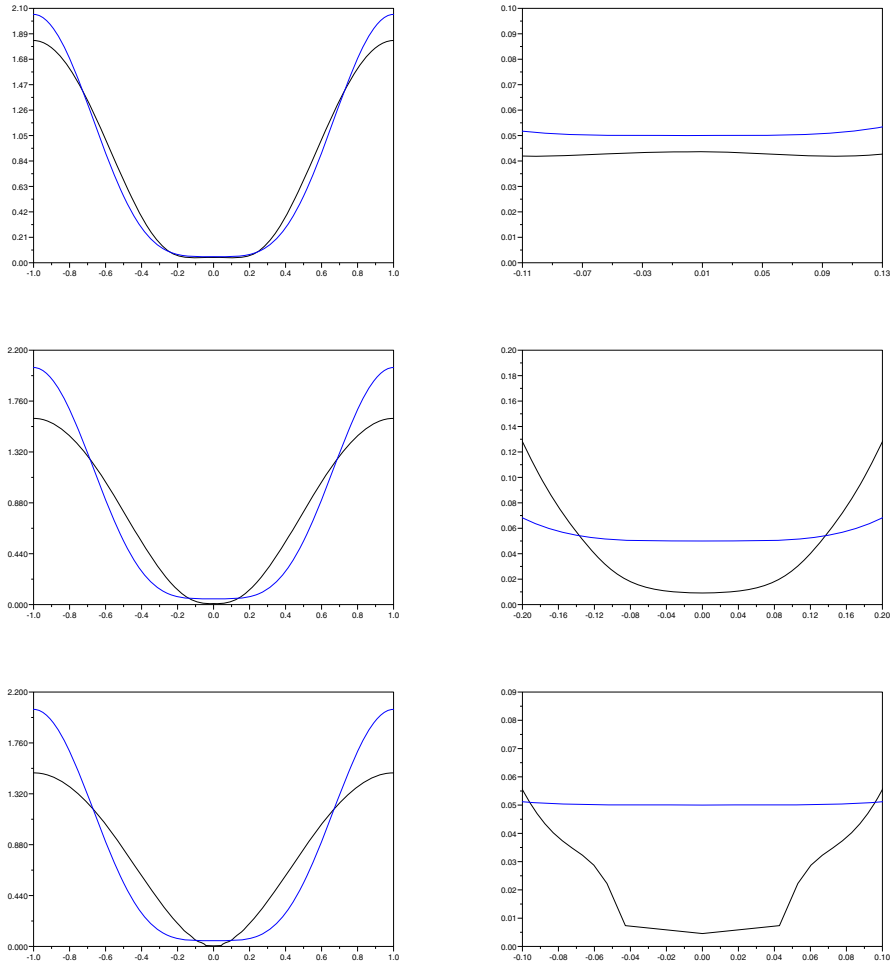

FIG. 5.1. *Evolution of the dead core for initial data* (5.5) *at times* $t = 0.001, 0.002, 0.0035$. *(top to bottom).*

ε): this comes from the loss of regularity at the vanishing points of the approximate solution. We note in passing that oscillations occured also for [29] and [34] where a post-processing has been implemented to plot only selected points (see p. 550). From our CFL condition (5.3), there is clearly a tradeoff between quality of rendering and CPU time for such a simulation. We plotted the initial data and the exact value of the source-type profile for the sake of completeness; an evolution of the temperature is also shown.

Remark 2. From the computations in section 5.1, one could formally extend the present approach to mobilities $a(u) \neq u$. In this direction, equation (5.1) would modify like,

$$\partial_t X(t,\varrho) - a\left(\frac{1}{\partial_\varrho X}\right)\partial_\varrho X(t,\varrho)\int_0^1 \Gamma_\varepsilon'''(X(t,\varrho) - X(t,\varrho')).d\varrho' = 0,$$

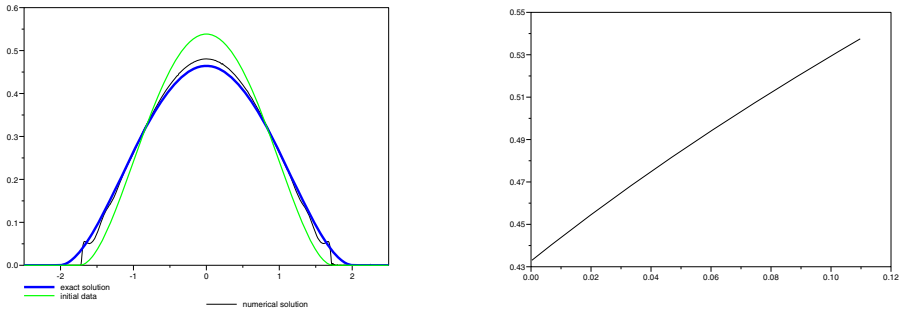

FIG. 5.2. *Spreading of the similarity solution* (5.6) *at time* $t = 0.11$ *with* $\tau = 0.5$ *(left) and evolution of its temperature* $t \mapsto \theta(t)$ *(right).*

and the numerical scheme (5.2), with obvious notation,

$$X_k^{n+1} = X_k^n + \tilde{a}\left(\frac{\delta\varrho_{k+\frac{1}{2}}}{\delta X^n_{k+\frac{1}{2}}}, \frac{\delta\varrho_{k-\frac{1}{2}}}{\delta X^n_{k-\frac{1}{2}}}\right)\Delta t \sum_{k'\in\mathcal{K}} |C_{k'}|\Gamma'''_\varepsilon(X_k^n - X_{k'}^n).$$

Hence the design of a discrete mobility $\tilde{a} \geq 0$ which could maintain monotonicity and energy estimates has to be tackled (as in [34]).

6. An efficient Fourier scheme for general thin film equations. As pointed out in Remark 2, the preceding approach can't be extended to general mobilities $a(u) = u^p$ in a straightforward manner. To partially fix this drawback, we wish to propose here a general way to build schemes for equation (1.3), based on repeated use of `fft` routines. However, we've been unable to establish any rigorous bound on this class of approximate solutions thus we just mention it here. Future work might be dedicated to these matters.

We assume solutions of (1.3) endowed with a $H^1(\mathbb{R})$ regularity in the space variable, see [16]; hence we introduce their Fourier transform which we shall denote $\hat{u}(t,\xi)$ for $\xi \in \mathbb{R}$. Obviously, there holds $\partial_t\hat{u}(t,\xi) = -i\xi\widehat{a(u)\partial_{xxx}u}$. Our strategy is based on the idea of computing correctly the nonlinear term relying on the unknown's smoothness. By repeated integrations by parts, one derives the following equality:

$$\widehat{a(u)\partial_{xxx}u}(t,\xi) = -i\xi\int_{\mathbb{R}}\Big(A(\partial_x u) - 2\partial_x u\partial_x a(u) - \xi^2 A(u)\Big)\exp(-ix\xi).dx,$$

where, for $a(u) = u^p$, $A(u) = \frac{u^{p+1}}{p+1}$. This leads to the general expression:

$$\partial_t\hat{u}(t,\xi) + \xi^2\left\{\widehat{B(u,\partial_x u)}(t,\xi) + \xi^2\widehat{A(u)}(t,\xi)\right\} = 0, \quad B(u,\partial_x u) = p(\partial_x u)^2u^{p-1} + A(\partial_x u). \tag{6.1}$$

The Hele-Shaw cell corresponding to $p = 1$ is actually a quite particular case as (6.1) simplifies and boils down to[1]:

$$\partial_t\hat{u}(t,\xi) + \xi^2\left\{\frac{3}{2}\widehat{|\partial_x u|^2}(t,\xi) + \frac{\xi^2}{2}\widehat{|u|^2}(t,\xi)\right\} = 0. \tag{6.2}$$

[1]This has to do with Theorem 1, Case 2 in [6].

It is easy to implement both (6.1) and (6.2) using e.g. the SciLab package. The derivative $|\partial_x u|^2$ is computed via Fourier transform too; this has the advantage of suppressing the wide-stencil problem inherent to (1.3). Indeed, both approaches in [21, 34] suffer from it. Moreover, we observed that (6.2) doesn't ask for an implicit time-integrator; like all the schemes proposed in this paper, it can be treated explicitly thus keeping the computational cost very low. A last observation is that *it seems* that the CFL condition for (6.2) is linear, like $\Delta t \leq \Delta x/20^3$ instead of $\Delta t = O(\Delta x^4)$ for finite differences; the factor 20 has been found experimentally, we cannot give any rigorous explanation of it. It is necessary to filter spurious oscillations when computing derivatives; a standard Gaussian filter has been used there.

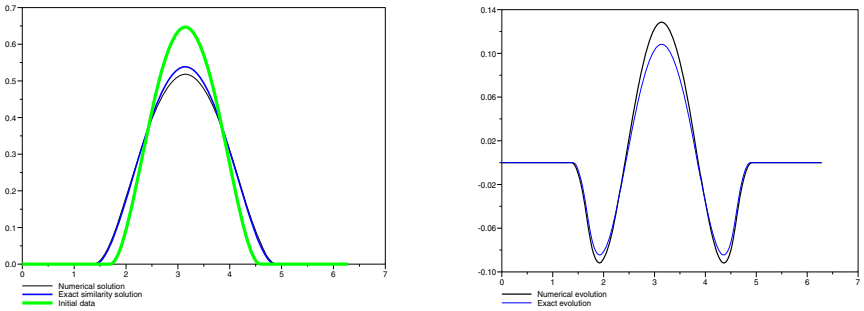

FIG. 6.1. *Similarity solution* (5.6) *at time* $t = 0.06$ *with* (6.2), $\tau = 0.2$ *(left); difference with initial data for exact solution and numerical (right).*

As a numerical validation, we selected again the similarity solution (5.6) with the same parameters, except a lower value of τ, and 256 discretization points in space (this asks for $\Delta t = 2.10^{-6}$). The results are displayed on Figure 6.1. One observes that despite a lower space resolution, the approximate solution is better than the one on Figure 5.2 as it doesn't show any oscillations at its tails. The speed of propagation is also correct. There is a particular structure associated to the differential

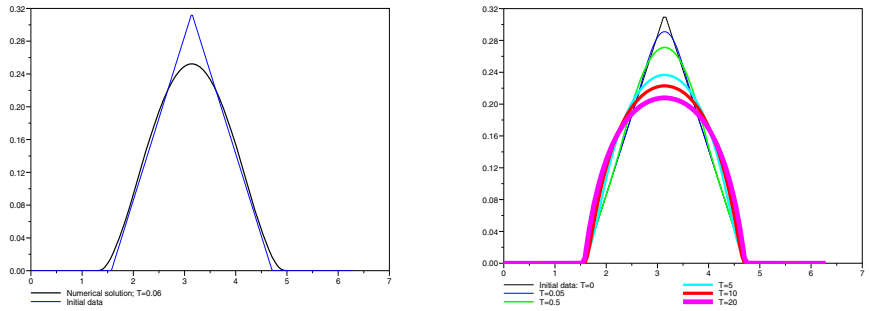

FIG. 6.2. *Evolution of Lipschitz initial data, for* $a(u) = |u|$ *(left) and* $a(u) = |u|^4$ *(right)*

equation (6.2); we present some computations on this in Appendix C. For the sake of completeness, we also include numerical results on a test-case proposed in [21]: it is a Lipschitz initial datum, $u_0(x) = 0.1 \max(0, \pi - 2|\pi - x|)$, for $x \in [0, 2\pi]$. This highlights the great importance of the exponent p in the dynamics of (1.3) since, for $p = 1$, it develops instantaneously a zero-angle contact on each side whereas for $p = 4$, the support doesn't spread *at all* and convergence onto a parabolic profile is observed.

Both features are successfully rendered by our Fourier approach (6.1), as can be seen on Figure 6.2. Similar parameters have been used and the scheme is fully explicit in time.

7. Conclusion and outlook. We have proposed in this paper to work out a rather wide class of 1D nonlinear diffusion equations arising from physics modelling by using numerical techniques borrowed from mass transportation theory, [32]. In cases it applies, this approach generally allows for the derivation of simple schemes and short proofs. Similar techniques could be applied in other areas of application, like e.g. the so–called flashing ratchet, [23]. However, several questions remain open; among them lies the design of an efficient strategy for the simulation of the full dissipative Boltzmann equation in the space t, x, v (see however [26]). The situation for the general thin films equation is even more challenging; especially, the case $a(u) = u^3$ is very much of interest from the physics perspective, see again [25]. For this case, $p = 3$ and (6.1) reads:

$$\partial_t \hat{u}(t,\xi) + \xi^2 \left\{ \frac{3}{2} \widehat{|\partial_x u^2|^2}(t,\xi) + \frac{1}{4} \widehat{|\partial_x u|^4}(t,\xi) + \frac{\xi^2}{4} \widehat{|u|^4}(t,\xi) \right\} = 0.$$

Appendix A. Proof of Theorem 2.1. We consider two possible solutions f, g and assume that as densities, they satisfy $\int_{\mathbb{R}} f(t,v).dv \equiv 1$ and $\int_{\mathbb{R}} v f(t,v).dv \equiv 0$. Then we insert equation (2.5) in the following expression to derive:

$$\begin{aligned} \tfrac{d}{dt} \int_0^1 |V(t,\varrho) - W(t,\varrho)|^2.d\varrho &= 2 \int_0^1 \partial_t (V(t,\varrho) - W(t,\varrho))(V(t,\varrho) - W(t,\varrho)).d\varrho \\ &= -2 \int_{(0,1)^2} \Big(\mathcal{I}'(V(t,\varrho) - V(t,p)) - \mathcal{I}'(W(t,\varrho) - W(t,p)) \Big) \\ &\quad \times (V(t,\varrho) - W(t,\varrho)).dp.d\varrho \end{aligned}$$

At this point, it is useful to recall from [24] that $\int_{\mathbb{R}} (v-w) d\mu(v,w) = 0$ for all density μ admitting f and g as marginals. This implies that $\int_0^1 V(t,\varrho) - W(t,\varrho).d\varrho = 0$; hence,

$$V(t,\varrho) - W(t,\varrho) = \int_0^1 (V(t,\varrho) - V(t,q)) - (W(t,\varrho) - W(t,q)).dq.$$

This is to be plugged in the previous equality in order to derive:

$$\begin{aligned} \frac{d}{dt} \int_0^1 |V(t,\varrho) - W(t,\varrho)|^2.d\varrho = -2 \int_0^1 \Big(\int_0^1 \mathcal{I}'(V(t,\varrho) - V(t,p)) - \mathcal{I}'(W(t,\varrho) - W(t,p)).dp \Big) \\ \times \Big(\int_0^1 (V(t,\varrho) - V(t,q)) - (W(t,\varrho) - W(t,q)).dq \Big).d\varrho \end{aligned}$$

Now, the trick is to observe that the roles of p and ϱ are symmetric, thus we can exchange them in the second integral part. The integral rewrites:

$$\begin{aligned} \frac{d}{dt} \int_0^1 |V(t,\varrho) - W(t,\varrho)|^2.d\varrho = 2 \int_{(0,1)^3} \Big(\mathcal{I}'(V(t,p) - V(t,\varrho)) - \mathcal{I}'(W(t,p) - W(t,\varrho)) \Big) \\ \times \Big((V(t,p) - V(t,q)) - (W(t,p) - W(t,q)) \Big).d\varrho.dp.dq. \end{aligned}$$

The sign changes because $\mathcal{I}'$ is odd. At this point, one computes half the difference of the preceding expressions in order to make the q-terms disappear. All this boils

down to

$$\frac{d}{dt}\int_0^1 |V(t,\varrho)-W(t,\varrho)|^2.d\varrho = -\int_{(0,1)^2}\Big((V(t,\varrho)-V(t,p))-(W(t,\varrho)-W(t,p))\Big) \times\Big(\mathcal{I}'(V(t,\varrho)-V(t,p))-\mathcal{I}'(W(t,\varrho)-W(t,p))\Big).dp.d\varrho \leq 0,$$

since $\mathcal{I}'$ is nondecreasing. Finally, we reach the contraction estimate which has been first proved in [24] for special potentials.

Appendix B. Proof of Theorem 4.1. We follow the same idea while aiming at computing $\frac{d}{dt}\int_0^1 |V(t,\varrho)-W(t,\varrho)|^2.d\varrho$ as in section A. Two terms appear on the right-hand side; the first one corresponds to the friction part and is to be treated exactly as in section A. For the second one, it is enough to observe that,

$$-\int_0^1 (V(t,\varrho)-W(t,\varrho))\partial_\varrho\left(\frac{1}{\partial_\varrho V(t,\varrho}-\frac{1}{\partial_\varrho W(t,\varrho)}\right).d\varrho = \int_0^1 (\partial_\varrho V(t,\varrho)-\partial_\varrho W(t,\varrho))\left(\frac{1}{\partial_\varrho V(t,\varrho)}-\frac{1}{\partial_\varrho W(t,\varrho)}\right).d\varrho \leq 0,$$

by monotonicity of the function $v\mapsto 1/v$. The boundary terms are zero thanks to:

$$\frac{1}{\partial_\varrho V(t,\varrho)} = f(t,V(t,\varrho)) = 0 \text{ for } \varrho = 0,1.$$

Appendix C. The structure of (6.2). It is interesting to rewrite in full detail the equation (6.2):

$$\partial_t\hat{u}(t,\xi) = -\frac{\xi^2}{8\pi^2}\int_{\mathbb{R}}\exp(-ix\xi)\left\{3\Big(\int_{\mathbb{R}} i\eta\hat{u}(t,\eta)\exp(ix\eta).d\eta\Big)^2+\xi^2\Big(\int_{\mathbb{R}}\hat{u}(t,\eta)\exp(ix\eta).d\eta\Big)^2\right\}.dx.$$

It trivially yields the conservation of mass since $\partial_t\hat{u}(t,0)=0$. The quantities between parentheses belong to $\mathbb{R}$. We use the fact that $a^2+b^2=(a+ib)(a-ib)$ and derive, after a simple change of variables:

$$\partial_t\hat{u}(t,\xi) = -\frac{3\xi^2}{8\pi^2}\int_{\mathbb{R}}\exp(-ix\xi)\Big(\int_{\mathbb{R}} i\eta\hat{u}\left(t,\eta-\frac{\xi}{\sqrt{3}}\right)\exp(ix\eta).d\eta\Big)\Big(\int_{\mathbb{R}} i\eta\hat{u}\left(t,\eta+\frac{\xi}{\sqrt{3}}\right)\exp(ix\eta).d\eta\Big).dx.$$

We can easily see that the terms between parentheses are now each other's complex conjugate since they read, e.g.,

$$\begin{aligned} M(x,\xi) &\overset{def}{=} \int_{\mathbb{R}} i\eta\hat{u}\left(t,\eta+\tfrac{\xi}{\sqrt{3}}\right)\exp(ix\eta).d\eta \\ &= \int_{\mathbb{R}^+} i\eta\left[\hat{u}\left(t,\eta+\tfrac{\xi}{\sqrt{3}}\right)\exp(ix\eta)-\overline{\hat{u}\left(t,\eta-\tfrac{\xi}{\sqrt{3}}\right)\exp(ix\eta)}\right].d\eta, \end{aligned}$$

where we have used that u being real, $\hat{u}(t,-\xi)=\overline{\hat{u}(t,\xi)}$ for all ξ. Moreover, the quantity appearing in the expression of $\partial_t\hat{u}(t,\xi)$, $\xi\mapsto M(x,\xi)\overline{M(x,\xi)}$, is an even function for any value of x. Using this while performing a backwards Fourier transform leads us to a simple expression for $\partial_t u(t,x)$,

$$\partial_t u(t,x)+\frac{1}{32\pi^3}\int_{\mathbb{R}^2}\xi^2\cos((y-x)\xi)M(x,\xi)\overline{M(x,\xi)}.dy.d\xi = 0,$$

which turns out to belong to $\mathbb{R}$. Hence the whole computation is coherent. This highlights a convolution structure inside the Hele-Shaw equation since one could rewrite it under the form:

$$\partial_t u(t,x) + \frac{1}{32\pi^3}\int_{\mathbb{R}^2} \xi^2 \Big(M(.,\xi)\overline{M(.,\xi)} * \cos(\xi x)\Big).d\xi = 0.$$

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