Kinetic Equations modelling Wealth Redistribution: A comparison of Approaches

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Kinetic equations modelling the redistribution of wealth in simple market economies is one of the major topics in the field of econophysics. We present a unifying approach to the qualitative study for a large variety of such models, which is based on a moment analysis in the related homogeneous Boltzmann equation, and on the use of suitable metrics for probability measures. In consequence, we are able to classify the most important feature of the steady wealth distribution, namely the fatness of the Pareto tail, and the dynamical stability of the latter in terms of the model parameters. Our results apply e.g. to the market model with risky investments [S. Cordier, L. Pareschi and G. Toscani, *J. Stat. Phys.* **120**, 253 (2005)], and to the model with *quenched saving propensities* [B.K. Chakrabarti, A. Chatterjee and S.S. Manna, *Physica A* **335**, 155 (2004)]. Also, we present results from numerical experiments that confirm the theoretical predictions.

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I. INTRODUCTION

In the rapidly growing field of *econophysics*, kinetic market models are presently of particular interest, see e.g. the various contributions in the recent books [6, 11, 33, 34], or the introductory articles [21, 31, 37]. The founding idea, dating back to the works of Mandelbrot [23], is that the laws of statistical mechanics govern the behavior of a huge number of interacting individuals just as well as that of colliding particles in a gas container. The classical theory for homogeneous gases is easily adapted to the new economic framework: molecules and their velocities are replaced by agents and their wealth, and instead of binary collisions, one considers trades between two individuals.

The model designer's input is the definition of rules on the *microscopic* level, i.e., the prescription of how money is exchanged in trades. Such rules are usually derived from plausible assumptions in an *ad hoc* manner. The corresponding output of the model are the *macroscopic* statistics of the wealth distribution in the society. It is commonly accepted that the wealth distribution approaches a stationary profile for large times, and that the latter exhibits a *Pareto tail.* Such overpopulated tails are a manifestation of the existence of an upper class of very rich agents, i.e. an indication of an unequal distribution of money. The various articles in [11] provide an overview over historical and recent studies on the shape of wealth distributions; see also [8] for a collection of relevant references.

A variety of models has been proposed and studied in view of the relation between parameters in the microscopic rules and the resulting macroscopic statistics. A typical ingredient on the microscopic level is a mechanism for saving, probably first introduced in [5]. It ensures that agents exchange at most a certain fraction of their wealth in each trade event; this is in contrast to the original molecular dynamics for gases. Moreover, randomness plays a rôle in virtually all available models, taking into account that many trades are risky, so that the exact amount of money changing hands is not known *a priori.* Depending on the specific choice of the saving mechanism and the stochastic nature of the trades, the studied systems produce wealth curves with the desired Pareto tail — or not.

In this paper, we analyze and compare a selection of recently developed models. The focus is on the class of models with *risky investments*, introduced by Cordier, Pareschi and one of the authors [15], and on variants of the model with *quenched saving propensities*, designed by Chakraborti, Chatterjee and Manna [9]. The applied analytical techniques, however, easily generalize to a broader class of economic games. Some alternative approaches, like the hydrodynamic limit, are briefly discussed.

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Our analysis is heavily based on specific results from the current mathematical literature [15, 17, 18, 24, 25, 28], where kinetic econophysics has been treated in the framework of Maxwell-type molecules, the Kac equation etc. These mathematical results are briefly reviewed, before they are applied to the specific models under consideration. In a separate section, our theoretical predictions are verified in a series of new numerical simulations.

The presented approach differs in several subtle points from the numerous theoretical and numerical studies that can be found in the recent physics literature on the subject. First, the analysis is entirely based on the spatially homogeneous *Boltzmann equation* associated to the microscopic trade rules of the respective model. Thus, we treat the agents on the market as a *continuum*, just like molecules in classical gas dynamics. Not only does this approach constitute the most natural generalization of the classical ideas to econophysics. But moreover, it clarifies that certain peculiar observations made in ensembles of finitely many agents and in numerical experiments — like the apparent creation of steady distributions of infinite average wealth, e.g. [8–10] — are genuine *finite size effects*. Second, we do not intend to derive explicit formulas for the solutions, nor do we investigate the wealth distribution for the poor agents. Instead, we provide relations that allow to calculate characteristic features, like the Pareto index of the steady money distribution, directly from the model parameters. Finally, apart from the shape of the steady states, we investigate their dynamical stability by estimating the speed of relaxation of transient solutions to stationarity. The 1-Wasserstein metric — briefly reviewed below — is used to estimate the distance between the wealth distribution at finite times, and the steady state.

II. PRELIMINARIES

A. Wealth distributions

In a closed ensemble of agents (i.e. a market), the *wealth distribution* $P(t; w)$ refers to the relative density of agents with wealth w at time $t \geq 0$. Debts are excluded in the models considered here, i.e. $P(t; w) = 0$ for $w < 0$, but concentration in $w = 0$ is allowed. The first moment of $P(t; w)$ yields the *average wealth per agent*,

$$M_1(t) = \int_0^\infty w\, P(t; w)\, dw. \tag{1}$$

In the models under consideration, the density $P(t; w)$ stabilizes at some *stationary wealth curve* $P_\infty(w)$ in the large-time limit $t \to \infty$. The central notion in the theory of wealth distributions is that of the celebrated *Pareto index* $\alpha \geq 1$. This number describes the size of the rich upper class in the considered ensemble of agents. Roughly, the smaller α is, the more of the total wealth is concentrated in the hands of a small group of individuals.

The stationary curve $P_\infty(w)$ satisfies the Pareto law [27] with index α, provided that P_∞ decays like an inverse power function for large w,

$$P_\infty(w) \propto w^{-(\alpha+1)} \quad \text{as } w \to +\infty. \tag{2}$$

More precisely, P_∞ has Pareto index $\alpha \in [1, +\infty)$ if the moments

$$M_s := \int_0^\infty w^s\, P_\infty(w)\, dw \tag{3}$$

are finite for all positive $s < \alpha$, and infinite for $s > \alpha$. If all M_s are finite (e.g. for a Gamma distribution), then P_∞ is said to possess a *slim tail*.

According to empirical data from ancient Egypt until today [8, 11], the wealth distribution among the population in a capitalistic country follows the Pareto law, with an index α ranging between 1.5 and 2.5. Slim tails are typical for societies with a rather uniform distribution of wealth. Intuitively, one may think of socialist countries.

B. Wasserstein distance

Since Monte Carlo simulations produce distributions of point masses instead of smooth curves, a good notion of *distance between measures* is important to quantify the convergence of numerical results to the continuous limit. The *Wasserstein distance* of two density functions $f_1(w)$, $f_2(w)$ is given by

$$\mathbf{W}[f_1, f_2] := \int \big|F_1(v) - F_2(v)\big|\, dv, \tag{4}$$

where the F_i denote the distribution functions,

$$F_i(v) = \int_v^\infty f_i(w)\,dw, \quad (i = 1, 2). \tag{5}$$

Equivalently, the Wasserstein distance is defined as the infimum of the costs for transportation [35],

$$\mathbf{W}[f_1, f_2] := \inf_{\pi\in\Pi} \int |v - w|\,d\pi(v, w). \tag{6}$$

Here Π is the collection of all measures in the plane $\mathbb{R}^2$ with marginal densities f_1 and f_2, respectively. The infimum is in fact a minimum, and is realized by some *optimal transport plan* π_{opt}. Convergence of densities $f(t; w)$ to a limit $f_\infty(w)$ in Wasserstein is equivalent to the weak convergence $f(t; w)dw \rightharpoonup f_\infty(w)dw$ in the sense of measures, and convergence of the first moments.

There is an intimate relation of Wasserstein to *Fourier metrics* [20], defined by

$$d_s[f_1, f_2] = \sup_k [|k|^{-s}|\hat{f}_1(k) - \hat{f}_2(k)|] \quad (s > 0). \tag{7}$$

For $s > 1$, the two are related [17] by

$$\mathbf{W}[f_1, f_2] \le C(d_s[f_1, f_2])^{-(s-1)/s(2s-1)}. \tag{8}$$

For further details, see e.g. [4].

Example: Two Dirac distributions have Wasserstein distance $\mathbf{W}[\delta_x, \delta_y] = |x - y|$. More generally, a density $f_1(v)$ and its translate $f_2(v) = f_1(v - z)$ have Wasserstein distance $\mathbf{W}[f_1, f_2] = |z|$. Thus, the Wasserstein distance provides a more sensible notion of "closeness" of densities than e.g. L^p-norms; observe that $\|\delta_x - \delta_y\|_{L^1} = 2$ unless $x = y$.

III. ONE-DIMENSIONAL MODELS — ANALYSIS

Here we consider a class of models in which agents are indistinguishable. Then, an agent's "state" at any instant of time $t \ge 0$ is completely characterized by his current wealth $w \ge 0$. When two agents encounter in a trade, their *pre-trade wealths* v, w change into the *post-trade wealths* v^*, w^* according to the rule

$$v^* = p_1 v + q_1 w, \quad w^* = q_2 v + p_2 w. \tag{9}$$

The *interaction coefficients* p_i and q_i are non-negative random variables. While q_1 denotes the fraction of the second agent's wealth transferred to the first agent, the difference $p_1 - q_2$ is the relative gain (or loss) of wealth of the first agent due to market risks. We assume that p_i and q_i have fixed laws, which are independent of v and w, and of time.

In one-dimensional models, the wealth distribution $P(t; w)$ of the ensemble is sufficient to describe the momentary configuration of the system. There is no need to distinguish between the wealth curve and agent density $f(t; w) = P(t; w)$. The latter satisfies the associated spatially homogeneous Boltzmann equation

$$\partial_t f + f = Q_+(f, f) \tag{10}$$

on the real half line, $w \ge 0$. The collisional gain operator Q_+ acts on test functions $\varphi(w)$ as

$$Q_+(f, f)[\varphi] := \int_0^\infty \varphi(w) Q_+\big(f, f\big)(w)\,dw = \frac{1}{2}\int_0^\infty \int_0^\infty \langle \varphi(v^*) + \varphi(w^*)\rangle f(v) f(w)\,dv\,dw, \tag{11}$$

with $\langle\cdot\rangle$ denoting the expectation with respect to the random coefficients p_i and q_i in (9). We restrict attention to models which conserve the average wealth of the society,

$$M := M_1(t) = \int_0^\infty w\, f(t; w)\,dw = const., \tag{12}$$

and we assume the value of M to be finite. In terms of the interaction coefficients, this is equivalent to $\langle p_1 + q_2\rangle = \langle p_2 + q_1\rangle = 1$.

A. Pareto tail of the wealth distribution

We introduce the characteristic function

$$\mathbf{S}(s) = \frac{1}{2}\Big(\sum_{i=1}^{2}\langle p_i^s + q_i^s\rangle\Big) - 1, \tag{13}$$

which is convex in $s > 0$, with $\mathbf{S}(0) = 1$. Also, $\mathbf{S}(1) = 0$ because of the conservation property (12). The results from [17, 24] imply the following. Unless $\mathbf{S}(s) \geq 0$ for all $s > 0$, any solution $f(t; w)$ tends to a steady wealth distribution $P_\infty(w) = f_\infty(w)$, which depends on the initial wealth distribution only through the conserved mean wealth $M > 0$. Moreover, exactly one of the following is true:

(PT) if $\mathbf{S}(\alpha) = 0$ for some $\alpha > 1$, then $P_\infty(w)$ has a *Pareto tail* of index α;

(ST) if $\mathbf{S}(s) < 0$ for all $s > 1$, then $P_\infty(w)$ has a *slim tail*;

(DD) if $\mathbf{S}(\alpha) = 0$ for some $0 < \alpha < 1$, then $P_\infty(w) = \delta_0(w)$, a *Dirac Delta* at $w = 0$.

To derive these results, one studies the evolution equation for the moments

$$M_s(t) := \int_0^\infty w^s\, f(t; w)\, dw, \tag{14}$$

which is obtained by integration of (10) against $\varphi(w) = w^s$,

$$\frac{d}{dt} M_s = Q_+[\varphi] - M_s. \tag{15}$$

Using an elementary inequality for $x, y \geq 0$, $s \geq 1$,

$$x^s + y^s \leq (x + y)^s \leq x^s + y^s + 2^{s-1}(xy^{s-1} + x^{s-1}y) \tag{16}$$

in (11), one calculates for the right-hand side of (15)

$$\mathbf{S}(s)M_s \leq Q_+[\varphi] - M_s \leq \mathbf{S}(s)M_s + 2^{s-2}\sum_{i=1}^{2}\langle p_i q_i^{s-1} + p_i^{s-1} q_i\rangle M M_s^{1-1/s}. \tag{17}$$

Solving (15) with (17), one finds that either $M_s(t)$ remains bounded for all times when $\mathbf{S}(s) < 0$, or it diverges like $\exp[t\mathbf{S}(s)]$ when $\mathbf{S}(s) > 0$, respectively.

In case (PT), exactly the moments $M_s(t)$ with $s > \alpha$ blow up as $t \to \infty$, giving rise to a Pareto tail of index α. We emphasize that $P(t; w)$ possesses finite moments of all orders at any finite time. The Pareto tail forms *in the limit* $t \to \infty$.

In case (ST), all moments converge to limits $M_s(t) \to M_s^*$, so the tail is slim. One can obtain additional information on the stationary wealth distribution $P_\infty(w)$ from the *recursion relation* for the principal moments,

$$-\mathbf{S}(s)M_s^* = \frac{1}{2}\sum_{k=1}^{s-1}\sum_{i=1}^{2}\binom{s}{k}\langle p_i^k q_i^{s-k}\rangle M_k^* M_{s-k}^* \quad (s = 2, 3, \ldots). \tag{18}$$

The latter is obtained by integration of (10) against $\varphi(w) = w^s$ in the steady state $\partial_t f = 0$.

In case (DD), all moments $M_s(t)$ with $s > 1$ blow up. The underlying process is a separation of wealth as time increases: while more and more agents become extremely poor, fewer and fewer agents possess essentially the entire wealth of the society. In terms of $f(t; w)$, one observes an accumulation in the pauper region $0 \leq w \ll 1$, while the density rapidly spreads into the region $w \gg 1$. The expanding support of $f(t; w)$ is balanced by a decrease in magnitude, since the average wealth is fixed. This induces a pointwise convergence $f(t; w) \to 0$ for all $w > 0$. Such a *condensation of wealth* has been observed and described in several contexts [2, 3, 12, 22] before.

An illustration of the solution's behavior in the (DD) case is provided by the "Winner takes all" dynamics, with rules

$$v^* = v + w, \quad w^* = 0. \tag{19}$$

In each trade, the second agent loses all of his wealth to the first agent. The solution for the initial condition $f(0; w) = \exp(-w)$ is explicit,

$$f(t; w) = \Big(\frac{2}{2+t}\Big)^2 \exp\Big(-\frac{2}{2+t}w\Big) + \frac{t}{2+t}\delta_0(w). \tag{20}$$

Note that the average wealth is conserved at all finite times $t \geq 0$, but vanishes in the limit.

B. Angel's model: strict wealth conservation

The first explicit description of a binary money exchange model dates back to Angel [1] (although the intimate relation to statistical mechanics was only described about one decade later [16, 22]): in each binary interaction, winner and loser are randomly chosen, and the loser yields a random fraction of his wealth to the winner. From here, Chakrabarti and Chakraborti [5] developed the class of *strictly conservative* exchange models, which preserve the total wealth in each individual trade,

$$v^* + w^* = v + w. \tag{21}$$

In its most basic version, the microscopic interaction is determined by one single parameter $\lambda \in (0,1)$, which is the global *saving propensity.* In interactions, each agent keeps the corresponding fraction of his pre-trade wealth, while the rest $(1-\lambda)(v+w)$ is equally shared among the two trade partners,

$$v^* = \lambda v + \frac{1}{2}(1-\lambda)(v+w), \quad w^* = \lambda w + \frac{1}{2}(1-\lambda)(v+w). \tag{22}$$

In result, all agents become equally rich eventually. Indeed, the stochastic variance of $f(t;w)$ satisfies

$$\frac{d}{dt}\int_0^\infty (w-M)^2 f(t;w)\,dw = -\frac{1}{2}(1-\lambda^2)\int_0^\infty (w-M)^2 f(t;w)\,dw. \tag{23}$$

The steady state $f_\infty(w) = \delta_M(w)$ is a Dirac Delta concentrated at the mean wealth, and is approached at the exponential rate $(1-\lambda^2)/2$.

More interesting, non-deterministic variants of the model have been proposed, where the amount $(1-\lambda)(v+w)$ is not equally shared, but in a stochastic way:

$$v^* = \lambda v + \epsilon(1-\lambda)(v+w), \quad w^* = \lambda w + (1-\epsilon)(1-\lambda)(v+w), \tag{24}$$

with a random variable $\epsilon \in (0,1)$. Independently of the particular choice of ϵ, the characteristic function

$$\mathbf{S}(s) = \frac{1}{2}\Big(\langle[\lambda+\epsilon(1-\lambda)]^s\rangle + \langle[1-\epsilon(1-\lambda)]^s\rangle + [\langle\epsilon^s\rangle + \langle(1-\epsilon)^s\rangle](1-\lambda)^s\Big) - 1 \tag{25}$$

is negative for all $s > 1$, hence case (ST) applies. Though the steady state f_∞ is no longer explicit — for approximations see [14, 30] — one concludes that its tail is slim. In conclusion, *no matter how sophisticated the trade mechanism is chosen, one-dimensional, strictly conservative trades always lead to narrow, "socialistic" distributions of wealth.*

C. CPT model: wealth conservation in the mean

Cordier et al. [15] have introduced the CPT model, which breaks with the paradigm of strict conservation. The idea is that money changes hands for a specific reason: one agent intends to *invest* his money in some asset, property etc. in possession of his trade partner. Typically, such investments bear some risk, and either provide the buyer with some additional wealth, or lead to the loss of wealth in a non-deterministic way. The random effect is chosen such that

- the total wealth of the trade partners increases or decreases in any individual interaction,
- gains and losses average out in the ensemble such that the mean wealth M is preserved.

An easy realization of this idea [24] consists in coupling the previously discussed rules (22) with some *risky investment* that yields an immediate gain or loss proportional to the current wealth of the investing agent,

$$v^* = \Big(\frac{1+\lambda}{2} + \eta_1\Big)v + \frac{1-\lambda}{2}w, \quad w^* = \Big(\frac{1+\lambda}{2} + \eta_2\Big)w + \frac{1-\lambda}{2}v, \tag{26}$$

The coefficients $\eta_1, \eta_2 \in (-\lambda, +\infty)$ are random parameters. Assuming that they are centered, $\langle\eta_i\rangle = 0$, the society's mean wealth is preserved on the average,

$$\langle v^* + w^*\rangle = (1+\langle\eta_1\rangle)v + (1+\langle\eta_2\rangle)w = v + w. \tag{27}$$

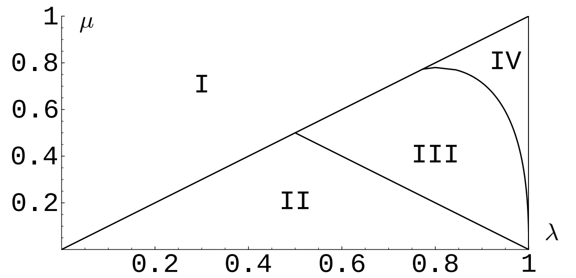

FIG. 1: Regimes for the formation of Pareto tails.

Various specific choices for the η_i have been discussed [24]. The easiest one leading to interesting results is $\eta_i = \pm\mu$, where each sign comes with probability 1/2. The factor $\mu \in (0, \lambda)$ should be understood as the *intrinsic risk* of the market: it quantifies the fraction of wealth agents are willing to gamble on. Figure 1 displays the various regimes for the steady state f_∞ in dependence of λ and μ, which follow from numerical evaluation of

$$\mathbf{S}(s) = \frac{1}{2}\Big[\Big(\frac{1+\lambda}{2} - \mu\Big)^s + \Big(\frac{1+\lambda}{2} + \mu\Big)^s\Big] + \Big(\frac{1-\lambda}{2}\Big)^s - 1. \tag{28}$$

In zone II, corresponding to low market risk, the wealth distribution shows again "socialistic" behavior with slim tails. Increasing the risk, one falls into "capitalistic" zone III, where the wealth distribution displays the desired Pareto tail. A minimum of saving ($\lambda > 1/2$) is necessary for this passage; this is expected since if money is spent too quickly after earning, agents cannot accumulate enough to become rich. Inside zone III, the Pareto index α decreases from $+\infty$ at the border with zone II to unity at the border to zone IV. Finally, in zone IV, the steady wealth distribution is a Delta in zero. Both risk and saving propensity are so high that a marginal number of individuals manages to monopolize all of the society's wealth. In the long-time limit, these few agents become infinitely rich, leaving all other agents truly pauper.

D. Rates of relaxation

In the cases (PT) and (ST), the transient solution $f(t;w)$ converges to the respective limit $f_\infty(w)$ exponentially fast in Wasserstein metric,

$$\mathbf{W}[f(t), f_\infty] \leq C \exp\Big(-\frac{(\bar{s}-1)\mathbf{S}(\bar{s})}{\bar{s}(2\bar{s}-1)}t\Big), \tag{29}$$

where $\bar{s} < 2$ can be any number with $\mathbf{S}(\bar{s}) < 0$. In the derivation [24], one first shows that $f(t)$ converges exponentially fast in Fourier metrics (7) with $s = \bar{s}$,

$$d_{\bar{s}}[f(t), f_\infty] \leq d_{\bar{s}}[f(0), f_\infty] \exp(-\mathbf{S}(\bar{s})t). \tag{30}$$

From (30), convergence in (29) follows by the relation (8). To verify (30), rewrite the Boltzmann equation (10) with kernel (13) in its *Fourier representation*,

$$\partial_t \hat{f} = \widehat{Q}_+(\hat{f}, \hat{f}) - \hat{f}, \quad \widehat{Q}_+(\hat{f}, \hat{f}) = \frac{1}{2}\sum_{i=1}^{2} \langle \hat{f}(p_i k)\hat{f}(q_i k)\rangle. \tag{31}$$

To conclude non-expansivity of $\widehat{Q}_+$, it suffices to observe

$$\begin{aligned} |k|^{-s}|\widehat{Q}_+(\hat{f}, \hat{f}) - \widehat{Q}_+(\hat{f}_\infty, \hat{f}_\infty)| &\leq \frac{1}{2}\sum_{i=1}^{2} |k|^{-s}\langle|\hat{f}(p_i k)\hat{f}(q_i k) - \hat{f}_\infty(p_i k)\hat{f}_\infty(q_i k)|\rangle \\ &\leq \frac{1}{2}\sum_{i=1}^{2}\langle p_i^s + q_i^s\rangle |k'|^{-s}|\hat{f}(k') - \hat{f}_\infty(k')| \\ &\leq [1 + \mathbf{S}(s)] d_s[f, f_\infty]. \end{aligned}$$

Here $k' = p_i k$ or $k' = q_i k$, respectively. In case (DD), the transient wealth distributions converge exponentially fast in Fourier metrics, and weakly in the sense of measures to the Delta. They *do not* converge in Wasserstein metrics, since the first moment equals to $M > 0$ at all finite times, but is zero in the limit.

IV. TWO-DIMENSIONAL MODELS — ANALYSIS

The *Chakrabarti-Chatterjee-Manna* (CCM) model constitutes another improvement of Angel's original game. Arguing that agents are not indistinguishable in reality, but have personal trading preferences, Chakrabarti et al. [9] introduced the concept of *quenched saving propensity.* Now λ is not a *global quantity,* but characterizes the agents. The current "state" of an agent is consequently described by *two* numbers, his wealth $w > 0$ *and* his personal saving propensity $\lambda \in (0, 1)$. We shall only discuss the case where λ does not change with time. Trade rules which allow the agents to adapt their saving strategy in time ("annealed saving") have been investigated [7, 9], but seemingly do not exhibit genuinely novel effects.

The configuration of the kinetic system is described by the *extended density* function $f(t; \lambda, w)$. The wealth distribution $P(t; w)$ is recovered from $f(t; \lambda, w)$ as marginal,

$$P(t; w) = \int_0^1 f(t; \lambda, w)\, d\lambda, \tag{32}$$

but is no longer sufficient to characterize the configuration completely. The other marginal yields the time-independent density of saving propensities,

$$\rho(\lambda) = \int_0^\infty f(t; \lambda, w)\, dw. \tag{33}$$

Clearly, $\rho(\lambda)$ is determined by the initial condition $f(0; \lambda, w)$, and should be considered as defining parameter of the model. The collision rules are the same as originally (24), but take into account the individual characteristics: two agents with pre-trade wealth v, w and saving propensities λ, μ, respectively, exchange wealth according to

$$v^* = \lambda v + \epsilon[(1 - \lambda)v + (1 - \mu)w], \tag{34}$$
$$w^* = \mu w + (1 - \epsilon)[(1 - \lambda)v + (1 - \mu)w]. \tag{35}$$

Clearly, money is strictly conserved, $v^* + w^* = v + w$, so the mean wealth M is constant in time. The Boltzmann equation (10) is now posed on a two-dimensional domain, $(\lambda, w) \in (0, 1) \times (0, \infty)$. The collisional gain operator Q_+ satisfies

$$Q_+(f, f)[\varphi](\lambda) = \int_0^\infty dv \int_0^\infty dw \int_0^1 d\mu \langle\varphi(v^*)\rangle f(\lambda, v) f(\mu, w) \tag{36}$$

after integration against a regular test function $\varphi(w)$. For simplicity, we assume that ϵ is symmetric around 1/2.

A. Pareto tail of the wealth distribution

Due to its two-dimensionality, the CCM model behaves very different from the strictly conservative model (24). In particular, $P_\infty(w)$ may possess a Pareto tail. In analogy to $\mathbf{S}(s)$ from (13), define the function

$$\mathbf{Q}(r) := \int_0^1 \frac{\rho(\lambda)}{(1 - \lambda)^r}\, d\lambda, \tag{37}$$

which determines the properties of the steady wealth distribution $P_\infty(w)$ as follows [25]:

(PT') if $\mathbf{Q}(1) < +\infty$, and $\alpha \in [1, +\infty)$ is the infimum of r for which $\mathbf{Q}(r) = +\infty$, then $P_\infty(w)$ has a Pareto tail of index α;

(ST') if $\mathbf{Q}(r) < +\infty$ for all $r \geq 1$, then $P_\infty(w)$ has a slim tail;

(DD') if $\mathbf{Q}(1) = +\infty$, then $P_\infty(w) = \delta_0(w)$.

To derive these results, it is useful to think of the global wealth distribution $P_\infty(w)$ as superposition of λ-specific steady wealth distributions $f_\infty(\lambda,w)/\rho(\lambda)$, i.e., the wealth distributions of all agents with a certain personal saving propensity λ. The individual λ-specific distributions are conjectured [9, 29] to resemble the wealth distributions associated to the one-dimensional model (24), but their features are so far unknown. However, they are conveniently analyzed in terms of the λ-specific moments

$$\hat{M}_s^*(\lambda) = \frac{1}{\rho(\lambda)} \int_0^\infty w^s\, f_\infty(\lambda,w)\, dw. \tag{38}$$

Integration of the stationary Boltzmann equation

$$f_\infty(\lambda,w) = Q_+(f_\infty, f_\infty) \tag{39}$$

against $\varphi(w) = w^s$ for a *non-negative integer* s gives

$$\hat{M}_s^*(\lambda) \;=\; \frac{1}{g(\lambda)} \int_0^\infty dv \int_0^1 d\mu \int_0^\infty dw \big\langle \big([\lambda+\epsilon(1-\lambda)]v + \epsilon(1-\mu)w\big)^s \big\rangle f_\infty(\lambda,v) f_\infty(\mu,w)$$

After simplifications,

$$(1-\lambda)\phi_s(\lambda)\hat{M}_s^*(\lambda) = \sum_{k=0}^{s-1} \binom{s}{k} \langle \epsilon^{s-k}[\lambda+\epsilon(1-\lambda)]^k \rangle \hat{M}_s^*(\lambda) \int_0^1 (1-\mu)^{s-k} \hat{M}_{s-k}^*(\mu) g(\mu) d\mu, \tag{40}$$

where $\phi_s(\lambda)$ is a polynomial with no roots in $[0,1]$. *The λ-specific steady wealth distributions have slim tails, and moments of arbitrary order can be calculated recursively from (40).* From

$$\hat{M}_0^*(\lambda) \equiv 1, \quad \hat{M}_1^*(\lambda) = \frac{M}{\mathbf{Q}(1)}(1-\lambda)^{-1}, \tag{41}$$

it follows inductively that

$$\hat{M}_s^*(\lambda) = r_s(\lambda)(1-\lambda)^{-s}, \tag{42}$$

and $r_s(\lambda)$ is a continuous, strictly positive function for $0 \le \lambda \le 1$. By Jensen's inequality, formula (42) extends from integers s to all real numbers $s \ge 1$. In conclusion, the total momentum

$$M_s^* = \int_0^1 \hat{M}_s^*(\lambda)\, d\lambda \propto \int_0^1 \frac{\rho(\lambda)}{(1-\lambda)^s} d\lambda \tag{43}$$

is finite exactly if $\mathbf{Q}(s)$ is finite.

Remark: $\mathbf{Q}(1) = +\infty$ would imply infinite average wealth per agent in the steady wealth distribution by formula (43). This clearly contradicts the conservation of the mean wealth at finite times. In reality, the first moment vanishes, and P_∞ is a Dirac distribution; see Sect. IV C.

We emphasize this fact since a noticeable number of theoretical and numerical studies has been devoted to the calculation of P_∞ for *uniformly distributed* λ, i.e. $\rho(\lambda) \equiv 1$, where clearly $\mathbf{Q}(1) = +\infty$. In the corresponding experiments [7, 9, 10, 13, 29] with finite ensembles of N agents, an almost perfect Pareto tail $P_\infty(w) = C_N w^{-2}$ of index $\alpha = 1$ has been observed over a wide range $w_N < w < W_N$. However, the "true" tail of $P_\infty(w)$ — for $w \gg W_N$ — is slim. As the systems size N increases, also $W_N \propto N$ increases and $C_N \propto 1/\log N \to 0$. In fact, one proves [25] weak convergence of $P_\infty(w)$ to $\delta_0(w)$ in the thermodynamic limit $N \to \infty$.

B. Rates of relaxation: Pareto tail

The discussion of relaxation is more involved than in one dimension, and we restrict our attention to the deterministic CCM model, $\epsilon \equiv 1/2$, in the case (PT') of Pareto tails of index $\alpha > 1$. In fact, it is believed [10] that the randomness introduced by ϵ has little effect on the large-time behavior of the kinetic system.

The stationary state of the deterministic CCM model is characterized by the *complete stop of wealth exchange*. This is very different from the steady states for the one-dimensional models, where the macroscopic wealth distribution is stationary despite the fact that money *is* exchanged on the microscopic level. Stationarity in (34) and (35) is achieved precisely if $v(1-\lambda) = w(1-\mu)$ for arbitrary agents with wealth v, w and saving propensities λ, μ, respectively. Correspondingly, the particle density concentrates in the plane on the curve

$$K_\infty = \{(\lambda, w) | (1-\lambda)w = M/\mathbf{Q}(1)\}, \tag{44}$$

and the steady wealth distribution is explicitly given by Mohanty's formula [26],

$$P_\infty(w) = \frac{M}{w^2}\rho\Big(1 - \frac{M}{w}\Big), \tag{45}$$

with the convention that $\rho(\lambda) = 0$ for $\lambda < 0$.

The conjectured [8, 13] time scale for relaxation of solutions is $t^{-(\alpha-1)}$,

$$\lim_{t\to\infty} \frac{\log \mathbf{W}[P(t;w), P_\infty(w)]}{-\log t} = \alpha - 1. \tag{46}$$

It has been proven [17] for all $\alpha > 1$ that the limit in (46) is *at most* $\alpha - 1$, i.e. relaxation cannot occur on a faster time scale. The complete statement (46), however, was made rigorous only for $1 < \alpha < 2$ so far [25].

The key tool for the analysis is the equation for the λ-specific mean wealth,

$$\frac{d}{dt}\hat{M}_1(t;\lambda) = -\frac{1-\lambda}{2}\hat{M}_1(t;\lambda) + \int_0^1 \frac{1-\mu}{2}\hat{M}_1(t;\mu)g(\mu)\,d\mu. \tag{47}$$

Intuitively, the slow algebraic relaxation is explained by the temporal behavior of the richest agents. By (47), the λ-specific average wealth $\hat{M}_1(t;\lambda)$ grows at most linearly in time,

$$\hat{M}_1(t;\lambda) \le t + \hat{M}_1(0;\lambda). \tag{48}$$

Thus, the tail of the wealth curve $P(t;w)$ becomes slim for $w \gg t$. The cost of transportation in (6) to "fill up" the fat tail $P_\infty(w) \propto w^{-(\alpha+1)}$ is approximately given by

$$\int_t^\infty w\,P_\infty(w;t)\,dw \propto \int_t^\infty w^{-\alpha}\,dw \propto t^{-(\alpha-1)}. \tag{49}$$

That equilibration works *no slower* than this (at least for $1 < \alpha < 2$) follows from a detailed analysis of the relaxation process. In [25], it has been proven that

$$\int_0^1 \Big|\hat{M}_1(t;\lambda) - \frac{M}{\lambda\mathbf{Q}(1)}\Big|\,\rho(\lambda)\,d\lambda \propto t^{-(\alpha-1)} \tag{50}$$

by relating (47) to the radiative transfer equation [19]. Moreover, the λ-specific variance

$$\hat{V}(t;\lambda) = \hat{M}_2(t;\lambda) - \hat{M}_1(t;\lambda)^2 \tag{51}$$

was shown to satisfy

$$\int_0^1 (1-\lambda)^2\hat{V}(t;\lambda)g(\lambda)\,d\lambda \propto t^{-\alpha} \tag{52}$$

provided $1 < \alpha < 2$. Combination of (50) and (52) leads to (46).

Moreover, relaxation may be decomposed into two processes. The first is concentration of agents at the λ-specific mean wealth $\hat{M}_1(t;\lambda)$; i.e., all agents with the same saving propensity become approximately equally rich. According to (52), this process happens on a time scale $t^{-\alpha/2}$. Second, the localized mean values tend towards their respective terminal values $M/\lambda\mathbf{Q}(1)$. Thus, agents of the same saving propensity simultaneously "adjust" their wealth. By (50), the respective time scale is $t^{-(\alpha-1)}$, which is indeed slower than the first provided $\alpha < 2$.

C. Rates of relaxation: Dirac delta

Finally, the deterministic CCM model is considered with a density $\rho(\lambda)$ where $\rho(1) > 0$, e.g. $\rho(\lambda) \equiv 1$. Clearly, $\mathbf{Q}(1) = +\infty$. An analysis of (47) provides [25] for $\lambda < 1$ the estimate

$$\frac{c}{1-\lambda} \le \log t \cdot \hat{M}_1(t;\lambda) \le \frac{C}{1-\lambda} \quad (t > T_\lambda), \tag{53}$$

with $0 < c < C < +\infty$, and $T_\lambda \to +\infty$ as $\lambda \to 1$. Convergence of $P(t; w)$ to a Delta in $w = 0$ is a direct consequence, since for each $0 \le \lambda < 1$, $\hat{M}_1(t; \lambda)$ tends to zero as $t \to \infty$.

Estimate (53) has a direct interpretation. Agents of very high saving propensity $\lambda \approx 1$ drain all wealth out of the remaining society as follows. At intermediate times $t \gg 1$, agents equilibrate in microscopic trades so that the product $(1 - \lambda)w$ becomes approximately a global constant $m(t)$. Agents with low saving propensity $\lambda < 1 - m(t)/t$ indeed satisfy $w \approx m(t)/(1 - \lambda)$. Agents with higher saving propensity, however, are in general far from this (apparent) equilibrium; their target wealth $m(t)/(1-\lambda)$ is very large, whereas their actual wealth is bounded by t on the average. Correspondingly, a "Pareto region" of the shape $P(t; w) \approx \rho(1)m(t)w^{-2}$ forms over a range $1 \ll w \le t$, whereas the tail of $P(t; w)$ for $w \gg t$ is slim. The average wealth per agent contained in the Pareto region amounts to

$$\int_1^t w\, P(t; w)\, dw \approx \rho(1)m(t) \log t. \tag{54}$$

By conservation of the average wealth, the global constant $m(t)$ tends to zero logarithmically in t and gives rise to (53).

V. OTHER APPROACHES

A. Dynamical rescaling

A crucial assumption made for the models considered in detail so far is the conservation (at least in a statistical sense) of the average wealth per agent, i.e. the first moment of the wealth distribution, over time. Wealth conservation sounds plausible on a microscopic level, whereas on a macroscopic level, it is arguable that the apparent conservation is in reality a mixture of two effects. On one hand, *wealth is created* through the production of goods, interests on savings etc. On the other hand, (monetary) *wealth is lost* through inflation.

Kinetic models which take these two effects into account, were proposed by Slanina [32], and were further developed by Pareschi et al. [28]. In order to incorporate the *creation* of wealth, the respective trade rules are designed to "reward" agents for trading activity. In the CPT model (26), this can be achieved by assuming that the market risk satisfies $\langle \eta_i \rangle = \varepsilon > 0$. In other words, the risky investment is more likely to create additional wealth, than to destroy existing wealth. *This is a genuine motivation for agents to engage in trades!* The effect of inflation is modelled by a *time-dependent rescaling* $f(t; w) \rightsquigarrow g(t; v)$ of the wealth distribution,

$$g(t; v) = e^{\varepsilon t} f(t; w), \quad w = e^{\varepsilon t} v, \tag{55}$$

chosen so that the mean wealth of $g(t; v)$ is kept constant. *The monetary unit is adapted in a way that people stay equally wealthy on the average.* The Boltzmann equation (10) is respectively modified by an additional drift term,

$$\partial_t g = Q_+(g, g) - g + \varepsilon \partial_v (v\, g). \tag{56}$$

Solutions to this equation have been proven [28] to converge to a steady state g_∞, which may or may not have a Pareto tail. Again, the evolution of moments can be analyzed, and leads to a classification of the tail size in terms of properties of the η_i.

B. Fokker-Planck equations

Apart from an investigation of moments, the Boltzmann equations (10) or (56) are hard to analyze, even in the stationary regime. The *grazing collision limit* provides a method to generate from the kinetic equation a *Fokker-Planck equation*, i.e. a parabolic differential equation of second order, which is better accessible.

Consider the CPT model (26) with saving propensity $\lambda = 1 - \beta^2$ and market risks $\beta\eta_i$,

$$v^* = (1 - \beta^2)v + \beta\eta_1 v + \beta^2 w, \quad w^* = (1 - \beta^2)w + \beta\eta_2 w + \beta^2 v. \tag{57}$$

where $\beta > 0$ is a small parameter, and η_1 and η_2 are two equally distributed, centered random variables with $\sigma^2 := \langle \eta_i^2 \rangle$.

Expand of the collisional operator in terms of β,

$$
\begin{aligned}
& Q_+(f,f)[\varphi] - \int_0^\infty \varphi(v)dv \\
= & \int_0^\infty dv \int_0^\infty dw \Big(\varphi'(v)[\beta^2(w-v) + \beta\langle\eta\rangle v] + \frac{1}{2}\varphi''(v)[\beta^2(w-v) + \beta\eta v]^2 + O(\beta^4) \Big) f(v)f(w) \\
= & \beta^2 \int_0^\infty \varphi(v) \Big(-\partial_v[(M-v)f(v)] + \frac{\sigma^2}{2}\partial_v^2[v^2 f(v)] \Big)\, dv + O(\beta^4).
\end{aligned}
$$

Finally, increase the collision frequency by rescaling $t \rightsquigarrow t/\beta^2$. In the limit $\beta \to 0$, the Boltzmann equation turns into the Fokker-Planck equation

$$
\partial_t f = \frac{\sigma^2}{2}\partial_v^2[v^2 f] - \partial_v[(M-v)f], \tag{58}
$$

which possesses an explicit stationary solution,

$$
f_\infty(v) = C_{\sigma,M} \exp(-\frac{2M}{\sigma^2 v}) v^{-(2+2/\sigma^2)}. \tag{59}
$$

The solution f_∞ constitutes an approximation of the steady state of the respective (kinetic) CPT model for sufficiently small $\beta > 0$ [28]. For instance, in agreement with results on the CPT model, more risky trades (larger σ) induce fatter Pareto tails (decreasing index $\alpha = 1 + 2/\sigma^2$).

C. Hydrodynamic limit

In [18], a two-dimensional model is proposed, where the density $f(t;x,w)$ depends both on the wealth w, and on the propensity to trade $0 < x < 1$ (morally, $x = 1 - \lambda$). Trade interactions work like in the CPT model (26). In addition, agents adjust their propensity x in time, in dependence of their current wealth w,

$$
\dot{x} = \Phi(x,w) = (w - \chi\bar{w})\mu(x). \tag{60}
$$

Here χ is a positive constant and $\bar{w}$ represent a suitable fixed value of the wealth. The choice of the function $\mu(x) = \vartheta x^\alpha (1-x)^\beta$ is motivated by recent results on opinion formation.

Assuming that the majority of trades takes place between agents of comparable propensity, the following *inhomogeneous* Boltzmann equation results:

$$
\partial_t f + \Phi(x,w)\partial_x f = \frac{1}{\tau}[Q_+(f,f) - f]. \tag{61}
$$

The relaxation time τ is related to the velocity of money circulation [36], and acts analogously to the Knudsen number.

In the regime of fast relaxation $\tau \to 0$, hydrodynamic equations are derived from (61), which are the Euler equations for the economic system. Integration of (61) against test functions $\varphi(w) \equiv 1$ and $\varphi(w) = w$, respectively, gives

$$
\partial_t \rho + \mu(x)\partial_x \Big[\rho\big(m - \chi\bar{w}\big)\Big] = 0, \tag{62}
$$

$$
\partial_t(\rho m) + \mu(x)\partial_x \left[\int_0^\infty w^2 f(t;x,w)\, dw - \chi\bar{w}\rho m\right] = 0. \tag{63}
$$

The implicitly defined macroscopic variables are the *local density* $\rho(t;x) = \int f(t;x,w)\, dw$ of agents with propensity x, and the *local mean* $m(t;x) = \rho(t;x)^{-1}\int w f(t;x,w)\, dw$.

Equation (63) contains the second moment of the density, that needs to be expressed in terms of $\rho(t;x)$ and $m(t;x)$. An appropriate *closure* is obtained replacing $f(t;x,w)$ by a *local equilibrium state* $M_f(t;x,w)$. The state M_f possesses the same local density ρ and momentum m as f, but in addition satisfies the stationary Boltzmann equation $Q_+(M_f) = M_f$ in w, at each time t and propensity x. Thus, the *unknown* stationary solution of the CPT model plays the same rôle as the local Maxwell distribution in the kinetic theory of rarefied gases.

Regardless of the fact that the exact shape of the local equilibrium is unknown, the second moment of $M_f(t;x,w)$ can be evaluated explicitly by means of the recursion relation (18), leading to

$$
M_2^*(t;x) = \kappa\rho(t;x)m(t;x)^2, \quad \kappa = \frac{2\langle p_1 q_1 + p_2 q_2\rangle}{2 - \langle p_1^2 + p_2^2 + q_1^2 + q_2^2\rangle}. \tag{64}
$$

The characteristics of the underlying kinetic model thus enter into the Euler equations only through the constant $\kappa > 0$. In conclusion, (63) becomes

$$\partial_t(\rho m) + \mu(x)\partial_x\Big[\rho m\Big(\kappa m - \chi\bar{w}\Big)\Big] \; = \; 0. \tag{65}$$

In analogy to the Euler equations, (62)&(65) form a symmetric hyperbolic system.

VI. NUMERICAL EXPERIMENTS

To verify the analytical results for the relaxation behavior, we have performed a series of kinetic Monte Carlo simulations for both the CPT and the CCM model. In these rather basic simulations, known as direct simulation Monte Carlo (DSMC) method or Bird's scheme, pairs of agents are randomly and non-exclusively selected for binary collisions, and exchange wealth according to the respective trade rules. One *time step* corresponds to $N/2$ such interactions, with N denoting the number of agents. In all experiments, every agent possesses unit wealth initially.

The state of the kinetic system at time $t > 0$ is characterized by the N wealth values $w_1(t), \dots, w_N(t)$ in the CPT simulations, and additionally by the saving propensities $\lambda_1, \dots, \lambda_N$ for CCM. The densities for the current wealth $P^{(N)}(t;w)$ and the steady state $P_\infty^{(N)}$ are each a collection of scaled Dirac Deltas at positions w_i. The associated distribution functions are build of a sequence of rectangles,

$$F^{(N)}(t;w) = \#\{\text{agents with wealth } w_i(t) > w\}/N,$$

and respectively for $F_\infty^{(N)}(w)$.

We monitor the convergence of the wealth distribution $P^{(N)}(t;w)$ to the approximate steady state $P_\infty^{(N)}(w)$ over time in terms of the Wasserstein-one-distance. This amounts to computing the area between the two distribution functions $F^{(N)}(t;w)$ and $F_\infty^{(N)}(w)$, which is performed as follows. We start with two arrays of length N, one containing the current wealth values $w_i(t)$, and one the steady state data $w_i(\infty)$. We concatenate these arrays, sort them in ascending order, and compute the array of differences between consecutive elements. This array represents the *widths* of the rectangles. To construct the array of the rectangles' *heights*, we concatenate two arrays of length N containing the entries $1/N$ and $-1/N$, respectively, into one, and permute it in the same way as the wealth vector in the step before. The absolute value of this array's cumulative sum represents the heights. The Wasserstein-one-distance is now readily obtained by evaluation of the scalar product of width and height vector.

A. CPT model

We investigate the relaxation behavior of the CPT model (26) when the random variables η_1, η_2 attain values $\pm\mu$ with probability 1/2 each. According to the analytical results, the shape of the steady state can be determined from Figure 1. We report results for zones II and III. Recall that zone I is forbidden by the constraint $|\mu| < \lambda$, whereas parameters in zone IV lead to wealth condensation (without convergence in Wasserstein metrics). For zones II and III we run simulations for systems consisting of $N = 500$, $N = 5000$ and $N = 50000$ agents, respectively.

The relaxation in the CPT model occurs exponentially fast. Though the system has virtually reached equilibrium after less than 10^2 time steps, simulations are performed for 10^4 time steps. In order to obtain a smooth result, the wealth distribution is averaged over another 10^3 time steps. The resulting reference state $P_\infty^{(N)}$ is used in place of the (unknown) steady wealth curve.

For zones II and III we have chosen a risk index of $\mu = 0.1$, and a saving propensity of $\lambda \equiv 0.7$ for zone II and $\lambda \equiv 0.95$ for zone III, respectively. The non-trivial root of $\mathbf{S}(s)$ in (13) is $\bar{s} \approx 12.91$ in the latter case. For each choice of N and each pair (μ, λ), we averaged over 100 simulations. Figure 2 shows the decay of the Wasserstein-one-distance of the wealth distribution to the approximate steady state over time. In both zones, we observe exponential decay. The reason for the residual Wasserstein distance of order 10^{-2} lies in the statistical nature of this model, which *never* reaches equilibrium in finite-size systems, due to persistent thermal fluctuations. Note that before these fluctuations become dominant, relaxation is extremely rapid. The exponential rate is independent of the number of agents N.

B. CCM model

The CCM model is expected to relax at an algebraic rate (46). As simulations indeed take much longer to reach equilibrium than in the case of CPT, the numerical experiments are carried out for about 10^5 time steps, and then

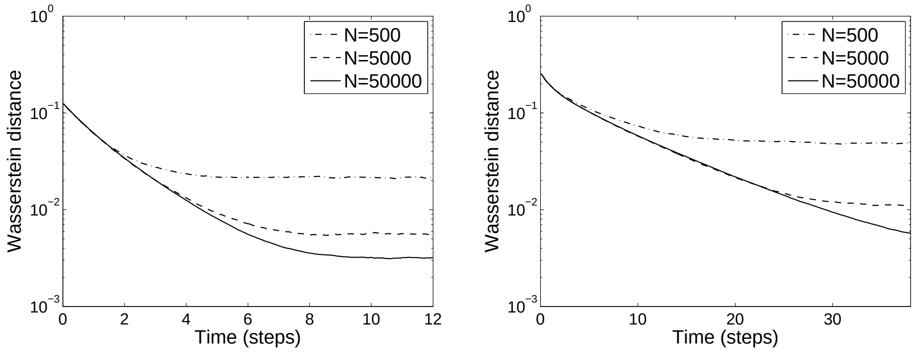

FIG. 2: CPT model: Decay of the Wasserstein distance to the steady state in zones II (left) and III (right).

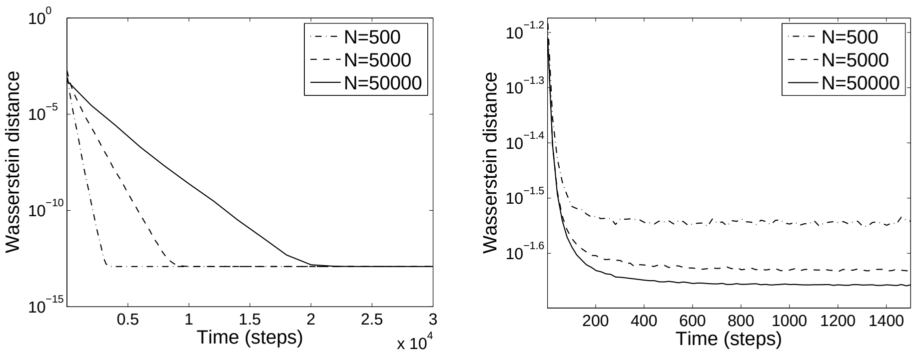

FIG. 3: CCM model: Decay of the averaged Wasserstein distance to the steady states for $\varepsilon \equiv 1/2$ and for $\varepsilon \in (0.4, 0.6)$ uniformly distributed.

the wealth distribution is averaged over another 10^4 time steps. Again, this reference state is used in place of the (unknown) steady wealth curve. The saving propensities for the agents are assigned at the beginning of each run and are kept fixed during this simulation. Agents are assigned the propensities $\lambda_j = 1 - \omega_j^{1/2.5}$, where the $\omega_j \in (0, 1)$ are realizations of a uniformly distributed random variable. Simulations are performed for the deterministic situation $\epsilon \equiv 1/2$ as well as for uniformly distributed $\epsilon \in (0.4, 0.6)$. In both situations, computations are carried out for systems consisting of $N = 500$, $N = 5000$ and $N = 50000$ agents, respectively.

The steady state reached in one simulation is typically non-smooth, and smoothness is only achieved by averaging over different simulations. However, in contrast to the CPT model, the steady states for CCM *do* depend on the initial conditions, namely through the particular realization of the distribution of saving propensities $\lambda_1, \ldots, \lambda_N$ among the agents. Consequently, there are two possibilities to calculate the relaxation rates. One can monitor *either* the convergence of the wealth distributions in one run to the steady distribution corresponding to that specific realization of the saving propensities, *or* the convergence of the transient distributions, obtained from averaging over several simulations, to the single smooth steady state that results from averaging the simulation-specific steady states.

Figure 3 shows the evolution of the Wasserstein-one-distance of the wealth distributions to the individual steady states, both in the purely deterministic setting $\epsilon \equiv 1/2$ (left), and for uniformly distributed $\epsilon \in (0.4, 0.6)$. (The curves in the figures represent *averages* of the Wasserstein distances calculated in the individual simulations.) In comparison, the distance of the simulation-averaged wealth distributions to the single (averaged) steady state is display in Figure 4. Again, results are shown for $\epsilon \equiv 1/2$ (left), and for uniformly distributed $\epsilon \in (0.4, 0.6)$, respectively.

Some words are in order to explain the results. The almost perfect exponential — instead of algebraic — decay displayed in Figure 3 obviously originates from the finite size of the system. The exponential rates decrease as the system size N increases. In the theoretical limit $N \to \infty$, one expects sub-exponential relaxation as predicted by the theory. We stress that, in contrast, the exponential decay rate for the CPT model in Figure 2 is independent of the system size.

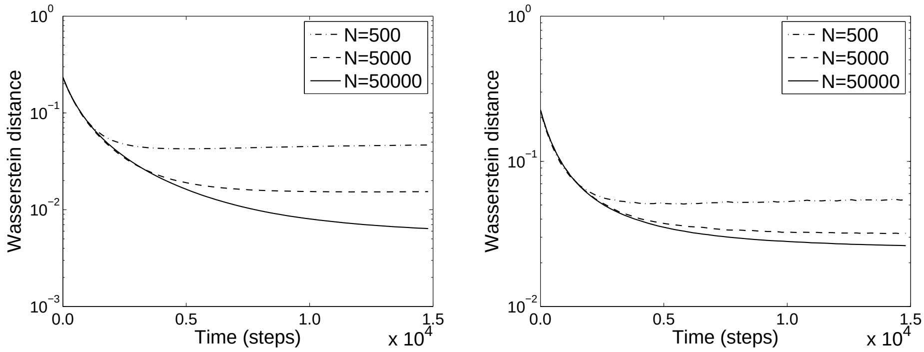

FIG. 4: CCM model: Decay of the Wasserstein distance to the averaged steady state for $\varepsilon \equiv 1/2$ (left) and for $\varepsilon \in (0.4, 0.6)$ uniformly distributed (right).

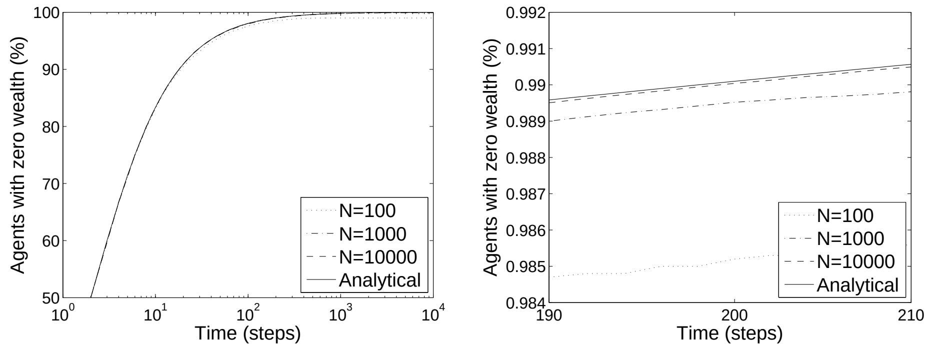

FIG. 5: "Winner takes all" model: Evolution of the fraction of agents with zero wealth (left) and blow up (right).

C. Winner takes all

Finally, the "Winner takes all" model (19) is simulated. As time evolves, all agents but one become pauper and give rise to a Dirac Delta at $w = 0$. We run $M = 100$ simulations for systems consisting of $N = 100$, $N = 1000$ and $N = 10000$ agents, respectively. Figure 5 displays the — simulation-averaged — fraction of the population with zero wealth. This fraction of pauper agents grows linearly until a saturation effect becomes visible. The blow up figure shows the improving approximation of the theoretically predicted rate for growing system size.

VII. CONCLUSIONS

We have reviewed and compared various approaches to model the dynamics of wealth distribution in simple market economies. The considered models were based on a kinetic description of the binary trade interactions between the agents, comparable to collisions between molecules in a homogeneous gas. The macroscopic statistics of the models display wealth distributions that are in agreement with empirical data.

The main focus has been on the *risky market* approach (CPT) by Cordier et al [15], and on the model with *quenched saving propensities* (CCM) by Chakrabarti et al [9]. Both constitute refinements of the original idea developed by Angel [1]. For CPT, randomness — related to the unknown outcome of risky investments — plays the pivotal rôle. In contrast to Angel's original model, the market risk is defined in a way that breaks the *strict conservation* of wealth in microscopic trades and replaces it by *conservation in the statistical mean.* The founding idea of CCM is to incorporate individual trading preferences by assigning *personal saving propensities* to the agents. For suitable choices of the respective model parameters, both approaches are able to produce realistic Pareto tails in the wealth distribution. In direct comparison, the CPT model appears more natural, since the dependence of the stationary wealth distribution on the system parameters is more robust, and the steady state is exponentially attracting in contrast to algebraic relaxation for CCM.

An important finding is that one must be careful with numerical simulations when delicate features like Pareto tails are concerned. The simulated ensembles in kinetic Monte Carlo experiments are necessarily of finite size, and the qualitative features of finite-size systems differ in essential points from those proven for the continuous limit. Most remarkably, the finite-size CCM model exhibits non-trivial steady states with (apparent) Pareto tail in situations where the continuous model produces a Dirac distribution. Also, the typical time scale for relaxation in the deterministic CCM model changes from exponential convergence (finite size) to algebraic convergence (continuous).

It is arguable which kind of approach — finite size or continuous — provides the better approximation to reality. However, it is important to notice that the predictions are qualitatively different. This should be kept in mind in the further development of these (currently over-simplistic) models.

VIII. ACKNOWLEDGEMENTS

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